COMPATIBILITY INDICES BETWEEN PRIORITY VECTORS

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ABSTRACT

In the Analytic Hierarchy Process priorities are derived from judgments. Different priority vectors, however, can be obtained in the solution of a problem. The main objective of this article is to present two different compatibility indices between priority vectors that have been suggested. A comparison between the indices is presented, following a mixed qualitative-quantitative research approach.

Keywords: Compatibility Index, Garuti Compatibility Index, Saaty Compatibility Index, Priority Vector, Analytic Hierarchy Process, multi-criteria decision theory

http://dx.doi.org/10.13033/ijahp.v4i2.130

1. Introduction

As observed by Saaty (2011), in the Analytic Hierarchy Process (AHP), for intangibles, “Priorities are derived from judgments in a special way. The process involves a composition of priority vectors given as the columns of a matrix according to certain rules”. Saaty (1977) first established the consistency index, $\mu = (\lambda_{max} - n) / (n - 1)$, “as a measure of the consistency or reliability of information by an individual”. It was also noted that it is desirable to have μ near zero, to obtain consistency. If they were not close, the judgments may be revised and the consistency index may be improved. But, “improving consistency does not mean getting an answer closer to the real life solution”. It only means that judgments are closer to being logically related than randomly chosen.

Different priority vectors can be obtained in the solution of a Multi-Criteria Decision Making (MCDM) problem. As a matter of fact, different MCDM methods may yield different results when applied to the same problem (Zanakis, et al. 1998). Still, a single method application, such as AHP, can lead to different priorities. This can be a result of different individuals providing judgments or lapses in time when collecting judgments.

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That is, different priority vectors are often obtained in Group Decision Making (GDM). This situation was previously studied mainly by focusing on the ranks that different priorities imply (Emond and Mason, 2002). The present article aims to extend the study from the ordinal focus to a cardinal one, where differences in the priorities are considered.

Even when vectors are not identical, they can sometimes be considered close to each other. According to Saaty (2005), "when two vectors are close, we say they are *compatible*". The Saaty Compatibility Index, *S*, was the first developed measure of compatibility between priority vectors. This index uses the concept of the Hadamard Product, the element-wise product of two matrices. A different index has been proposed by Garuti (2007), the Garuti Compatibility Index, *G*, based on a physical interpretation of the Inner Product that includes the concept of vector projection.

The main objective of this article is to present the two compatibility indices between priority vectors that have been proposed: *G* and *S*. A comparison of these indices is presented based on the hypothetical and classical examples of priority vectors obtained with the AHP. That is, we intend to achieve the objective with a non-exhaustive number of examples. This is a mixed qualitative-quantitative research approach (Bryman and Bell, 2007).

Compatibility indices are a new theme in AHP theory and practice, and this is the first article about them to be published in a journal. Section 2 on theory presents the calculation procedures. Section 3 presents some examples of the usage of *G* and *S*. Section 4 ends this article with concluding remarks and possible uses for compatibility indices to be investigated in future works.

2. Theory Background

The Saaty Compatibility Index, *S*, between vectors **x** and **y** is obtained with Equation 1, where *n* is the number of elements of the vectors, **e** is a column-matrix with all elements equal to 1,$a_{ij} = x_i/x_j$, $b_{ij} = y_i/y_j$, and $\bullet$ is the Hadamard Product operator.

$$S = (1/n^2)\mathbf{e}^T\mathbf{A}\bullet\mathbf{B}^T\mathbf{e} \qquad (1)$$

The calculation procedure of *S* is explained in Saaty and Peniwati (2007, p. 148) as: "Given two sets of positive numbers, form the matrix of ratios of all the numbers in one set; then also form the matrix of ratios of all the numbers in the second set. Take the transpose of the second matrix. Multiply the two matrices element-wise (that is perform the Hadamard Product). Add all the numbers and divide by n^2."

