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Code–checkable group rings

Research Article

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Abstract: A code over a group ring is defined to be a submodule of that group ring. For a code C over a group ring RG, C is said to be checkable if there is $v \in RG$ such that $C = \{x \in RG : xv = 0\}$. In [6], Jitman et al. introduced the notion of code-checkable group ring. We say that a group ring RG is code-checkable if every ideal in RG is a checkable code. In their paper, Jitman et al. gave a necessary and sufficient condition for the group ring $\mathbb{F}G$, when $\mathbb{F}$ is a finite field and G is a finite abelian group, to be code-checkable. In this paper, we give some characterizations for code-checkable group rings for more general alphabet. For instance, a finite commutative group ring RG, with R is semisimple, is code-checkable if and only if G is π'-by-cyclic π; where π is the set of noninvertible primes in R. Also, under suitable conditions, RG turns out to be code-checkable if and only if it is pseudo-morphic.

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1. Introduction

A code over a group ring is originally defined to be an ideal in the group algebra $\mathbb{F}G$, where $\mathbb{F}$ is a finite field and G is a finite group. When G is cyclic, this concept characterizes the classical cyclic codes over $\mathbb{F}$ as, in this case, the ideals of $\mathbb{F}G \cong F[x]/ < x^n - 1 >$. This concept has been first introduced by F. MacWilliams [7] in 1969. In general when G is abelian, they are called abelian codes.

Later on 2007, Hurley [4] introduced new techniques for constructing codes from encoding in group rings, for arbitrary group ring RG where R is a ring with unity and G is a finite group. Codes from group-ring encoding are basically defined by considering a left R-submodule W of the group ring RG and any element u of RG, the right group-ring code C generated by u relative to W is the code defined by $C = \{xu : x \in W\}$. When the element u is zero-divisor (resp. unit), C is called zero-divisor (resp. unit-derived) code. This method allows us to produce codes from every zero-divisor and every unit in the group ring. A zero-divisor code C is called checkable if there exists $v \in RG$ such that $C = \{y \in RG : yv = 0\}$, that is $y \in C$ if and only if $yv = 0$. In this case we say that v is a check element for the code C. It is

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easy to see that every cyclic code is a checkable code, but the converse is not true. This means that the concept of checkable codes is, somehow, a generalization for cyclic codes.

In 2010, Jitman et al [6] introduced the notion of code-checkable group rings, where a group ring is said to be code-checkable if every ideal in that group ring is a checkable code. So, the main question was about a characterization for code-checkable group rings. In the same paper [6], such a characterization for RG was given in the special case when R is a finite field and G is an abelian group. In this paper, we give a necessary and sufficient condition for a group ring RG to be code-checkable in a more general setting, when R is a finite commutative semisimple ring and G is any finite abelian group. We were also able to get rid of the condition that R has to be semisimple and give a characterization for RG to be code-checkable for any finite commutative ring R.

The paper is organized as follows: In section 2, we present basic results and tools that will be used in the following sections. We start with basic structural properties for group rings specially; the relation between the group-ring elements and the matrix ring. Some characterizations for principal ideal group rings are presented. We also present some notions concerning generalized morphic rings.

In section 3, the notion of codes from group ring encodings, due to Hurley [4], is presented. We focus on checkable codes, which are special case of codes from group ring encodings. A zero-divisor code is said to be checkable if it is a left annihilator for some element in the group ring. We also present the result due to Jitman et al [6] in characterizing code-checkable group algebras.

Our two main results Theorem 4.1 and Theorem 4.5 are presented in the last section. We give necessary and sufficient conditions for group rings to be code-checkable. Also, in the last section we provide Example 4.6 that shows the necessity of one of the hypothesis of Theorem 4.2.

Note: All rings are considered to be unitary.

2. Preliminaries

In this section we are going to describe the structure of the group rings. We also introduce some basic concepts and necessary terminologies that will be used later in this paper.

2.1. Group rings and matrices

In this section we describe the very important isomorphism between a group ring RG and a subring of the matrix ring $M_{n\times n}(R)$; where n is the number of element of G. This isomorphism plays a basic role in studying the generator and the parity check matrices of certain types of codes that are going to be mentioned later.

