ALIGNING THE MEASUREMENT OF TANGIBLES WITH INTANGIBLES AND NOT THE CONVERSE

Thomas L. Saaty
University of Pittsburgh
saaty@katz.pitt.edu

It is known in the AHP that when dealing with intangible alternatives they are compared with respect to the criteria and the resulting priorities are multiplied by the priorities of the respective criteria and then synthesized in either the distributive or the ideal mode. When the criteria are tangible the alternatives need to be measured on ratio scales. Since ratio scales of measurement differ, they must be standardized and then weighted by the priorities of the criteria to trade off a unit of one scale against a unit of another. Of course ratio scales suffer from the defect that they use a unit of \measurement uniformly whether the measurements are large or small. But the meaning and significance of these measurements may not reflect our actual preference for them because our ability to appreciate values differs when they are very large or very small. We need to obtain these measurements in relative form through prioritization and comparisons. If we insist on using them as they are, we must convert them to relative priorities by dividing each value by the total values. But we cannot do that without noting that the weights of the criteria in this case depend on the measurements of the alternatives with respect to these criteria.

The Analytic Hierarchy Process was developed to deal with models with intangible criteria or with both tangible and intangible criteria. Sometimes critics, in trying to show that the AHP is not a valid theory, use a model with only tangible criteria, and, because they need this for their validation against some real world result obtained by applying an arithmetic formula, they choose tangible criteria with the same ratio scale. They then normalize the results for the alternatives under each criterion, assume the weights of the criteria are equal and get the wrong answer.

How to Combine Normalized Measurements of Tangibles

When the alternatives are tangible they can be dealt with in the manner described below. The eventual concern is to combine them with intangibles in a mathematically plausible way. If there are several criteria all measured on the same tangible scale, these criteria depend on the measurements of the alternatives for their priority and cannot be treated as if they are independent and assigned priority weights in advance. To reduce the overall weights of the alternatives to relative priority form, first the criteria are given relative weights computed from the sum of the readings under each one to the total readings (on the same scale) under the other criteria. These weights are used to weigh the normalized readings under each criterion. Then the weighted relative readings are summed for each alternative with respect to all the criteria to obtain the overall priority of that alternative. In this manner the several criteria with different measurements for the alternatives under each are reduced to a single overall criterion with the alternatives having a normalized overall priority. That overall criterion is then compared in the usual way with other intangible criteria or overall tangible ones obtained in a similar way to the given overall tangible criterion. First here is an involving two tangible criteria C1 and C2 and three alternatives A1, A2 and A3 with respect to a single scale using dollar values.

Table 1
Unnormalized criteria and alternative weights from measurements in the same dollar scale for both criteria

Alternatives	Criterion C_1 Unnormalized weight = 1.0	Criterion C_2 Unnormalized weight = 1.0	Weighted Sum Unnormalized	Normalized or Relative values
A_1	200	150	350	350/1300=.269
A_2	300	50	350	350/1300=.269
A_3	500	100	600	600/1300=.462
Totals	1000	300	1300	1

Relative values require that criteria be examined as to their relative importance with respect to each other (on the average or on the whole). What is the relative importance of a criterion, or what numbers should the tangible criteria be assigned that reflect their relative importance? In the AHP when the values of the alternatives are measured on the same scale for several criteria, it is necessary that these criteria have priorities that reflect the proportion of the sum of the values under them to the total under all criteria. Multiplying the relative values of the alternatives by the relative values of the criteria and adding gives the final column of Table 2. The outcome coincides with the last column of Table 1.