One desirable property of a consistency index is that it should indicate that a vector is completely compatible with itself. For identical vectors, $S = 1$. It can be observed that this is true by substituting $\mathbf{x} = \mathbf{y}$, in the variables involved in creating Equation 1:

$$S = (1/n^2)(1 \quad 1 \quad \ldots \quad 1)\begin{pmatrix} x_1/x_1 & x_1/x_2 & \ldots & x_1/x_n \\ x_2/x_1 & x_2/x_2 & \ldots & x_2/x_n \\ \ldots & \ldots & & \ldots \\ x_n/x_1 & x_n/x_1 & \ldots & x_n/x_n \end{pmatrix} \bullet \begin{pmatrix} x_1/x_1 & x_2/x_1 & \ldots & x_n/x_1 \\ x_1/x_2 & x_2/x_2 & \ldots & x_n/x_2 \\ \ldots & \ldots & & \ldots \\ x_1/x_n & x_2/x_n & \ldots & x_n/x_n \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ \ldots \\ 1 \end{pmatrix} =$$

$$= (1/n^2)(1 \quad 1 \quad \ldots \quad 1)\begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \ldots & \ldots & & \ldots \\ 1 & 1 & \ldots & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ \ldots \\ 1 \end{pmatrix} = (1/n^2)\ (1 + 1 + \ldots + 1) = (1/n^2)\ (n^2) = 1$$

So when the two vectors are the same, $S = 1$ as shown above.

If $S \leq 1.1$ the two vectors are said to be compatible; otherwise, not (Saaty and Peniwati, 2007).

Table 1 presents three hypothetical priority vectors obtained with judgments provided by three different judges. Corresponding elements of Vectors 1 and 2 appear to be close to each other based on a cursory examination of the differences between them. Elements of Vectors1 and 3 do not even appear to be close. So, for Vectors 1 and 2, $S = 1.03$, indicating that they are indeed compatible. This result was expected. For Vectors 1 and 3, $S = 3.13$, indicating incompatibility between them.

Table 1

Examples of compatible and incompatible vectors

	Vector 1	Vector 2	Vector 3
Element 1	0.50	0.52	0.10
Element 2	0.40	0.41	0.60
Element 3	0.10	0.07	0.30

Table 2 presents two more priority vectors that appear, a priori, to be compatible. Vectors 4 and 5 have elements that appear close to each other in terms of differences between them, but their compatibility index $S = 1.63$ indicates incompatibility. This result was not expected since the vectors appeared close to each other.

Table 2
Another example of compatible vectors

	Vector 4	Vector 5
Element 1	0.45	0.49
Element 2	0.30	0.30
Element 3	0.20	0.20
Element 4	0.05	0.01

The Garuti Compatibility Index, *G*, between vectors **x** and **y** is obtained with Equation 2. This index is based on a physical interpretation of the inner product of two vectors, ‹**x**, **y**›, given by |**x**||**y**| cos α, where α is the angle between vectors **x** and **y**. For identical normalized vectors, $\alpha = 0$ and ‹**x**, **y**› = 1. For perpendicular (orthogonal) vectors, $\alpha = 90^o$ and ‹**x**, **y**› = 0. For identical normalized vectors, *G* = 1, meaning total compatibility. The minimum possible value is *G* = 0, indicating total incompatibility.

$$G = \sum_{i=1}^{n} \left[\frac{\min(x_i, y_i)}{\max(x_i, y_i)} \frac{(x_i + y_i)}{2} \right] \tag{2}$$

If *G*< 0.9, Garuti (2007) proposes that **x** and **y** should be considered as not compatible.

For Vectors 1 and 2 from Table 1, *G* = 0.95; for Vectors 1 and 3, *G* = 0.46. This way, using 0.9 as the lower limit for compatibility, *G* indicates compatibility between Vectors 1 and 2 and incompatibility between Vectors 1 and 3. For Vectors 4 and 5 (Table 2), *G* = 0.94. That is, contrary to the *S* index for this case, *G* indicates compatibility between these vectors. It is possible that the small element of 0.01 in Vector 5 may be the cause of the problem with the *S* index. It seems that *G* does not have the sensitivity to small element vectors that *S* does, i.e. *G* is not affected by small elements in the vector.

