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The bicyclic semigroup as the quotient inverse semigroup by any gauge inverse submonoid

Research Article

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Abstract: Every gauge inverse submonoid (including Jones-Lawson's gauge inverse submonoid of the polycyclic monoid P_n) is a normal submonoid. In 2018, Alyamani and Gilbert introduced an equivalence relation on an inverse semigroup associated to a normal inverse subsemigroup. The corresponding quotient set leads to an ordered groupoid. In this note we shall show that this ordered groupoid is inductive if the normal inverse subsemigroup is a gauge inverse submonoid and the corresponding quotient inverse semigroup by any guage inverse submonoid is isomorphic either to the bicyclic semigroup or to the bicyclic semigroup with adjoined zero.

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1. Introduction

An equivalence relation $\simeq_N$ on an inverse semigroup S associated to a normal inverse subsemigroup N is introduced in [1]. Usually, it is not a congruence on S. Following [1] the quotient set $S/\simeq_N$ (also denoted by $S/\!\!/N$) leads to an ordered groupoid [1, Theorem 3.6]. If this ordered groupoid is inductive then the set of all morphisms, that is $S/\!\!/N$, equipped with the "pseudoproduct" $\otimes$ ([3, page 112]) forms an inverse semigroup (see [3, Proposition 4.1.7 (1)]), and we say, by abuse of language (since $\simeq_N$ is not necessary a congruence), that this inverse semigroup $(S/\!\!/N, \otimes)$ is the quotient inverse semigroup of S by the normal inverse subsemigroup N.

The gauge inverse monoid G_M is a special submonoid of such a combinatorial bisimple (0-bisimple) inverse monoid $\mathbb{S}(M)$ for which the submonoid M of right units is an ℓ-RILL monoid (see [5]). Any gauge inverse submonoid is normal ([5, Proposition 5.6]). Jones-Lawson's gauge inverse monoid is the gauge inverse submonoid (denoted by G_n) of the polycyclic monoid P_n ([2, Section 3]).

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The case of the polycyclic monoid P_n is examined in Example 3.11 from [1]. The conclusion of this examination is that $P_n /\!\!/ G_n$ is isomorphic to the Brandt semigroup on the set of non-negative integers. In fact the product "$[(u,v)]_{G_n}[(s,t)]_{G_n} = [(u,t)]_{G_n}$" considered at the end of Section 3 in [1] is the composition of two morphisms (if it is defined) in the corresponding ordered groupoid and it is not the pseudoproduct $\otimes$ which defines the quotient inverse semigroup $P_n /\!\!/ G_n$.

The aim of this note is to show that for any gauge inverse submonoid G_M, the quotient inverse semigroup $(\mathbb{S}(M) /\!\!/ G_M, \otimes)$ is isomorphic either to the bicyclic semigroup B or to the bicyclic semigroup with adjoined zero B^0.

In the next section, we will survey the background results, particularly from [3] (Subsection 2.1), [1] (Subsection 2.2) and [5] (Subsection 2.3), needed to understand this paper. The symbol $\circ$ is used only for composition (from right to left) of two morphisms.

2. Background. Ordered groupoids, normal inverse subsemigroups and gauge inverse submonoids

2.1. Ordered groupoids

A *groupoid* $\mathcal{G}$ is a small category in which every morphism is an isomorphism, meaning that for any morphism $f : X \to Y$ there is a morphism $f^{-1} : Y \to X$ such that $f^{-1} \circ f = 1_X$ and $f \circ f^{-1} = 1_Y$, where 1_X and 1_Y are the identity morphisms of X and Y, respectively. A groupoid $\mathcal{G}_{\mathcal{X}}$ is said to be *connected simple system* on the set $\mathcal{X}$ (or simplicial groupoid on $\mathcal{X}$) if the set of objects $Ob\mathcal{G}_{\mathcal{X}} = \mathcal{X}$ and there is exactly one morphism between any two objects. We call the groupoid $\mathcal{G}^0_{\mathcal{X}}$ obtained from $\mathcal{G}_{\mathcal{X}}$ by adjoining an extra object 0 such that the set of morphisms from X to Y is empty if either $X = 0, Y \neq 0$ or $X \neq 0, Y = 0$ and it is a singleton if $X = Y = 0$, the *connected simple system with adjoined* 0.