Table 2
Normalized criteria weights and normalized alternative weights from measurements on the same ratio scale (additive synthesis)

Alternatives	Criterion C_1 Normalized weight = 1000/1300=0.7692	Criterion C_2 Normalized weight = 300/1300=0.2308	Weighted Sum
A_1	200/1000	150/300	350/1300=.269
A_2	300/1000	50/300	350/1300=.269
A_3	500/1000	100/300	600/1300=.462

It is clear that the following computations are involved in the process of finding the necessary weights for the criteria. We must solve a system of simultaneous equations

$$\sum_j a_{ij} x_j / \sum_i a_{ij} = \sum_j a_{ij} / \sum_i \sum_j a_{ij}$$

from which we get

$$x_j = \sum_j a_{ij} / \sum_i \sum_j a_{ij}$$

Comparing Tangibles that Require Arithmetic Formulas

This example of expecting to rank rectangles by perimeter length by making pairwise comparisons of the respective lengths and widths, and not getting the right answer, is due to Claudio Garuti of Chile. The problem is that one expects the AHP hierarchy to know the formula P = L × W for the perimeter of rectangles as they relate to the length L and width W without providing the right data. In a case like this, AHP can give the right answer, but information must be provided by setting up criteria for length and width and weighing the criteria by the proportion of the total linear measure they control.

How to Combine Tangibles with Intangibles

Assume that a family is considering buying a house and there are three houses to consider. Four factors dominate their thinking as in the hierarchy of Figure 1: the price of the house, the remodeling costs, the size of the house as reflected by its footage and the style of the house which is an intangible. They have looked at three houses with data shown in Table 3 below on the quantifiables.

Goal Choosing the Best House

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Criteria Price Remodeling Size Style

Alternatives A1 A2 A3

Figure 1 Three level hierarchy

Table 3
Values of the alternatives with respect to the criteria; the first two are in thousands of dollars, the third in square feet and the forth in house styles to be prioritized

	Price	Remodeling	Size	Style
A	200	150	3000	Colonial
B	300	50	2000	Ranch
C	500	100	5500	Split Level

First we normalize each of the columns of quantifiable factors as shown in Table 4.

Table 4
Normalized Numerical Columns

	Price	Remodeling	Size	Style
A	200/1000	150/300	3000/10500	Colonial
B	300/1000	50/300	2000/10500	Ranch
C	500/1000	100/300	5500/10500	Split Level

Next we combine the two factors with the common dollar scale by multiplying the numbers in the first column by 1000/1300 (the weight of C1) and those in the second by 300/1300 (the weight of C2) and adding as in Table 5.

Table 5
Combined First Two Columns that were Measured in Dollars

	Cost	Size	Style
A	350/1300	3000/10500	Colonial
B	350/1300	2000/10500	Ranch
C	600/1300	5500/10500	Split Level

Next we establish priorities for the criteria and also for the three different styles through paired comparisons as in the usual way. The resulting priorities are used to weigh the normalized values of the alternatives as they are given here.

Observation: We can alternatively, and better, perform paired comparisons on the measurements of the alternatives themselves and then weigh the four criteria through comparisons. The detailed approach illustrated above, would be probably used by a disinterested party for whom the dollar costs are simply numbers of no personal significance, although they might have substantially different significance to the family. To treat dollars (or any other scale measurements) directly without comparisons can yield misleading results. In this example, tangible and intangible factors had to be compared. Their priorities are used to weigh the priorities of the alternatives. These priorities are obtained by converting measurements to priorities directly through normalization (seldom justified) or by interpreting their relative importance through judgment (essential where no measurements are possible).

There are two other cases to consider. The first is when gambles and expected values are involved between criteria measured on the same scale which are then combined accordingly into a single criterion for that scale. The other may involve relations between subcriteria from a common scale and intangible criteria. The tangible subcriteria must be normalized with the other subcriteria as above and then combined with the intangibles accordingly. The other subcriteria from the given scale would be combined into a single scale criterion as above and one proceeds in the usual way to complete the prioritization process.

