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Note on the permutation group associated to E-polynomials*

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Abstract: This is a continuation of our project which focuses on E-polynomials and the related combinatorics. A pair of groups appearing in the definition of E-polynomials yields the permutation group. In this paper, we determine the multi-matrix structures of the centralizer algebras of the tensor representations of this permutation group.

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1. Introduction

The purpose of this note is to investigate the centralizer algebras of the tensor representation of the finite group G that arises from E-polynomials in genus 1. We also discuss the association scheme which appears as the first case. Our results can be regarded as combinatorial properties of E-polynomials.

The study of Eisenstein series is of interest because of its importance in number theory. We defined and studied finite analogue of Eisenstein series from combinatorial point of view. We call it an E-polynomial in some of our previous papers [8, 10, 11].

Let g be a positive integer. Then, an action of $\Gamma_g = Sp(2g, \mathbf{Z})$ on the theta series induces a finite subgroup H_g of $GL(2^g, \mathbf{C})$. It is known that the invariant ring of H_g is closely related to the ring of modular forms for Γ_g. As to this direction in higher genus, see [9, 14]. There exist some results based on the fact that the weight enumerator of a self-dual and doubly even binary code is invariant under the action of H_g [3, 4, 13, 15]. The correspondence existing among the invariant theory of the finite groups,

the theory of modular forms, and combinatorics such as coding theory motivates the notion and study of E-polynomials.

Let K_g be a subgroup of H_g fixing x_0. An E-polynomial of degree ℓ in degree g is defined by

$$\varphi_\ell(x_a : \ a \in \mathbf{F}_2^g) = \frac{1}{|H_g|} \sum_{K_g \backslash H_g \ni \sigma} (\sigma x_0)^\ell.$$

The rings generated by E-polynomials have been studied in [10, 11] (*cf.* [8]). Moreover, the representation theory of H_1 is investigated and the centralizer rings of the tensor representation of H_1 are determined in [6]. Since the character table of H_1 is used in this study, it is reproduced at the end of this paper from [6].

As usual, let $\mathbf{C}$ denote the complex number field. We denote by $\mathcal{M}_d$ the matrix algebra of degree d over $\mathbf{C}$. For simpliticy, let $n\mathcal{M}_d$ denote $\underbrace{\mathcal{M}_d \oplus \cdots \oplus \mathcal{M}_d}_{n}$.

2.   Preliminaries

We follow the notations in [6].

Let H_1 be the group of order 96 generated by $\frac{1+i}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ and $K_1 = \langle \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \rangle$ be its subgroup of orer 4. A set of representatives of $K_1 \backslash H_1$ is given by

$$1, T, T^2, T^3, T^4, T^5, T^6, T^7, TD, TD^2, TD^3, T^3D, T^3D^2, T^3D^3,$$
$$T^5D, T^5D^2, T^5D^3, T^7D, T^7D^2, T^7D^3, TD^2T, T^3D^2T, T^5D^2T, T^7D^2T$$

and we name each of the classes $1, 2, \ldots, 24$, for example, $24 = K_1 T^7 D^2 T$. Now, we set $\Omega = \{1, 2, \ldots, 24\}$, then the action of H_1 on Ω gives a transitive permutation representation of H_1. We observe that T and D correspond to

$$t = (1, 2, 3, 4, 5, 6, 7, 8)(9, 14, 12, 17, 15, 20, 18, 11)(10, 21, 13, 22, 16, 23, 19, 24),$$

and

$$d = (2, 9, 10, 11)(4, 12, 13, 14)(6, 15, 16, 17)(8, 18, 19, 20)(21, 22, 23, 24),$$

respectively. Consequently, we obtain a faithful permutation representation $G = \langle t, d \rangle$ of H_1. The group G is an imprimitive permutation group of order 96 on the 24 points. Under the correspondence

$$g \mapsto (\delta_{\alpha^g, \beta})_{\alpha, \beta \in \Omega},$$

we may regard G as a matrix group.

3.   Results

We follow the argument presented in the papers [6, 7]. Let us denote by χ the permutation character of G. It is known that $\chi(g)$ is the number of $\alpha \in \Omega$ which is fixed by g.

Proposition 3.1. $\chi = \chi_1 + \chi_5 + \chi_9 + \chi_{10} + \chi_{12} + \chi_{13} + \chi_{14} + \chi_{15} + \chi_{16}$.