A groupoid $\mathcal{G}$ is said to be *ordered* if the set of all morphisms $Mor(\mathcal{G})$ of $\mathcal{G}$ is equipped with a partial order $\preceq$ such that:

(O_1) $f \preceq g$ implies $f^{-1} \preceq g^{-1}$;

(O_2) If $f \preceq g$, $f' \preceq g'$ and $f \circ f'$ and $g \circ g'$ are defined then $f \circ f' \preceq g \circ g'$;

(O_3) If $1_Z \preceq 1_X$ and $f : X \to Y$ then there exists a unique morphism $f|_Z : Z \to \bullet$ called the *restriction* of f to Z such that $f|_Z \preceq f$;

(O_4) If $1_Z \preceq 1_Y$ and $f : X \to Y$ then there exists a unique morphism $f|^Z : \bullet \to Z$ called the *corestriction* of f to Z such that $f|^Z \preceq f$;

The axiom (O_4) is a consequence of axioms $(O_1) - (O_3)$.

An inverse semigroup S (i.e. a semigroup S in which every element $s \in S$ has a unique inverse $s^{-1} \in S$ in the sense that $s = ss^{-1}s$ and $s^{-1} = s^{-1}ss^{-1}$) can be considered as an ordered groupoid $\mathcal{G}(S)$ in which the set of objects is the set of idempotents $E(S)$ of S, the set of morphisms from e to f is the set $\{s \in S | s^{-1}s = e \text{ and } ss^{-1} = f\}$ and the composition $s \circ t$ of two morphisms s and t

$$t^{-1}t \xrightarrow{t} tt^{-1} = s^{-1}s \xrightarrow{s} ss^{-1}$$

is the usual product st in S (i.e., the composition is just the restriction of the multiplication of S to composable pairs). The partial order on the set of all morphisms of $\mathcal{G}(S)$ is the natural partial order $\leq$ on the inverse semigroup S, i.e. $s \leq t \Leftrightarrow s = ss^{-1}t$ (or equivalently $s = ts^{-1}s$). In the ordered groupoid $\mathcal{G}(S)$ the partially ordered set of identities forms a meet-semilattice. If S is the Brandt semigroup $\mathcal{B}_\omega$ whose set of elelements is $\{(m,n) \mid m,n \in \omega = \{0,1,2,\cdots\}\} \cup \{0\}$ with the multiplication defined by:

$$(m,n) \cdot (m',n') = \begin{cases} (m,n') & \text{if} \quad n = m' \\ 0 & \text{if} \quad n \neq m' \end{cases} \quad \text{and} \quad 0 \cdot (m,n) = (m,n) \cdot 0 = 0 \cdot 0 = 0,$$

then $\mathcal{G}(\mathcal{B}_\omega)$ is category isomorphic to the connected simple system with adjoined 0: $\mathcal{G}_\omega^0$. But $\mathcal{G}(\mathcal{B}_\omega)$ is an ordered groupoid and the order $\le_{\mathcal{B}_\omega}$ on $\mathcal{G}(\mathcal{B}_\omega)$ (that is the natural partial order on $\mathcal{B}_\omega$) induces a partial order $\le_{\mathcal{B}_\omega}$ on $Mor(\mathcal{G}_\omega^0)$ given by: $1_0 \le_{\mathcal{B}_\omega} f$ for all $f \in Mor(\mathcal{G}_\omega^0)$, and $f \le_{\mathcal{B}_\omega} g$ iff $f = g$, otherwise. Note that $\mathcal{G}_\omega^0$ (and $\mathcal{G}_\omega$) can be equipped as an ordered groupoid in many other way.

Now, an ordered groupoid in which the set of identities forms a meet-semilattice (like in the case of the ordered groupoids $\mathcal{G}(S)$) is called *inductive*. If $f : X \to Y$ and $f' : X' \to Y'$ are two morphisms of an inductive groupoid $\mathcal{G}$ and $1_X \wedge 1_{Y'} = 1_Z$ then the pseudoproduct $\otimes$:

$$f \otimes f' = f|_Z \circ f'|^Z$$

defines a binary operation on the set $Mor(\mathcal{G})$ such that $(Mor(\mathcal{G}), \otimes)$ is an inverse semigroup ([3, Proposition 4.1.7 (1)]). Note that if we denote this semigroup by $\mathcal{S}(\mathcal{G})$, then $\mathcal{S}(\mathcal{G}(S)) = S$ ([3, Proposition 4.1.7 (3)]), $\mathcal{G}(\mathcal{S}(\mathcal{G}, \preceq)) = (\mathcal{G}, \preceq)$ ([3, Proposition 4.1.7 (2)]), and $\mathcal{S}(\mathcal{G}_\omega^0, \le) \cong \mathcal{B}_\omega$ only if $\le$ is the induced order $\le_{\mathcal{B}_\omega}$ on $Mor(\mathcal{G}_\omega^0)$ considered above.