Starting with a finite group G and a ring R, let $\{g_1, g_2, ..., g_n\}$ be a fixed listing of elements of the group G. Then every element u in RG is written as $u = \sum_{i=1}^{n} u_{g_i} g_i$, where $u_{g_i} \in R$. For $u = \sum_{i=1}^{n} u_{g_i} g_i \in RG$ define the matrix $U \in M_{n\times n}(R)$ by:

$$U = \begin{bmatrix} u_{g_1^{-1}g_1} & u_{g_1^{-1}g_2} & \cdots & u_{g_1^{-1}g_n} \\ u_{g_2^{-1}g_1} & u_{g_1^{-1}g_2} & \cdots & u_{g_2^{-1}g_n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{g_n^{-1}g_1} & u_{g_n^{-1}g_2} & \cdots & u_{g_n^{-1}g_n} \end{bmatrix}$$

Definition 2.1. *Define the map $\sigma : RG \to M_{n\times n}(R)$ by $u \mapsto U$, for every $u \in RG$. The map σ is called* ***left regular representation*** *of RG.*

The left regular representation of RG is a monomorphism of rings. This means that restricting the codomain of σ on $Im(\sigma)$ will yield an isomorphism between the group ring RG and a subring of the matrix ring $M_{n\times n}(R)$. This subring is denoted by $RG(M_n)$ and matrices in $RG(M_n)$ are called RG-matrices.

Theorem 2.2. *The group ring RG is isomorphic to $RG(M_n)$ as rings.*

When the group G is cyclic, $G = \{g^n, g^1, ..., g^{n-1}\}$. Any element $u \in RG$ is written as $u = \sum_{i=1}^{n} u_i g^i$, then the associated matrix to u is of the form:

$$U = \begin{bmatrix} u_{g^n} & u_{g^1} & \cdots & u_{g^{n-1}} \\ u_{g^{n-1}} & u_{g^n} & \cdots & u_{g^{n-2}} \\ \vdots & \vdots & \ddots & \vdots \\ u_{g^1} & u_{g^2} & \cdots & u_{g^n} \end{bmatrix}$$

Because of the isomorphism σ, the elements of the group ring RG inherit many concepts and properties of matrices which turn out to be very useful in our work.

Definition 2.3. *The **transpose** of an element $u = \sum_{g\in G} u_g g$ in RG is $u^T = \sum_{g\in G} u_g g^{-1}$, or equivalently, $u^T = \sum_{g\in G} u_{g^{-1}} g$.*

Definition 2.4. *We say that $u \in RG$ is **symmetric** if and only if $u^T = u$.*

Note that this definition is consistent with the matrix definition of transpose. If we take an element $u \in RG$, then the transpose of the RG-matrix of u is again an RG-matrix and is associated to group ring element u^T. That is $\sigma(u^T) = \sigma(u)^T = U^T$.

2.2. Principal ideal group rings

In this section we are interested in a characterization of principal ideal group rings. In the case when the ring is a finite field, the principal ideal group rings are characterized in [3]. Another characterization in [2] is established in a more general setting.

Definition 2.5. *A ring R is said to be **principal ideal ring** (for short: **PIR**), if every two-sided ideal in R is a principal ideal.*

Definition 2.6. *A prime number p is said to **invertible** in a ring R if $p.1$ is an invertible element in R. Otherwise p is called **noninvertible**.*

To be able to see a characterization of PIR group rings, we first need the following notions about finite groups.

Definition 2.7. *Let p be a prime and G be a finite group of order n.*

- *We say that G is a **p-group** if n is a power of p.*
- *We say that G is a **p'-group** (here p' does not mean another prime p') if $(n, p) = 1$.*

The above definition can be easily generalized if we replace the prime p by a finite set of primes. That is, for a finite set of primes π and a finite group G with order n, G is said to be **π-group** if n is a power of primes from π while G is **π'-group** if n is coprime with every prime in π.

For any two classes of groups; $\mathcal{A}$ and $\mathcal{B}$, we say that a group G is $\mathcal{A}$-by-$\mathcal{B}$ if there exists $N \lhd G$ such that $N \in \mathcal{A}$ and $G/N \in \mathcal{B}$. So, a finite group G is called π'-by-cyclic π, if there is $H \lhd G$ such that H is a π'-group and G/H is cyclic and a π-group.

Now we are ready to introduce two results concerning principal ideal group rings. In fact, the property that the group ring RG is a PIR depends on the relation between the set of noninvertible primes in R and the number of elements of G, as we shall see in Theorem 2.8 and Theorem 2.9. Notice that, the only noninvertible prime in a finite field $\mathbb{F}$ is the characteristic of $\mathbb{F}$. We have the following two theorems for the characterization of principal ideal group rings.