Proof.   Let

$$\chi = m_1\chi_1 + m_2\chi_2 + \cdots + m_{16}\chi_{16}.$$

Here we remind that χ_1 is the identity character of G. Since the characters depend on the conjugacy classes, $\chi(\mathfrak{C})$ denotes the value of χ at a conjugacy class $\mathfrak{C}$. Then, we have that

$$\big(\chi(\mathfrak{C}_1) \ \chi(\mathfrak{C}_2) \ \ldots \ \chi(\mathfrak{C}_{16})\big) = \big(m_1 \ m_2 \ \ldots \ m_{16}\big)\mathbf{X},$$

where $\mathbf{X}$ denotes the character table of G presented at the end of this paper. If we know the explicit values $\chi(\mathfrak{C}_i)$'s on the left-hand side, we multiply $\mathbf{X}^{-1}$ on both sides from the right to get the result.

Suppose $g \in \mathfrak{C}$ for a conjugacy class $\mathfrak{C}$, then $\chi(\mathfrak{C})$ is the number of $\alpha \in \Omega$ which is fixed by g. Therefore the values of $\chi(\mathfrak{C}_i)$'s can be given by

$$\big(\chi(\mathfrak{C}_1) \ \chi(\mathfrak{C}_2) \ \ldots \ \chi(\mathfrak{C}_{16})\big) = (24, 0, 0, 0, 0, 0, 4, 0, 0, 0, 0, 0, 4, 0, 4, 0).$$

This completes the proof of Proposition 3.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Before proceeding further, we shall interpret this proposition from the perspective of Wielandt [17]. Let $\mathfrak{A}$ be the centralizer ring of the matrix group G. Then Proposition 3.1 shows that $\mathfrak{A}$ is commutative and of dimension 9 and a basis can be obtained as follows. Let G_1 be the stabilizer of a point $1 \in \Omega$ in G. Then G_1 has 9 orbits

$$\{1\}, \{3\}, \{5\}, \{7\}, \{2, 9, 10, 11\}, \{4, 12, 13, 14\}, \{6, 15, 16, 17\}, \{8, 18, 19, 20\}, \{21, 22, 23, 24\}.$$

With each orbit Δ we associate a matrix $\mathfrak{V}(\Delta) = (v^\Delta_{\alpha,\beta})$ by

$$v^\Delta_{\alpha,\beta} = \begin{cases} 1 & {}^\exists g \in G \text{ such that } \ 1^g = \beta \text{ and } \alpha^{g^{-1}} \in \Delta, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for $\Delta = \{1\}$, we have that

$$\mathfrak{V}(\{1\}) = I$$

where I is the 24×24 identity matrix. Then the matrices $\mathfrak{V}(\Delta)$'s form a basis for $\mathfrak{A}$.

Next we shall connect $\mathfrak{A}$ with the theory of association schemes ([1]. *cf.* [16]). Under the ordering of the orbits given above, we denote the matrices $\mathfrak{V}(\Delta)$ by $A_0 = I, A_1, \ldots, A_8$. We observe that the matrices A_0, A_2, A_8 are symmetric and

$${}^tA_1 = A_3, {}^tA_4 = A_7, {}^tA_5 = A_6.$$

Then the set of matrices $\mathcal{X} = \{A_0, A_1, \ldots, A_8\}$ forms a non-symmetric commutative association scheme of class 8. The algebra $\mathfrak{A}$ is also called the Bose-Mesner algebra of the association scheme $\mathcal{X}$. With regards to the commutativity and the normality of A_i's, $\mathcal{X}$ has another basis $E_0 = \frac{1}{24}J, E_1, \ldots, E_8$ of the primitive idempotents, where J is a 24×24 matrix with every entry 1. The transformation matrix P is defined by

$$\big(A_0 \ A_1 \ \ldots \ A_8\big) = \big(E_0 \ E_1 \ \ldots \ E_8\big)P$$

which can also be called the first eigenmatrix of $\mathcal{X}$. In our case, the first eigenmatrix is