In justifying the theory in a mathematical context, the invariance principle is the most important idea to remember. Priorities are derived from judgments in a special way. The process involves a composition of priority vectors given as the columns of a matrix according to certain rules. For example, in additive composition, priority vectors are weighed (multiplied) by the priority of the element with respect to which they were derived and then summed over all such elements to obtain an overall composite priority vector. In particular, the columns of a judgment matrix are themselves priority vectors with respect to each element represented at the top of the matrix, and their composition, by multiplying by the priority of that element and adding across the elements, yields the composite which can be written as priority vector of all the elements. In the additive case, what we just described above $\sum_{j=1}^{n} a_{ij} w_j = c w_i$ i=1,...,n where c > 0 (which may be taken as c = 1) is a constant of proportionality so that the derived vector is a similarity transformation of the original vector and thus both belong to the same ratio scale. This is the principle of invariance of priorities. The foregoing problem has a unique solution in eigenvectors.

AHP model and ANP model for perimeter of rectangles

The implicit assumption the critics make is that somehow the hierarchical structure is sufficient to give an expected answer regardless of the criteria weights. Having the proper criteria weights is essential in AHP for it is in weighing the alternative priorities by the criteria priorities that the priorities are converted to commensurate absolute scales so they can be combined using addition in the synthesis process. To get the right weights for the criteria one must make the weight of each criterion proportionate to the amount of the total resource it controls, or alternatively, combine the measures using whatever arithmetic formula is being used to combine separate measures in the validation example, before normalizing, thus ending up with a single set of measures which can then be normalized. The example given here shows how to reproduce priority results for measures of tangibles involving formulas, even when it is not a simple add and weight formula as in the example below involving perimeters of rectangles that was suggested by Claudio Garuti of Chile.

Validating AHP and ANP results in situations involving tangibles and arithmetic formulas

We have four alternative rectangles with dimensions of length and width given as shown in Table 6. We wish to prioritize the rectangles by the lengths of their perimeters. This is an example of tangibles that are being combined using formulas and the question is whether AHP results match what would be expected using arithmetic. We show here that both AHP and ANP give valid answers.

Table 6
Four rectangles and their perimeters

Alternatives	Length	Width	Perimeter	Perimeter Normalized
Rectangle 1	9	1	20	.25
Rectangle 2	8	2	20	.25
Rectangle 3	7	3	20	.25
Rectangle 4	6	4	20	.25
SUM	30	10	20	

In attempting to validate the AHP people often set up a model of criteria and alternatives where measurements of the alternatives are made on both criteria using some existing ratio scale. They then give the criteria default equal priorities, reasoning that as both measurements on both criteria are made in the same ratio scale, they should be equally weighed. They then normalize the alternative measurements under each criterion, wrongly synthesize by weighing the normalized measures by the criteria weights and adding and do not get the expected priorities as shown in Table 7. This example will show why. The expected answers are that the rectangles should have equal priority for perimeter length because we have an arithmetic formula that says $P = 2L + 2W$ and applying this formula gives the answers for the perimeter, which we can then normalize and note that this is equivalent to priority and as all the rectangles have the same perimeter measurement, they *should* have equal priority.

Table 7
Priorities obtained by normalizing then synthesizing with equal criteria weights

Alternatives	Criterion 1 Length .5	Criterion 2 Width .5	Priorities
Rectangle 1	9/30	1/10	=9/30×.5+1/10×.5=.2
Rectangle 2	8/30	2/10	=8/30×.5+2/10×.5=.2333
Rectangle 3	7/30	3/10	=7/30×.5+3/10×.5=.2666
Rectangle 4	6/30	4/10	=6/30×.5+4/10×.5=.3000
SUM	30	10	

The correct way to set up the problem is to give the criteria their proper importance by weighing them by the proportion of the resource they control as shown in Table 8. The total linear measure involved is 40 units. Criterion 1 controls 30/40 or .75 of the total resource, the number of units, and Criterion 2 controls 10/40 or .25, so these are the proper weights to use and the priorities are correct as shown below.

