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Collocation Approach for the Computational Solution Of Fredholm-Volterra Fractional Order of Integro-Differential Equations

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Abstract

In this work, a collocation technique is used to determine the computational solution to fractional order Fredholm-Volterra integro-differential equations with boundary conditions using Caputo sense. We obtained the linear integral form of the problem and transformed it into a system of linear algebraic equations using standard collocation points. The matrix inversion approach is adopted to solve the algebraic equation and substituted it into the approximate solution. We established the uniqueness and convergence of the method and some modelled numerical examples are provided to demonstrate the method's correctness and efficiency. It is observed that the results obtained by our new method are accurate and performed better than the results obtained in the literature. The study will be useful to engineers and scientists. It is advantageous because it addresses the difficulty in tackling fractional order Fredholm-Volterra integro-differential problems using a simple collocation strategy. The approach has the advantage of being more accurate and reducing computer running time.

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1. Introduction

Fractional calculus is an aspect in mathematics that studies the properties of integrals combined with noninteger order derivatives.The concept and method of solving differential equations containing fractional derivatives of unknown functions are covered in this field (fractional differential equations). Many prominent mathematicians, such as Liouville, Grunwald, Riemann, Euler, Langrange, Heaviside, Fourier, Abel, and others, established fractional calculus on a formal foundation[1]. Very recently, scholars developed huge interest in fractional calculus due to its relevance and application to many areas of scientific endeavors. Fractional differential equations, or those containing real and complex order derivatives have been more significant in describing the most broad fields of science and technology with peculiar dynamics of various processes involving complex systems [2].

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Many different approaches have been adopted to investigate the solution of fractional integrodifferential equations, such as Adomian decompositions method [3-5], collocation method [6, 7], Laplace decomposition method [8, 9], Taylor expansion method [10], Least square method [9], differential transform method [11], homotopy perturbation method [12-18], sinc-collocation method [15-17] and variational iteration method [12, 13]. Linna *et al.* [20] considered a numerical method to solve fractional variational problem. They simplified the fractional variational problems by the operational matrices. The operational matrices are based on the Chelyshkov polynomials. The fractional variational problem was transformed into a set of algebraic equations. The unknown coefficients were solved using the Lagrange multiplier techniques. Avcı & Mahmudov [19] proposed a numerical method based on the fractional Taylor vector for solving multi-term fractional differential equations. The main idea of this method was to reduce the given problems to a set of algebraic equations by utilizing the fractional Taylor operational matrix of fractional integration. Some numerical examples were given to demonstrate the accuracy and applicability. The results show that the presented method was efficient and applicable. Fadugba [21] presented the Mellin transform for the solution of the fractional order equations. The Mellin transform approach occurs in many areas of applied mathematics and technology. The Mellin transform of fractional calculus of different flavours; namely the Riemann-Liouville fractionalderivative, Riemann-Liouville fractional integral, Caputo fractional derivative and the Miller-Ross sequential fractional derivative were obtained.

In this study, we consider Fredholm-Volterra Integrodifferential equation of fractional order of the form:

$$\begin{aligned} &D^{\alpha}_{x}y(x) - P_1 y^{'}(x) - P_2 y^{''}(x) - P_0 y(x) \\ &= g(x) + \lambda_1 \int_0^x k_1(x,t)\, y(t)\, dt \\ &+ \lambda_2 \int_0^1 k_2(x,t)\, y(t)\, dt, \end{aligned} \tag{1}$$

with the given boundary conditions

$$y(a) = 0,\ y(b) = 0,\ a < x < b, \tag{2}$$

where $y(x)$ is to be determined, D^{α}_{x} is the Caputo's derivative, $k_1(x,t)$ and
$k_2(x,t)$ are the Fredholm and Volterra integral kanel function respectively. P_j, P_α, λ_j are known constants. $g(x)$ is the known function

2. Basic Definitions

Under this section, we present some definitions and basic concepts of fractional calculus for the formulation of the given problem

Definition 2.1: The Caputo derivative with order $\alpha > 0$ of the given function $f(x),\ x \in (a,b)$ is defined as [Litfi, Dehghan and Yousefi, 2011]

$${}^{C}_{x}D^{\alpha}_{a}y(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-s)^{m-\alpha-1} y^{(m)}(s)ds, \tag{3}$$

where $m-1 \le \alpha \le m, m \in N, x > 0$

Definition 2.2: Let $(a_n), n \ge 0$ be a sequence of real numbers. The power series in x with coefficients a_n is an expression [Edward, Ford and Simpson, 2002]

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots a_N x^N \\ &= \sum_{n=0}^{N} a_n x^n = \phi(x)\,\mathbf{A}, \end{aligned} \tag{4}$$

where $\phi(x) = [1 \ \ x \ \ x^2 \ \cdots \ x^N]$, $\mathbf{A} = [a_0 \ \ a_1 \ \cdots \ a_N]^T$, then $y(x,n) = x^n\mathbf{A},\ n = 0(1)N,\ n \in Z^+$.