2.2. Normal inverse subsemigroup and the corresponding ordered groupoid

An inverse subsemigroup N of an inverse semigroup S is called *normal* if $E(S) = E(N)$ and if $s^{-1}Ns \subseteq N$ for all $s \in S$. A normal inverse subsemigroup N of an inverse semigroup S together with the defining concepts ($\le$ and $\circ$) of the ordered groupoid $\mathcal{G}(S)$ determine a preorder $\le_N$ on $S = Mor\mathcal{G}(S)$, as follows:

$$s \le_N t \;\Leftrightarrow$$

there exist two morphisms a, b of $\mathcal{G}(S)$ such that $a, b \in N$, the compositions $a \circ s$ and $s \circ b$ are both defined, and

$$a \circ s \circ b \le t.$$

Since $\le_N$ is a preorder on the set S then it defines an equivalence relation $\simeq_N$ on S by $s \simeq_N t \Leftrightarrow s \le_N t$ and $t \le_N s$, and a partial order on the set of equivalence classes $S/\simeq_N$. In [1] this quotient set is denoted by $S/\!\!/N$ and the $\simeq_N$- class of $s \in S$ by $[s]_N$. The equivalence relation $\simeq_N$ needs not be a congruence on S. However, the quotient set $S/\!\!/N$ leads us to an ordered groupoid $\overline{\mathcal{G}}(S/\!\!/N)$: the objects are the classes $[e]_N$ where $e \in E(S)$, and $Mor(\overline{\mathcal{G}}(S/\!\!/N) = S/\!\!/N$ with $[s]_N$ being a morphism from $[s^{-1}s]_N$ to $[ss^{-1}]_N$. The composition of two morphisms $[s]_N \circ [t]_N$ (if $[s^{-1}s]_N = [tt^{-1}]_N$) is given by $[s]_N \circ [t]_N = [sat]_N$, where $a \in N$ such that $a^{-1}a = tt^{-1}$ and $aa^{-1} \le s^{-1}s$; and $[s]_N \preceq_N [t]_N \Leftrightarrow s \le_N t$, is the partial order of $\overline{\mathcal{G}}(S/\!\!/N)$. Now, if this ordered groupoid $\overline{\mathcal{G}}(S/\!\!/N)$ is inductive then $S/\!\!/N = Mor(\overline{\mathcal{G}}(S/\!\!/N))$ forms an inverse semigroup $(S/\!\!/N, \otimes)$ (where $\otimes$ is the pseudoproduct) called here the quotient inverse semigroup of S by the normal inverse subsemigroup N.

2.3. Gauge inverse submonoids

Following [5], a nontrivial right cancellative monoid M is a RILL monoid if 1_M is indecomposable and any two elements $s, t \in M$ that admit a common left multiple admit a least common left multiple $s \vee t$. In the RILL monoid M, we shall denote $s \ll t$ if t is a left multiple of s, $t = rs$, and by $\frac{t}{\triangleright s}$ the "left quotient" r. Since M is right cancellative and 1 is indecomposable, the "right divisibility" relation $\ll$ is a partial order on M. A length function on the RILL monoid M is a monoid homomorphism $\ell : M \to (\mathbb{N}, +)$ such that $\ell^{-1}(0) = 1_M$. A non-trivial monoid with a length function is atomic (every non-units element is a product of finitely many atoms). A length function ℓ is said to be normalized if $\ell(s) = 1 \Leftrightarrow s$ is an atom. An ℓ-RILL monoid is a RILL monoid equipped with a normalized length function ℓ.

If M is an ℓ-RILL monoid then the set

$$\mathbb{S}(M) = \begin{cases} M \times M & \text{if} \quad Ms \cap Mt \neq \emptyset \text{ for any } s, t \in M \\ (M \times M) \cup \{\theta\} & \text{if there exist } s, t \in M \text{ such that } Ms \cap Mt = \emptyset \end{cases}$$

(that is $M \times M$, adjoining an extra element θ if necessary), together with the product $\odot$ defined by

$$(s,t) \odot (s',t') = \begin{cases} (\frac{t \vee s'}{\rhd t} s, \frac{t \vee s'}{\rhd s'} t') & \text{if } t \text{ and } s' \text{ admit a common left multiple} \\ \theta & \text{otherwise} \end{cases}$$

and

$$\theta \odot (s,t) = (s,t) \odot \theta = \theta \odot \theta = \theta \quad \text{(if necessary)},$$

is an inverse monoid (the inverse of (s,t) is (t,s); the element (s,t) is an idempotent if and only if $s = t$, and $(1_M, 1_M)$ is the identity element). The submonoid of $\mathbb{S}(M)$:

$$G_M = \begin{cases} \{(s,t) \in M \times M |\ \ell(s) = \ell(t)\} & \text{if} \quad \mathbb{S}(M) = M \times M \\ \{(s,t) \in M \times M |\ \ell(s) = \ell(t)\} \cup \{\theta\} & \text{if} \quad \mathbb{S}(M) = (M \times M) \cup \{\theta\} \end{cases}$$

is the gauge inverse submonoid of $\mathbb{S}(M)$ induced by the ℓ-RILL monoid M. This submonoid of $\mathbb{S}(M)$ is a normal submonoid ([5, Proposition 5.6]).