The sensitivity of *S* to small elements is due to its calculation procedure. That is, in this procedure one element x_i of a vector interacts with all elements from the vector, **x**. These interactions are the ratios that form the matrix **A**, $a_{ij} = x_i/x_j$. So, for instance, for Vectors 4 and 5, we have

$a_{41} = 0.05/0.45 = 1/9$

and

$b_{41} = 0.49/0.01 = 49$.

The multiplication of these elements give $a_{14}\, b_{14} = 49/9 \approx 5.444$ which is relatively greater than 1. As the sum of all elements of **A•B** must be close to n^2, if one element is much greater than 1, that forces S to be greater than the upper limit of 1. We also should note that these numbers represent priorities. Therefore it should be expected that a small priority, like 0.01 or 0.05, may not have a great influence in the assessment of the index. The better behavior of G cannot be generalized based only on two hypothetical examples, so in Section 3 we compute the G and S indices for two classical AHP validation exercises.

3. Results

Table 3 is an estimate of the remoteness of cities from Philadelphia; for each pair of cities the judgment that is entered is an estimate of how many times farther the more distant is from Philadelphia than the nearer one (Saaty, 1977).

Table 3

Distance from Philadelphia (adapted from Saaty, 1977)

	CAI	TYO	ORD	SFO	LGW	YMX	Eigenvector	Distance [miles]
Cairo (CAI)	1	1/3	8	3	3	7	0.263	5,729
Tokyo (TYO)	3	1	9	3	3	9	0.397	7,449
Chicago (ORD)	1/8	1/9	1	1/6	1/5	2	0.033	660
San Francisco (SFO)	1/3	1/3	6	1	1/3	6	0.116	2,732
London (LGW)	1/3	1/3	5	3	1	6	0.164	3,658
Montreal (YMX)	1/7	1/9	1/2	1/6	1/6	1	0.027	400

The Saaty Compatibility Index between the eigenvector and the vector of distances is S = 1.02. This is an indication that these vectors are compatible. It is interesting to note that the elements of the Eigenvector and Distance vector in Table 3 do not have the same unit. That is, elements from the eigenvector sum to 1, since the normalized eigenvector gives the relative priority of its elements, but the elements from the Distance vector sum to 20,628 miles, the total distance from all the cities to Philadelphia. As the computation procedure of S is based on the ratios between two elements, the vectors do not need to first be transformed by normalizing to have their compatibility measured by this index. In other words, if the elements of the eigenvector were multiplied by 20,628 miles, one will still obtain S = 1.02 for the index, the same as between the eigenvector and the vector of distances. The compatibility between the eigenvector and the vector of distances can be visually checked. That is, Tokyo is the most remote city from Philadelphia, followed by Cairo, London, and San Francisco, and so on, in that order.

To use G, the vector of distances, [0.278, 0.361, 0.032, 0.132, 0.177, 0.019], must be normalized. Then, the Garuti Compatibility Index between the eigenvector and the normalized vector of distances will be

G = 0.92 >0.9

Thus, *G* also shows the two vectors are compatible.

Table 4 presents comparisons done by students in an electrical engineering class estimating the consumption of electricity of common household appliances (Whitaker, 2007).

Table 4
Relative electricity consumption of household appliances (adapted from Whitaker, 2007)

	A	B	C	D	E	F	G	Eigenvector	Actual
Electric range (A)	1	2	5	8	7	9	9	.393	.392
Refrigerator (B)	1/2	1	4	5	5	7	9	.261	.242
TV (C)	1/5	1/4	1	2	5	6	8	.131	.167
Dishwasher (D)	1/8	1/5	1/2	1	4	9	9	.110	.120
Iron (E)	1/7	1/5	1/5	1/4	1	5	9	.061	.047
Radio (F)	1/9	1/7	1/6	1/9	1/5	1	5	.028	.028
Hairdryer (G)	1/9	1/9	1/8	1/9	1/9	1/5	1	.016	.003