Table 8
Priorities obtained by weighing criteria proportionately to the total resource controlled

Alternatives	Criterion 1 Length 30/40=.75	Criterion 2 Width 10/40=.25	Priorities
Rectangle 1	9/30	1/10	9/30×.75+1/10×.25=.25
Rectangle 2	8/30	2/10	8/30×.75+2/20×.25=.25
Rectangle 3	7/30	3/10	7/30×.75+3/10×.25=.25
Rectangle 4	6/30	4/10	6/30×.75+4/10×.25=.25
SUM	30	10	

A second method that gives one the right answer is to use an ANP model that involves dependence, feedback and input of the raw data directly in the supermatrix as shown in Table 9. The SuperDecisions software can be used to set up a supermatrix as follows, first using the direct data, then normalizing the columns that are the rectangle measures with respect to the criteria and the criteria values with respect to the rectangles.

Table 9
ANP supermatrix with raw data

	C_1 (length)	C_2 (width)	Rectangle1	Rectangle2	Rectangle3	Rectangle4
C_1 (length)	0	0	9	8	7	6
C_2 (width)	0	0	1	2	3	4
Rectangle1	9	1	0	0	0	0
Rectangle2	8	2	0	0	0	0
Rectangle3	7	3	0	0	0	0
Rectangle4	6	4	0	0	0	0

The SuperDecisions software for the ANP will automatically convert the raw input data into priorities by normalizing it as shown in Table 10.

Table 10
ANP supermatrix of priorities

	C_1 (length)	C_2 (width)	Rectangle1	Rectangle2	Rectangle3	Rectangle4
C_1 (length)	0	0	0.9	0.8	0.7	0.6
C_2 (width)	0	0	0.1	0.2	0.3	0.4
Rectangle1	0.3	0.1	0	0	0	0
Rectangle2	0.26666667	0.2	0	0	0	0
Rectangle3	0.23333333	0.3	0	0	0	0
Rectangle4	0.2	0.4	0	0	0	0

Raising the matrix to powers until it stabilizes in the limit supermatrix is shown in Table 11. In this example, the powers of the matrix are actually cycling between two steady states that give the same final priorities. The supermatrix automatically gives the priorities of both the criteria and the rectangles without any special intercession on the part of the user.

Table 11
The limit supermatrix of the ANP model with final priorities

	C_1 (length)	C_2 (width)	Rectangle1	Rectangle2	Rectangle3	Rectangle4
C_1 (length)	0.0000	0.0000	0.7500	0.7500	0.7500	0.7500
C_2 (width)	0.0000	0.0000	0.2500	0.2500	0.2500	0.2500
Rectangle1	0.2500	0.2500	0.0000	0.0000	0.0000	0.0000
Rectangle2	0.2500	0.2500	0.0000	0.0000	0.0000	0.0000
Rectangle3	0.2500	0.2500	0.0000	0.0000	0.0000	0.0000
Rectangle4	0.2500	0.2500	0.0000	0.0000	0.0000	0.0000

As shown in Table 11, the priorities are 0.75 for the length criterion, 0.25 for width and 0.25 for each of the rectangles meaning their perimeter lengths are equal. These results match the expected results in Table 6.

Rating Scales

Rating scale values of different alternatives must be treated as ratio scale readings of tangibles because their already derived priorities belong to a scale of normalized values of an absolute scale. Rating scale numbers may be the same for two criteria but their interpretation is different as priorities and they cannot be combined directly like numbers from the same ratio scale like money can. They must be weighed by the importance of the criteria, then combined.

Summarizing

As a final comment we note that scales of measurement are very recent in human history and that before the last millennium people used their biological talent to make comparisons and that talent will not disappear just because we invented scales. In fact, even with scales, comparisons are used to determine how good the results are. In addition, no matter what scheme people use to evaluate alternatives with respect to criteria by rating or by comparison, comparisons will always be needed to prioritize criteria because their importance varies from one decision to another (they are of essence intangible) and also because their parent criterion or goal is an intangible. Thus, our thinking needs to always make the particular or tangible a *specia*l case of the general or intangible and not the converse as some people have been inclined to do.

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