Theorem 2.8. *[3] Let G be a finite abelian group and $\mathbb{F}$ a finite field of characteristic p. Then $\mathbb{F}G$ is a PIR if and only if a Sylow p-subgroup of G is cyclic.*

Theorem 2.9. *[2] Let R be a finite semisimple ring and G a finite group. Then RG is PIR if and only if G is π'-by-cyclic π, where π is the set of noninvertible primes in R.*

2.3. Generalized morphic rings

A young topic in ring theory is being studied for the last decade, namely rings that satisfy the dual of the isomorphism theorem. That is, $\frac{R}{Ra} \cong Ann_l(a)$ for every $a \in R$, in this case R is called a left morphic ring [8]. It turns out that R being morphic is equivalent to say that for all $a \in R$ there exists $b \in R$ such that $Ra = Ann_l(b)$ and $Ann_l(a) = Rb$. Later on, the notions of quasi-morphic rings and pseudo-morphic rings have been introduced by relaxing the condition on morphic rings, see Definition 2.11. For more details, we refer to [8] and [1].

Definition 2.10. *An ideal I of a ring R is said to be (left)* ***annihilator*** *if $I = Ann_l(a) = \{r \in R : ra = 0\}$,*
for some $a \in R$.

Definition 2.11. *Let R be an arbitrary ring.*

1) *R is called (left)* ***morphic*** *if for all $a \in R$ there exists $b \in R$ such that $Ra = Ann_l(b)$ and $Ann_l(a) = Rb$.*
2) *R is called (left)* ***quasi-morphic*** *if $\{Ra : a \in R\} = \{Ann_l(b) : b \in R\}$. Means that every (left) principal ideal is (left) annihilator ideal and vise versa.*
3) *R is called (left)* ***generalized morphic*** *if $\{Ann_l(b) : b \in R\} \subset \{Ra : a \in R\}$. Means that every (left) annihilator ideal is a (left) principal ideal.*
4) *R is called (left)* ***pseudo-morphic*** *if $\{Ra : a \in R\} \subset \{Ann_l(b) : b \in R\}$. Means that every (left) principal ideal is a left annihilator ideal.*

We get similar definitions by replacing each "left" by "right", if we drop the word "left" it means that the property is satisfied for only two-sided ideals. In our work, we found that pseudo-morphic group rings characterizes code-checkable group rings, when the group ring is finite commutative. See Theorem 4.5.

3. Codes from group ring encoding

In this section we are going to present a construction of codes from encoding in group rings. This construction is due to Hurley in [5] and it leads to two new types of codes, namely zero-divisor codes and unit-derived codes. Many important codes like BCH and Reed-Solomon turn out to be special kinds of zero-divisor codes. In the following, RG denotes a group ring, W a free left submodule of RG and u is a fixed element of RG.

Definition 3.1. *A* ***right group ring encoding*** *is a map $f : W \to RG$ defined by $x \mapsto xu$, for every $x \in W$. (The left group ring encoding maps x to ux).*

Definition 3.2. *For a right group ring encoding f, the code C derived from f is defined by*

$$C := Im(f) = \{xu : x \in W\}.$$

We say that the code C is ***generated by*** *u* ***relative to*** *W. When u is a zero-divisor, the code C is called* ***zero-divisor code*** *and when u is a unit, the code C is called* ***unit-derived code****.*

Here W plays the role of an information set and u plays the role of the encoder. That is, u encodes the message x to the codeword xu.

Of course, one of the most important properties of codes is to have a way from the code C back to the information set, which is what we call a decoding algorithm. In our case, the existence of this algorithm depends on the choice of the information set W. For unit-derived codes, there is complete freedom in the choice of W, while zero-divisor codes have some restrictions placed on W.

3.1. Checkable Codes

Checkable codes are special kind of zero-divisor codes. They have one of the most important properties for a code which is to have a check matrix or a check element. A zero-divisor code C is said to be checkable if C has a single check element. That is $C = \{x \in RG : xv = 0\}$ for some $v \in RG$. The notion of checkable codes has been discussed by Hurley in [5]. Where, the checkable codes have been characterized in terms of the properties of the generator element of the code. However, the name checkable codes and the notion of code-checkable group rings were first established in [6] by Jitman et al. They have studied checkable codes in terms of the properties of the group ring RG.

Definition 3.3. *Let C be a zero-divisor code in the group ring RG. C is said to be a **checkable code** if there exists $v \in RG$ such that $C = \{x \in RG : xv = 0\}$. In other words, C is the left annihilator, denoted by $Ann(v)$, of an element v of RG.*

We are interested in finding alphabets where all its two-sided ideals are checkable codes. Here is the definition of such alphabets.