In [5] the first example of a gauge inverse submonoid is the submonoid of idempotents $E(B)$ of the bicyclic semigroup B. The bicyclic semigroup B is the monoid of all pairs of non-negative integers equipped with the multiplication defined by:

$$(m,n) \cdot (m',n') = \begin{cases} (m, n - m' + n') & \text{if} \quad n \geq m' \\ (m - n + m', n') & \text{if} \quad n \leq m'. \end{cases}$$

In this paper $(B^0, \cdot)$ denotes the bicyclic semigroup with adjoined zero 0.

3. Main results. The quotient inverse monoid $\mathbb{S}(M) /\!\!/ G_M$

Let M be an ℓ-RILL monoid and $(\mathbb{S}(M), \odot)$ the corresponding inverse monoid.

Proposition 3.1. *The natural partial order $\leq$, the preorder $\leq_{G_M}$ and the equivalence relation $\simeq_{G_M}$ on $\mathbb{S}(M)$ are given by:*

(i) $(s,t) \leq (s',t') \;\Leftrightarrow\; s' \ll s,\ t' \ll t$ *and* $\frac{s}{\rhd s'} = \frac{t}{\rhd t'}$ *([4, Proposition 2.6 (1)])*
($\theta \leq x$ *for any* $x \in \mathbb{S}(M)$ *if* $\mathbb{S}(M) = (M \times M) \cup \{\theta\}$*);*

(ii) $(s,t) \leq_{G_M} (s',t') \;\Leftrightarrow\;$ *there exists* $(p,q) \in \mathbb{S}(M)$ *such that* $\ell(p) = \ell(s)$, $\ell(q) = \ell(t)$ *and* $(p,q) \leq (s',t')$
($\theta \leq_{G_M} x$ *for any* $x \in \mathbb{S}(M)$ *if* $\mathbb{S}(M) = (M \times M) \cup \{\theta\}$*);*

(iii) $(s,t) \simeq_{G_M} (s',t') \;\Leftrightarrow\; \ell(s) = \ell(s')$ *and* $\ell(t) = \ell(t')$
(*if* $\mathbb{S}(M) = (M \times M) \cup \{\theta\}$ *then the* $\simeq_{G_M}$*-class* $[\theta]_{G_M}$ *is a singleton*).

Proof. (i). We have

$$(s,t) \leq (s',t') \Leftrightarrow (s,t) = (s,t) \odot (t,s) \odot (s',t') \Leftrightarrow (s,t) = (s,s) \odot (s',t') \Leftrightarrow$$

$$(s,t) = (\frac{s \vee s'}{\rhd s} s, \frac{s \vee s'}{\rhd s'} t') \Leftrightarrow (s,t) = (s \vee s', \frac{s \vee s'}{\rhd s'} t') \Leftrightarrow s' \ll s \text{ and } \frac{s}{\rhd s'} t' = t$$

$$\Leftrightarrow s' \ll s,\ t' \ll t \text{ and } \frac{s}{\rhd s'} = \frac{t}{\rhd t'}.$$

(ii). We have

$$(s,t) \leq_{G_M} (s',t') \ \Leftrightarrow \ \text{there exist } (p,u),(v,q) \in G_M \text{ such that}$$

$$(p,u)^{-1} \odot (p,u) = (s,t) \odot (s,t)^{-1}, \ (s,t)^{-1} \odot (s,t) = (v,q) \odot (v,q)^{-1}$$

$$\text{and } \ (p,u) \odot (s,t) \odot (v,q) \leq (s',t').$$

Since

$$(p,u)^{-1} \odot (p,u) = (u,u) \text{ and } (s,t) \odot (s,t)^{-1} = (s,s),$$

it follows $u = s$. Analogously, $v = t$. Now, we have:

$$(p,u) \odot (s,t) \odot (v,q) = (p,s) \odot (s,t) \odot (t,q) = (p,q)$$

and taking into account that $(p,s),(t,q) \in G_M$ we obtain:

$$(s,t) \leq_{G_M} (s',t') \ \Leftrightarrow \ \text{there exist } p,q \in M \text{ such that}$$