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 4 & 4 & 4 & 4 & 4 \\
1 & 1 & 1 & 1 & -2 & -2 & -2 & -2 & 4 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -4 \\
1 & i & -1 & -i & 2+2i & -2+2i & -2-2i & 2-2i & 0 \\
1 & i & -1 & -i & -1-i & 1-i & 1+i & -1+i & 0 \\
1 & -i & -1 & i & 2-2i & -2-2i & -2+2i & 2+2i & 0 \\
1 & -i & -1 & i & -1+i & 1+i & 1-i & -1-i & 0 \\
1 & -1 & 1 & -1 & 2i & -2i & 2i & -2i & 0 \\
1 & -1 & 1 & -1 & -2i & 2i & -2i & 2i & 0
\end{pmatrix}.$$

It turns out that $\mathcal{X}$ is isomorphic to No.349 in the list [5].

Summing up, we have obtained the following theorem.

Theorem 3.2. *The centralizer ring of the transitive permutation group G_1 on the 24 points is a commutative association scheme of degree 8 which is isomorphic to No.349 in [5].*

We proceed to the centralizer algebra of the tensor representation of G. Let $\mathfrak{A}^{(k)}$ be the centralizer algebra of the k-th tensor representation of G. Notice $\mathfrak{A} = \mathfrak{A}^{(1)}$. We set

$$\chi^{\otimes k} = \sum_{i=1}^{16} d_i^{(k)} \chi_i$$

and

$$\overrightarrow{d^{(k)}} = (d_1^{(k)}, d_2^{(k)}, \ldots, d_{16}^{(k)}).$$

We are going to determine the coefficients $d_i^{(k)}$ explicitly. By Proposition 3.1, we already knew

$$\overrightarrow{d^{(1)}} = (1, 0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1).$$

We shall consider the matrix A such that

$$\begin{pmatrix} \chi\chi_1 \\ \chi\chi_2 \\ \vdots \\ \chi\chi_{16} \end{pmatrix} = A \begin{pmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_{16} \end{pmatrix}.$$

The matrix A can be calculated explicitly as

$$
\begin{aligned}
A &= \mathbf{X}.\mathrm{diagonal}([\chi(\mathfrak{C}_1), \chi(\mathfrak{C}_2), \ldots, \chi(\mathfrak{C}_{16})]).\mathbf{X}^{-1} \\
&= \mathbf{X}.\mathrm{diagonal}([24, 0, 0, 0, 0, 0, 4, 0, 0, 0, 0, 0, 4, 0, 4, 0]).\mathbf{X}^{-1} \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
0 & 0 & 1 & 1 & 0 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 \\
0 & 1 & 1 & 0 & 1 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 2 & 1 & 2 & 2 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 0 & 2 & 1 & 1 & 2 & 2 & 2 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\
1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 2 \\
0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 3 & 2 & 2 & 2 & 3 & 3 \\
1 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 3 & 2 & 2 & 3 & 3 \\
1 & 1 & 1 & 0 & 2 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 3 & 2 & 3 & 3 \\
1 & 1 & 0 & 1 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 \\
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4
\end{pmatrix}.
\end{aligned}
$$

We have thus gotten

$$\overrightarrow{d^{(k)}} = \overrightarrow{d^{(k-1)}} A \quad (k \geq 2).$$

Then we have only to apply linear algebra. The matrix A is diagonalizable and we get[1]

$$P^{-1}AP = \mathrm{diagonal}([4, 4, 4, 24, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0])$$

---

[1] *The matrices $\mathbf{X}$, A, and P can be found at [12].*

for some non-singular matrix P. We apply this calculation to

$$\overrightarrow{d^{(k)}} = \overrightarrow{d^{(1)}} P A^{k-1} P^{-1}$$
$$= (a_k, b_k, b_k, b_k, c_k, d_k, d_k, d_k, e_k, e_k, f_k, g_k, g_k, g_k, h_k, h_k),$$

where

$$a_k = 6 \cdot 24^{k-2} + 3 \cdot 4^{k-2},$$
$$b_k = 6 \cdot 24^{k-2} - 4^{k-2},$$
$$c_k = 12 \cdot 24^{k-2} + 2 \cdot 4^{k-2},$$
$$d_k = 12 \cdot 24^{k-2} - 2 \cdot 4^{k-2},$$
$$e_k = 12 \cdot 24^{k-2} + 2 \cdot 4^{k-2},$$
$$f_k = 18 \cdot 24^{k-2} - 3 \cdot 4^{k-2},$$
$$g_k = 18 \cdot 24^{k-2} + 4^{k-2},$$
$$h_k = 24^{k-1}.$$

We have thus obtained the following multi-matrix structures of $\mathfrak{A}^{(k)}$.