Definition 3.4. *A group ring RG is said to be **code-checkable** if every two-sided ideal in RG is a checkable code.*

Of course if a unit-derived code is an ideal, then it will be the whole space RG. Thus, to determine whether RG is code-checkable, it suffices to consider all zero-divisor codes C where $C = FGu$.

The following proposition and theorem which characterize when a group algebra $\mathbb{F}G$ is code-checkable were proved in [6], when G is a finite abelian group. Proposition 3.5 may be proved along similar lines to Proposition 4.1. Its proof is omitted and it can be found in [[6], Proposition 3.1].

Proposition 3.5. *Let G be a finite abelian group and $\mathbb{F}$ be a finite field. Then $\mathbb{F}G$ is code-checkable if and only if it is a principal ideal group ring.*

Theorem 3.6. *Let G be a finite abelian group and $\mathbb{F}$ be a finite field of characteristic p. Then the group algebra $\mathbb{F}G$ is code-checkable if and only if a Sylow p-subgroup of G is cyclic.*

Proof. Follows immediately from Theorem 2.8 and Proposition 3.5. □

4. Code-checkable group rings

The last theorem in the previous section gives a complete characterization for a group algebra to be code-checkable. Our main result in this section is to give a characterization for more general alphabet group ring. If R is a semisimple commutative ring and G is a finite abelian group, we characterize when RG is a code-checkable group ring. We also give another characterization for any finite commutative group ring to be code-checkable.

Proposition 4.1. *Let R be a finite commutative ring and G a finite abelian group. Then RG is code-checkable if and only if it is a PIR.*

Proof. Assume that RG is code-checkable. Let I be an ideal of RG. If I is $\{0\}$ or RG, then it is principal. Assume that I is non-trivial. Then there exists a zero-divisor $v \in RG$ such that $I = \{x : xv = 0\} = Ann(v)$. Define

$$f : RG/Ann(v) \to RGv, \text{where } x + Ann(v) \mapsto xv, \text{forall } x \in RG.$$

It can easily be seen that f is well-defined and bijection. Then $|RG/Ann(v)| = |RGv|$.

Now, RGv is a non-trivial ideal then, from our assumption, $RGv = Ann(u)$ for some $u \in RG$. Then $v = 1.v \in Ann(u)$, so $vu = 0$. Commutativity of RG implies that uv also equals zero.

We claim that $Ann(v) = RGu$. Let $y = xu \in RGu$, then $yv = xuv = 0$. Hence $RGu \subseteq Ann(v)$. We also have, $|RG/Ann(v)| = |RGv|$ and $|RG/Ann(u)| = |RGu|$, thus

$$|RGu| = |RG/Ann(u)| = |RG|/|Ann(u)| = |RG|/|RGv| = |Ann(v)|.$$

Hence, $I = Ann(v) = RGu$ is a principal ideal.

Conversely, assume that RG is a PIR. Let $\mathfrak{J}$ denote the set of all non-trivial ideals of RG. From the finiteness of RG, it follows that $|\mathfrak{J}|$ is finite. Let $\sigma : \mathfrak{J} \to \mathfrak{J}$ be defined by:

$$RGa \mapsto Ann(a).$$

Using that R is commutative, we can show that $Ann(RGb) = Ann(b)$ for every $b \in RG$. If $RGa = RGb$, then

$$Ann(a) = Ann(RGa) = Ann(RGb) = Ann(b).$$

This implies that σ is well-defined. To show that σ is injective, assume that $\sigma(RGa) = \sigma(RGb)$, i.e. $Ann(a) = Ann(b)$. Since RG is PIR, then $Ann(a) = Ann(b) = RGv$, for some $v \in RG$. Hence, by the first part of the proof, we have $RGa = Ann(v) = RGb$.

Since $|\mathfrak{J}|$ is finite and σ is injective, then σ is surjective. This implies that every non-trivial ideal of RG is a checkable code. □

We will use a strong result, Theorem 2.9, of Dorsey [[2], Theorem 4.4] to complete our characterization. It gives a characterization for a group ring RG to be PIR, when R is a semisimple ring and G is any finite group. Using this, the complete result follows in the following theorem.