$$\ell(p) = \ell(s), \ \ell(q) = \ell(t) \text{ and } (p,q) \leq (s',t').$$

(iii). The assertion follows from (i) and (ii). □

Remark 3.2. *The equivalence relation $\simeq_{G_M}$ is not necessarily a congruence on $\mathbb{S}(M)$. For example, if M is the multiplicative ℓ-RILL monoid of positive integers $(\mathbb{Z}^+,\cdot)$ ([5, Example 4.2]), where $\ell(1) = 0$ and $\ell(n) =$the total number of prime divisors of n counted with their multiplicities if $n > 1$, then $\mathbb{S}(\mathbb{Z}^+)$ is the multiplicative analogue of the bicyclic semigroup:*

$$\mathbb{S}(\mathbb{Z}^+) = \mathbb{Z}^+ \times \mathbb{Z}^+; \qquad (m,n)\cdot(m',n') = (\frac{[n,m']}{n}m, \frac{[n,m']}{m'}n'),$$

$[n,m']$ being the least common multiple of n and m'. Now, if p and q are two distinct primes then $(p,q) \simeq_{G_M} (p,q)$ and $(p,q) \simeq_{G_M} (q,p)$ (since $\ell(p) = \ell(q) = 1$), but $(p,q)\cdot(p,q) = (p^2,q^2)$ and $(p,q)\cdot(q,p) = (p,p)$, that is $(p,q)\cdot(p,q) \not\simeq_{G_M} (p,q)\cdot(q,p)$. Thus $\simeq_{G_M}$ is not a congruence on the multiplicative analogue of the bicyclic semigroup.

The $\simeq_{G_M}$-class

$$[(s,t)]_{G_M} = \{(u,v) \in \mathbb{S}(M) |\ \ell(u) = \ell(s) \text{ and } \ell(v) = \ell(t)\}$$

is a morphism in the ordered groupoid $\overline{\mathcal{G}}(\mathbb{S}(M)/\!\!/G_M)$ from $[(t,t)]_{G_M}$ to $[(s,s)]_{G_M}$. If $[(s,t)]_{G_M}$ and $[(s',t')]_{G_M}$ are two morphisms of $\overline{\mathcal{G}}(\mathbb{S}(M)/\!\!/G_M)$ such that $\ell(s') = \ell(t)$ (that is $[(s',s')]_{G_M} = [(t,t)]_{G_M}$),

$$[(t',t')]_{G_M} \overset{[(s',t')]_{G_M}}{\longrightarrow} [(s',s')]_{G_M} = [(t,t)]_{G_M} \overset{[(s,t)]_{G_M}}{\longrightarrow} [(s,s)]_{G_M},$$

then the composition of these two morphisms, $[(s,t)]_{G_M} \circ [(s',t')]_{G_M}$ is given by

$$[(s,t)]_{G_M} \circ [(s',t')]_{G_M} = [(s,t) \odot (a,b) \odot (s',t')]_{G_M},$$

where $(a,b) \in G_M$ such that $(a,b)^{-1} \odot (a,b) = (s',t') \odot (s',t')^{-1}$ and $(a,b) \odot (a,b)^{-1} \leq (s,t)^{-1} \odot (s,t)$. We choose $(a,b) = (t,s')$ which is an element of G_M since $\ell(t) = \ell(s')$. Thus the composition $[(s,t)]_{G_M} \circ [(s',t')]_{G_M}$ in $\overline{\mathcal{G}}(\mathbb{S}(M)/\!\!/G_M)$ such that $\ell(t) = \ell(s')$ is given by:

$$[(s,t)]_{G_M} \circ [(s',t')]_{G_M} = [(s,t) \odot (t,s') \odot (s',t')]_{G_M} = [(s,t')]_{G_M}.$$

The ordering $\preceq_{G_M}$ of $\simeq_{G_M}$-classes in the ordered groupoid $\overline{\mathcal{G}}(\mathbb{S}(M)/\!\!/G_M)$ is given by:

$$[(s,t)]_{G_M} \preceq_{G_M} [(s',t')]_{G_M} \;\Leftrightarrow\; \text{there exists } (p,q) \in [(s,t)]_{G_M} \text{ such that}$$

$$s' \ll p,\; t' \ll q \text{ and } \frac{p}{\triangleright s'} = \frac{q}{\triangleright t'}.$$

and

$$[\theta]_{G_M} \preceq_{G_M} [x]_{G_M} \;\; \text{for any morphism } [x]_{G_M} \text{ of } \overline{\mathcal{G}}(\mathbb{S}(M)/\!\!/G_M)$$

$$\text{if } \mathbb{S}(M) = (M \times M) \cup \{\theta\}.$$

Remark 3.3. *The objects of $\overline{\mathcal{G}}(\mathbb{S}(M)/\!\!/G_M)$ other than $[\theta]_{G_M}$ (that is the $\simeq_{G_M}$-classes $[(s,s)]_{GM}$) can be indexed by non-negative integers (namely $[(s,s)]_{GM}$ by $\ell(s)$), then the set of morphisms from m to n is a singleton (for any pair (m,n) of non-negative integers) and, it goes without saying the composition of two morphisms.*