The Saaty Compatibility Index between the eigenvector and the normalized vector of actual consumption is *S* = 1.46. This is an indication that these vectors are not compatible. The elements of these vectors have the same digit in the first decimal place, and the vectors have the same ordinal rank: [1, 2, 3, 4, 5, 6, 7], so because of this the vectors could be considered to be compatible with each other in spite of the value of *S*. Again *S* seems not to be a good indicator when one of the vectors has a much smaller element, .003 for the Hairdryer in the Actual vector. The Garuti Compatibility Index between the eigenvector and the normalized vector of actual consumption gives *G* = 0.92. Thus the Garuti Compatibility Index shows that the vectors are indeed compatible and it does not have any problem handling the small element in the vector.

4. Conclusions

This article presents two possible compatibility indices: *G* and *S*. The Saaty Compatibility Index was developed first and uses the concept of the Hadamard Product. When two vectors are identical, *S* = 1. The threshold $S \leq 1.1$ was established as an upper limit for compatible vectors. However, we presented two examples with *S*> 1.1 between two vectors that appeared to be relatively compatible. In the examples we have given, it seems that this compatibility index has a strong sensitivity to vectors with small elements. It is important to remember that those elements represent priorities. So, a small element corresponds to small priority and it should have a small influence over the index.

The Garuti Compatibility Index is also equal to 1 for identical priority vectors, and seems to better represent the influence of the element according to its size. That is, a small element will have a small influence over the index, while a big one will have a big influence. This appears to be an advantage of *G* when compared to *S*.

As the calculation process of *S* only uses ratios between the elements, the vectors do not need to have the same unit. That advantage of *S* against *G* can be particularly useful when comparing results obtained with AHP to real values or values obtained with other MCDM methods. However, priority is a concept that does not require a dimensional meaning so the normalization process required by *G* should not present a problem. More generally, the normalization process cannot alter the cardinality of a well-defined set of cardinal priorities.

The first theme for future research is a more in depth study of the threshold of 10% suggested for both indices of compatibility. Perhaps for small element vectors *S* should have an incompatibility index limit higher than 1.1; however, initial studies show that *S* may present a trend to divergence in the presence of small numbers. As in *G*'s computational procedure the min/max ratio between elements is multiplied by their arithmetic mean, and values for this index can only vary from zero to one. Thus, the image of the *G* function is the closed interval [0, 1] and the image of the *S* function is the left-bounded interval $[1, +\infty]$. A mathematical simulation should be carried out in order to study if $G < 0.9$ can be tolerated for vectors with higher *n*, as was done for the consistency index.

Other interesting themes for research connect the index *G* with AHP/ANP models. For instance, directly using the weights of the criteria obtained from the models to measure compatibility. One can also study the membership of an alternative for a given set, based on the compatibility (closeness) between the alternative and the set, for instance, by assessing patterns of behavior against some level of perturbation.

The main practical use of compatibility indices may be for Group Decision Making (GDM). The use of *G* and *S* can facilitate the effort to come to a consensus by quantifying and qualifying differences between priority vectors obtained from different experts. Since priority vectors are often different for the members of the group in GDM, the use of the compatibility index can numerically express how far, or how incompatible, a priority vector provided by one person is from the aggregated priority vector. So the use of *G* and *S* may be considered a scientific way to provide consensus between conflicting parts.

There are many possible fields of application for compatibility indices that include: diagnoses pattern recognition in medicine, psychiatry and psychology; measuring the degree of matching, or closeness, between buyers and sellers: Does the offer match the customer's need? How well does it match the need?; and conflict resolution. In GDM, compatibility indices can be used to measure the closeness between different value systems from the participants, assess the compatibility between different ways of thinking, clearly establish where they agree (or disagree) and by how much, making it easier to achieve a consensus.

A compatibility index makes it possible to assess the closeness between complex profiles in weighted environments and answer the important question of when close really means close.

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