Theorem 4.2. *Let G be a finite abelian group, R a finite commutative semisimple ring and π the set of noninvertible primes in R. Then the group ring RG is code-checkable if and only if G is π'-by-cyclic π.*

Proof. Follows immediately from Theorem 2.9 and Proposition 4.1. □

4.1. The eight P-conditions

In order to present our last result, we first need to recall some notions about rings. A partially ordered set P is said to satisfy the **ascending chain condition** (ACC) if every strictly ascending sequence of elements terminates. Similarly, P is said to satisfy the **descending chain condition** (DCC) if every strictly descending sequence of elements terminates. Those two conditions are often called the finiteness conditions for the partially ordered set. A ring R is called **Artinian** if R satisfies the DCC on the set of all ideals of R. Another kind of finiteness conditions on a ring is called the eight P-conditions. Here are the eight P-conditions:

(i) ACC on $\{Ann_l(b) : b \in R\}$.
(ii) ACC on $\{Ann_r(b) : b \in R\}$.
(iii) ACC on $\{Ra : a \in R\}$.
(iv) ACC on $\{aR : a \in R\}$.
(v) DCC on $\{Ann_l(b) : b \in R\}$.
(vi) DCC on $\{Ann_r(b) : b \in R\}$.
(vii) DCC on $\{Ra : a \in R\}$.
$(viii)$ DCC on $\{aR : a \in R\}$.

It turns out that the eight P-conditions have some symmetry when the ring is pseudo-morphic. That is [1], if R is pseudo-morphic, then the eight P-conditions are all equivalent.

Since we work only on finite rings, then we don't have to worry about any finiteness condition. In fact, any finiteness condition is satisfied for all finite rings.

The following theorem by Camillo and Nicholson [[1], Theorem 6.3.] presents the relation between R being pseudo-morphic, quasi-morphic and principal ideal ring, when R satisfies some finiteness conditions.

Theorem 4.3. *The following conditions are equivalent for a ring R:*

(i) R is pseudo-morphic and satisfies any of the eight P-conditions.

(ii) R is quasi-morphic and satisfies any of the eight P-conditions.

(iii) R is an artinian principal ideal ring.

If the ring R is finite, then it follows immediately that R is both artinian and satisfies the eight P-conditions. The following corollary follows from (i) and (iii) from the last theorem.

Corollary 4.4. *Let R be a finite ring. R is pseudo-morphic if and only if R is a PIR.*

This allows us to present our last result for this paper.

Theorem 4.5. *Let R be a finite commutative ring and G a finite abelian group. The following are equivalent:*

1) RG is code-checkable.

2) RG is a PIR.

3) RG is pseudo-morphic.

Proof. Direct consequence of Proposition 4.1 and Corollary 4.4 □

Example 4.6. *Consider the group ring $\mathbb{Z}_4C_2$, where C_2 is the cyclic group of order three. Write C_2 as $C_2 = \{e, a\}$, then*

$$\mathbb{Z}_4C_2 = \{0, 1, 2, 3, a, 2a, 3a, 1+a, 2+a, 3+a, 1+2a, 2+2a, 3+2a, 1+3a, 2+3a, 3+3a\}.$$

Using [[1], Example 3.3], the group ring $\mathbb{Z}_4C_2$ is not a pseudo-morphic ring. Therefore, by Theorem 4.5, $\mathbb{Z}_4C_2$ is neither code-checkable nor PIR, we show this in the following. Naively, we are going to construct the multiplication table of $\mathbb{Z}_4C_2$ as in Table 1.

Consider the ideal $I = \langle 2+2a \rangle$ generated by $2+2a$. By the multiplication table, $I = \{0, 2+2a\}$. So I is a checkable code if there is a check element $x \in \mathbb{Z}_4C_2$ such that $xy = 0$ iff $y \in I$, for every $y \in \mathbb{Z}_4C_2$. Since I has only one nonzero element, this means that x is a check element of I implies that $xy = 0$ only when $y = 2+2a$ or $y = 0$. By looking at the multiplication table, there is no such an element x in $\mathbb{Z}_4C_2$. Therefore, I is not checkable and hence $\mathbb{Z}_4C_2$ is not code-checkable.

Also, we can show that $\mathbb{Z}_4C_2$ is not a PIR. Consider the ideal

$$J = \langle 1+a, 1+3a \rangle = \{0, 2, 2a, 1+a, 3+a, 2+2a, 1+3a, 3+3a\}.$$

Since J does coincide with any row of our multiplication table, then J is not a principal ideal, and hence $\mathbb{Z}_4C_2$ is not a PIR.