It follows:

Theorem 3.4. *The (ordered) groupoid $\overline{\mathcal{G}}(\mathbb{S}(M)/\!\!/G_M)$ is category isomorphic either to the connected simple system $\mathcal{G}_{\mathbb{N}}$ (if $\mathbb{S}(M) = M \times M$) or to the connected simple system with adjoined 0: $\mathcal{G}^0_{\mathbb{N}}$ (if $\mathbb{S}(M) = M \times M \cup \{\theta\}$).*

Theorem 3.5. *The ordered groupoid $\overline{\mathcal{G}}(\mathbb{S}(M)/\!\!/G_M)$ is inductive.*

Proof. It is straightforward to see that in the set of identities of $\overline{\mathcal{G}}(\mathbb{S}(M)/\!\!/G_M)$ we have:

$$[(s,s)]_{G_M} \preceq_{G_M} [(t,t)]_{G_M} \;\Leftrightarrow\; \ell(t) \leq \ell(s).$$

It follows that the partially ordered set of identities of $\overline{\mathcal{G}}(\mathbb{S}(M)/\!\!/G_M)$ forms a meet-semilattice:

$$[(s,s)]_{G_M} \wedge [(t,t)]_{G_M} = \begin{cases} [(s,s)]_{G_M} & \text{if } \ell(s) \geq \ell(t) \\ [(t,t)]_{G_M} & \text{if } \ell(s) \leq \ell(t) \end{cases}$$

and

$$[\theta]_{G_M} \wedge [x]_{G_M} = [\theta]_{G_M} \;\; \text{for any identity morphism } [x]_{G_M} \text{ of } \overline{\mathcal{G}}(\mathbb{S}(M)/\!\!/G_M)$$

$$\text{if } \mathbb{S}(M) = (M \times M) \cup \{\theta\}.$$

Therefore the ordered groupoid $\overline{\mathcal{G}}(\mathbb{S}(M)/\!\!/G_M)$ is inductive. □

Theorem 3.6. *The corresponding inverse semigroup $(\mathbb{S}(M)/\!\!/G_M, \otimes)$ is isomorphic either to the bicyclic semigroup $(B,\cdot)$ (if $\mathbb{S}(M) = M \times M$) or to the bicyclic semigroup with adjoined zero $(B^0,\cdot)$ (if $\mathbb{S}(M) = (M \times M) \cup \{\theta\}$).*

Proof. Let $[(s,t)]_{G_M}, [(s',t')]_{G_M} \in \mathbb{S}(M)/\!\!/G_M$. As morphisms of $\overline{\mathcal{G}}(\mathbb{S}(M)/\!\!/G_M)$, we have:

$$[(s,t)]_{G_M} : [(t,t)]_{G_M} \;\to\; [s,s]_{G_M} \quad \text{and} \quad [(s',t')]_{G_M} : [(t',t')]_{G_M} \;\to\; [s',s']_{G_M}.$$

Since

$$[(t,t)]_{G_M} \wedge [(s',s')]_{G_M} = \begin{cases} [(t,t)]_{G_M} & \text{if } \ell(t) \geq \ell(s') \\ [(s',s')]_{G_M} & \text{if } \ell(t) \leq \ell(s'). \end{cases}$$

we shall consider two cases:

1) $[(t,t)]_{G_M} \wedge [(s',s')]_{G_M} = [(t,t)]_{G_M}$. Then the restriction $[(s,t)]_{G_M}|_{[(t,t)]_{G_M}}$ of $[(s,t)]_{G_M}$ to $[(t,t)]_{G_M}$ is just $[(s,t)]_{G_M}$. The corestriction $[(s',t')]_{G_M}|^{[(t,t)]_{G_M}}$ of $[(s',t')]_{G_M}$ to $[(t,t)]_{G_M}$ is the morphism $[(t,y)]_{G_M} : [y,y]_{G_M} \to [t,t]_{G_M}$, where $y \in M$ such that

$$\ell(y) = \ell(t) - \ell(s') + \ell(t'),$$

since $[(t,y)]_{G_M} \preceq_{G_M} [(s',t')]_{G_M}$.