Theorem 3.3. *We have that*

$$\mathfrak{A}^{(k)} \cong \begin{cases} 9\mathcal{M}_1 & k = 1, \\ \mathcal{M}_{a_k} \oplus 3\mathcal{M}_{b_k} \oplus \mathcal{M}_{c_k} \oplus 3\mathcal{M}_{d_k} \oplus 2\mathcal{M}_{e_k} \oplus \mathcal{M}_{f_k} \oplus 3\mathcal{M}_{g_k} \oplus 2\mathcal{M}_{h_k} & k \geq 2, \end{cases}$$

where $a_k, b_k, \ldots, h_k$ are given above.

Corollary 3.4. (1) $\mathfrak{A}^{(k)}$ *is commutative if and only if $k = 1$.*

(2) $\dim \mathfrak{A}^{(k)} = 3456 \cdot 24^{2(k-2)} + 48 \cdot 4^{2(k-2)}$ $(k \geq 1)$.

The second assertion of Corollary 3.4 is obtained by taking the sum of the dimensions of the simple components. It would be interesting if we interpret this number from combinatorial point of view, see [6]. This paper is concluded with a small table of $\dim \mathfrak{A}^{(k)}$.

k	1	2	3	4	5	6
$\dim \mathfrak{A}^{(k)}$	9	3504	1991424	1146630144	660452081664	380420288937984

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H_1	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{C}_4$	$\mathfrak{C}_5$	$\mathfrak{C}_6$	$\mathfrak{C}_7$	$\mathfrak{C}_8$	$\mathfrak{C}_9$	$\mathfrak{C}_{10}$	$\mathfrak{C}_{11}$	$\mathfrak{C}_{12}$	$\mathfrak{C}_{13}$	$\mathfrak{C}_{14}$	$\mathfrak{C}_{15}$	$\mathfrak{C}_{16}$
	1	T	T^2	T^3	T^4	T^6	D	DT	DT^2	DT^3	DT^4	DT^5	DT^6	DT^7	D^2	D^2T^2
order	1	8	4	8	2	4	4	6	4	12	4	3	4	12	2	4
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	1	1	1
χ_3	1	$-i$	-1	i	1	-1	i	1	$-i$	-1	i	1	$-i$	-1	-1	1
χ_4	1	i	-1	$-i$	1	-1	$-i$	1	i	-1	$-i$	1	i	-1	-1	1
χ_5	2	0	2	0	2	2	0	-1	0	-1	0	-1	0	-1	2	2
χ_6	2	0	-2	0	2	-2	0	-1	0	1	0	-1	0	1	-2	2
χ_7	2	0	$-2i$	0	-2	$2i$	$-1+i$	1	$1+i$	$-i$	$1-i$	-1	$-1-i$	i	0	0
χ_8	2	0	$2i$	0	-2	$-2i$	$-1-i$	1	$1-i$	i	$1+i$	-1	$-1+i$	$-i$	0	0
χ_9	2	0	$-2i$	0	-2	$2i$	$1-i$	1	$-1-i$	$-i$	$-1+i$	-1	$1+i$	i	0	0
χ_{10}	2	0	$2i$	0	-2	$-2i$	$1+i$	1	$-1+i$	i	$-1-i$	-1	$1-i$	$-i$	0	0
χ_{11}	3	1	3	1	3	3	-1	0	-1	0	-1	0	-1	0	-1	-1
χ_{12}	3	-1	3	-1	3	3	1	0	1	0	1	0	1	0	-1	-1
χ_{13}	3	i	-3	$-i$	3	-3	i	0	$-i$	0	$-i$	0	$-i$	0	1	-1
χ_{14}	3	$-i$	-3	i	3	-3	$-i$	0	i	0	i	0	i	0	1	-1
χ_{15}	4	0	$-4i$	0	-4	$4i$	0	-1	0	i	0	1	0	$-i$	0	0
χ_{16}	4	0	$4i$	0	-4	$-4i$	0	-1	0	$-i$	0	1	0	i	0	0