Finally, this example shows that the semisimplicity condition is necessary for the conclusion of Theorem 4.2. In fact, the noninvertible primes of $\mathbb{Z}_4$ is the singleton $\pi = \{2\}$. Now, if we take the trivial subgroup $H = \{e\} \leq C_2$, then clearly H is normal in C_2 with $(|H|, 2) = 1$, so H is a π'-group. Also, $C_2/H = C_2$ is cyclic and has exactly 2^1 elements, which means that C_2/H is a cyclic π-group. Therefore, C_2 is π'-by-cyclic π, however $\mathbb{Z}_4C_2$ is not code-checkable. The reason that the conclusion in Theorem 4.2 is not satisfied here, is because that the ring $\mathbb{Z}_4$ is not semisimple.

Table 1. **Multiplication table of** $\mathbb{Z}_4C_2$.

.	0	1	2	3	a	$2a$	$3a$	$1+a$	$2+a$	$3+a$	$1+2a$	$2+2a$	$3+2a$	$1+3a$	$2+3a$	$3+3a$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	a	$2a$	$3a$	$1+a$	$2+a$	$3+a$	$1+2a$	$2+2a$	$3+2a$	$1+3a$	$2+3a$	$3+3a$
2	0	2	0	2	$2a$	0	$2a$	$2+2a$	$2a$	$2+2a$	2	0	2	$2+2a$	$2a$	$2+2a$
3	0	3	2	1	$3a$	$2a$	a	$3+3a$	$2+3a$	$1+3a$	$3+2a$	$2+2a$	$1+2a$	$3+a$	$2+a$	$1+a$
a	0	a	$2a$	$3a$	1	2	3	$1+a$	$1+2a$	$1+3a$	$2+a$	$2+2a$	$2+3a$	$3+a$	$3+2a$	$3+3a$
$2a$	0	$2a$	0	$2a$	2	0	2	$2+2a$	2	$2+2a$	$2a$	0	$2a$	$2+2a$	2	$2+2a$
$3a$	0	$3a$	$2a$	a	3	2	1	$3+3a$	$3+2a$	$3+a$	$2+3a$	$2+2a$	$2+a$	$1+3a$	$1+2a$	$1+a$
$1+a$	0	$1+a$	$2+2a$	$3+3a$	$1+a$	$2+2a$	$3+3a$	$2+2a$	$3+3a$	0	$3+3a$	0	$1+a$	0	$1+a$	$2+2a$
$2+a$	0	$2+a$	$2a$	$2+3a$	$1+2a$	2	$3+2a$	$3+3a$	1	$3+a$	a	$2+2a$	$3a$	$1+3a$	3	$1+a$
$3+a$	0	$3+a$	$2+2a$	$1+3a$	$1+3a$	$2+2a$	$3+a$	0	$3+a$	$2+2a$	$1+3a$	0	$3+a$	$2+2a$	$1+3a$	0
$1+2a$	0	$1+2a$	2	$3+2a$	$2+a$	$2a$	$2+3a$	$3+3a$	a	$1+3a$	1	$2+2a$	3	$3+a$	$3a$	$1+a$
$2+2a$	0	$2+2a$	0	$2+2a$	$2+2a$	0	$2+2a$	0	$2+2a$	0	$2+2a$	0	$2+2a$	0	$2+2a$	0
$3+2a$	0	$3+2a$	2	$1+2a$	$2+3a$	$2a$	$2+a$	$1+a$	$3a$	$3+a$	3	$2+2a$	1	$1+3a$	a	$3+3a$
$1+3a$	0	$1+3a$	$2+2a$	$3+a$	$3+a$	$2+2a$	$1+3a$	0	$1+3a$	$2+2a$	$3+a$	0	$1+3a$	$2+2a$	$3+a$	0
$2+3a$	0	$2+3a$	$2a$	$2+a$	$3+2a$	2	$1+2a$	$1+a$	3	$1+3a$	$3a$	$2+2a$	a	$3+a$	1	$3+3a$
$3+3a$	0	$3+3a$	$2+2a$	$1+a$	$3+3a$	$2+2a$	$1+a$	$2+2a$	$1+a$	0	$1+a$	0	$3+3a$	0	$3+3a$	$2+2a$

5. Conclusion

Throughout the paper, we generalized the characterization by Jitman et al [6], for a group ring RG to be code-checkable, by relaxing some of the conditions on the ring R and the group G. Also, we have given Example 4.6 which shows that the semisimplicity of the ring R is necessary for our characterization. In section 4, we have shown that for a group ring RG, under certain conditions, being code-checkable is equivalent to being pseudo-morphic, which is a relatively new concept for rings.

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