In this case,

$$[(s,t)]_{G_M} \otimes [(s',t')]_{G_M} = [(s,t)]_{G_M}|_{[(t,t)]_{G_M}} \circ [(s',t')]G_M|^{[(t,t)]_{G_M}} =$$

$$[(s,t)]_{G_M} \circ [(t,y)]_{G_M} = [(s,y)]_{G_M}.$$

2) $[(t,t)]_{G_M} \wedge [(s',s')]_{G_M} = [(s',s')]_{G_M}$. Then the restriction $[(s,t)]_{G_M}|_{[(s',s')]_{G_M}}$ of $[(s,t)]_{G_M}$ to $[(s',s')]_{G_M}$ is the morphism $[(x,s')]_{G_M} : [(s',s')]_{G_M} \to [(x,x)]_{G_M}$, where $x \in M$ such that

$$\ell(x) = \ell(s') - \ell(t) + \ell(s)$$

since $[x,s']_{G_M} \preceq_{G_M} [(s,t)]_{G_M}$. The corestriction $[(s',t')]_{G_M}|^{[(s',s')]_{G_M}}$ of $[(s',t')]_{G_M}$ to $[(s',s')]_{G_M}$ is just $[(s',t')]_{G_M}$. So, in this case, the product $[(s,t)]_{G_M} \otimes [(s',t')]_{G_M}$ is given by:

$$[(s,t)]_{G_M} \otimes [(s',t')]_{G_M} = [(s,t)]_{G_M}|_{[(s',s')]_{G_M}} \circ [(s',t')]G_M|^{[(s',s')]_{G_M}} =$$

$$[(x,s')]_{G_M} \circ [(s',t')]_{G_M} = [(x,t')]_{G_M}.$$

(If $\mathbb{S}(M) = (M \times M) \cup \{\theta\}$) then it is straightforward to check that $[\theta]_{G_M}$ is the zero element of $(\mathbb{S}(M)/\!\!/G_M, \otimes)$.)

Now, a careful examination shows that

$$\overline{\ell} : (\mathbb{S}(M)/\!\!/G_M, \otimes) \;\to\; (B, \cdot) \quad \text{if } \; \mathbb{S}(M) = M \times M$$

$$(\overline{\ell} : (\mathbb{S}(M)/\!\!/G_M, \otimes) \;\to\; (B^0, \cdot) \;\; \text{if } \; \mathbb{S}(M) = (M \times M) \cup \{\theta\})$$

defined by

$$\overline{\ell}([(s,t)]_{G_M}) = (\ell(s), \ell(t))$$

$$(\text{and } \; \overline{\ell}([\theta]_{G_M}) = 0 \;\; \text{if } \; \mathbb{S}(M) = (M \times M) \cup \{\theta\})$$

is a monoid isomorphism. □

Remark 3.7. *What is happening if the ℓ-RILL monoid M is the additive monoid of non-negative integers ? (that is if the monoid $(\mathbb{S}(M), \odot)$ is the bicyclic semigroup B ?) The gauge inverse submonoid of B is the semilattice of idempotents $E(B)$ ([5, Example 4.1]). It is straightforward to check that $\simeq_{E(B)}$ is the trivial relation (the equality) on B and of course $B/\!\!/E(B) = B$ (and $\overline{\mathcal{G}}(B/\!\!/E(B)) = \mathcal{G}(B)$).*

Now, since for any inverse semigroup S the relation $\simeq_{E(S)}$ is the trivial relation on S ([1, Proposition 3.4 (g)]), it follows that

Corollary 3.8. *The bicyclic semigroup is the only combinatorial bisimple inverse monoid for which the gauge inverse submonoid is the semilattice of idempotents.*

Remark 3.9. *The ordered groupoid $\overline{\mathcal{G}}(\mathbb{S}(M)/\!\!/G_M)$ is isomorphic either to the ordered groupoid $\mathcal{G}(B)$ or to the ordered groupoid $\mathcal{G}(B^0)$. Of course, the groupoids $\overline{\mathcal{G}}(P_n/\!\!/G_n)$ and $\mathcal{G}(\mathcal{B}_\omega)$ are also isomorphic as two categories (since both are category isomorphic to the connected simple system with adjoined 0: $\mathcal{G}^0_\omega$), but they are not isomorphic as two ordered groupoids due to the two partial orders $\preceq_{G_n}$ and $\leq_{\mathcal{B}_\omega}$ on $\overline{\mathcal{G}}(P_n/\!\!/G_n)$ and $\mathcal{G}(\mathcal{B}_\omega)$, respectively.*

4. Supplements. The quotient group $\mathbb{S}(M)/G_M$

If ρ is a relation on an inverse semigroup S, the kernel $ker\rho$ is the set

$$ker\rho = \{s \in S \mid s\rho e \text{ for some } e \in E(S)\}.$$

If ρ is a group congruence on S then we agree to write $S/ker\rho$ for the quotient group S/ρ.

In what follows assume that $\mathbb{S}(M) = M \times M$ (that is, $Ms \cap Mt \neq \emptyset$ for any $s, t \in M$). We have:

Proposition 4.1. *The relation $\approx_M$ on $\mathbb{S}(M)$ defined by*

$$(x, y) \approx_M (x', y') \quad \textit{if and only if} \quad \ell(x) - \ell(y) = \ell(x') - \ell(y'),$$

is a group congruence on $\mathbb{S}(M)$. The gauge inverse submonoid G_M is the kernel of $\approx_M$, and it is the identity element of the quotient group $\mathbb{S}(M)/G_M$ $(= \mathbb{S}(M)/\approx_M)$. This quotient group is isomorphic to the additive group of integers $(\mathbb{Z}, +)$.

Proof. The relation $\approx_M$ is an equivalence relation on $\mathbb{S}(M)$. Obviously, G_M is the kernel of $\approx_M$. If $(s,t) \approx_M (s',t')$ and $(u,v) \approx_M (u',v')$, then $(s,t) \odot (u,v) = (\frac{t \vee u}{\rhd t} s, \frac{t \vee u}{\rhd u} v)$, $(s',t') \odot (u',v') = (\frac{t' \vee u'}{\rhd t'} s', \frac{t' \vee u'}{\rhd u'} v')$ and

$$\ell(\frac{t \vee u}{\rhd t} s) - \ell(\frac{t \vee u}{\rhd u} v) = \ell(t \vee u) - \ell(t) + \ell(s) - (\ell(t \vee u) - \ell(u) + \ell(v)) =$$

$$\ell(s) - \ell(t) + \ell(u) - \ell(v) = \ell(s') - \ell(t') + \ell(u') - \ell(v') = \ell(\frac{t' \vee u'}{\rhd t'} s') - \ell(\frac{t' \vee u'}{\rhd u'} v').$$

It follows that $\approx_M$ is a congruence relation on $\mathbb{S}(M)$. The quotient monoid $\mathbb{S}(M)/\approx_M$ is again an inverse monoid. Since G_M is the only idempotent of $\mathbb{S}(M)/\approx_M$ it follows that this inverse monoid is a group (the quotient group $\mathbb{S}(M)/G_M$). The map $\overline{\ell} : \mathbb{S}(M)/G_M \to \mathbb{Z}$ defined by

$$([x, y]_{\approx_M} \in \mathbb{S}(M)/\approx_M) \qquad \overline{\ell}([x, y]_{\approx_M}) = \ell(x) - \ell(y)$$

is an isomorphism from the group $\mathbb{S}(M)/G_M$ onto the additive group of integers $(\mathbb{Z}, +)$. □

Remark 4.2. *It is straightforward to see that the kernel of $\simeq_{G_M}$ is also the gauge inverse submonoid G_M. However, the differences between the relations $\simeq_{G_M}$ and $\approx_M$ are significant:*

(a) *in general, the equivalence relation $\simeq_{G_M}$ is not a congruence on $\mathbb{S}(M)$ (Remark 3.2), but $\approx_M$ is a group congruence on $\mathbb{S}(M)$;*

(b) *the gauge inverse submonoid G_M is not a $\simeq_{G_M}$-equivalence class in $\mathbb{S}(M)$, but it is an $\approx_M$-equivalence class in $\mathbb{S}(M)$;*

(c) *there is not a $\simeq_{G_M}$-equivalence class $[(s,t)]_{G_M}$ such that $E(\mathbb{S}(M)) \subseteq [(s,t)]_{G_M}$, but the $\approx_M$-equivalence class G_M contains the set of all idempotents of $\mathbb{S}(M)$;*

(d) *the group $\mathbb{S}(M)/G_M$ is equipped with the product $\odot$ via the inverse monoid $\mathbb{S}(M)$; the product in the inverse monoid $\mathbb{S}(M)/\!\!/G_M$ is the pseudoproduct $\otimes$ via the inductive groupoid $\overline{\mathcal{G}}(\mathbb{S}(M)/\!\!/G_M)$;*

(e) *the following inclusion holds: $\simeq_{G_M} \subset \approx_M$.*

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References

[1] N. Alyamani, N. D. Gilbert, Ordered groupoid quotients and congruences on inverse semigroups, Semigroup Forum 96 (2018) 506–522.
[2] D. G. Jones, M. V. Lawson, Strong representations of the polycyclic inverse monoids: Cycles and atoms, Period. Math. Hung. 64 (2012) 54–87.
[3] M. V. Lawson, Inverse semigroups: the theory of partial symmetries, World Scientific, Singapore (1998).
[4] E. D. Schwab, Möbius monoids and their connection to inverse monoids, Semigroup Forum 90 (2015) 694–720.
[5] E. D. Schwab, Gauge inverse monoids, Algebra Colloq. 27(2) (2020) 181–192.