Definition 2.3: Standard Collocation Method (SCM). This approach is used to find the collocation points that are desired within a certain interval. i.e [a,b] and is given by

$$x_i = a + \frac{(b-a)i}{N}, i = 1,2,3,...N. \tag{5}$$

Definition 2.4: A metric on a set M is a function $d : M \times M \longrightarrow \mathbb{R}$ with the following properties. For all $x, y \in M$

(a) $d(x,y) \ge 0$;
(b) $d(x,y) = 0 \Longleftrightarrow x = y$
(c) $d(x,y) = d(y,x)$
(d) $d(x,y) \le d(x,z) + d(x,y)$

If d is a metric on M, then the pair (M,d) is called a metric space.

Definition 2.5: Let (X,d) be a metric space, A mapping $T : X \longrightarrow X$ is Lipschitzian if $\exists$ a constant $L > 0$ such that $d(Tx,Ty) \le Ld(x,y)\ \forall\ x,y \in X$.

Definition 2.6: Let $y(x)$ be a continuous function, then

$${}_0I^{\beta}_{x}\left({}^{C}_{0}D^{\beta}_{x}y(x)\right) = y(x) - \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k, \tag{6}$$

where $m-1 < \beta < 1$.

Let $p(s)$ be an integrable function, then

$${}_0I^{\beta}_{x}(p(s)) = \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} p(s)ds. \tag{7}$$

3. Mathematical Background

In this section, combination of collocation method and power series approximation is employed for the computational solution of (FVID) of fractional order.

Theorem 3.0: Banach's fixed point theorem Let (X,d) be a complete metric space, then each contraction mapping $T : X \longrightarrow X$ has a unique fixed point x of T in X, such that $T(x) = x$

Lemma (3.1). Let $y(x)$ be the solution to (1) subject to (2), the integral form

$$y(x) = W(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(P_2 y^{''}(s)\right) ds \tag{8}$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(P_1 y'(s)\right)ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(P_0 y(s)\right)ds$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\lambda_1\int_0^s k_1(s,t)\,y(t)\,dt\right.$$
$$\left.+\lambda_2\int_0^1 k_2(s,t)\,y(t)\,dt\right]$$

where

$$W(x)=\sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x^k-\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}g(s)ds.$$

Multiply equation (1) by ${}_0I_x^{\beta}(.)$ gives

$$\begin{aligned}{}_0I_x^{\alpha}\left(D^{\alpha}y(x)\right)=\ &{}_0I_x^{\beta}\left(P_2y''(x)\right)+{}_0I_x^{\beta}\left(P_1y'(s)\right)\\&+{}_0I_x^{\beta}\left(P_0y(s)\right)-{}_0I_x^{\beta}\left(g(x)\right)\\&-{}_0I_x^{\beta}\left[\lambda_1\int_0^s k_1(s,t)\,y(t)\,dt\right.\\&\left.+\lambda_2\int_0^1 k_2(s,t)\,y(t)\,dt\right].\end{aligned}\tag{9}$$

Using(6) on equation (9)gives

$$\begin{aligned}y(x)=&\sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x^k+{}_0I_x^{\beta}\left(P_2y''(x)\right)\\&+{}_0I_x^{\beta}\left(P_1y'(s)\right)+{}_0I_x^{\beta}\left(P_0y(s)\right)\\&{}_0I_x^{\beta}\left(g(x)\right)+{}_0I_x^{\beta}\left(\int_0^b k_1(x,t)y(t)dt\right)\\&+{}_0I_x^{\beta}\left(\int_0^s k_2(s,t)y(t)dt\right)\end{aligned}\tag{10}$$

Applying equation (4) and (7) to (10) gives

$$\begin{aligned}y(x)=&\sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x^k\\&+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}P_2\left(\phi(s)''\right)ds\mathbf{A}\\&+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}P_1\left(\phi(s)'\right)ds\mathbf{A}\\&+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}P_0\left(\phi(s)\right)ds\mathbf{A}\\&-\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(g(s)\right)ds\\&-\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\lambda_1\int_0^s k_1(s,t)\,\phi(t)\,dt\right.\\&\left.+\lambda_2\int_0^1 k_2(s,t)\,\phi(t)\,dt\right]ds\mathbf{A}\quad .\end{aligned}\tag{11}$$

3.1. Method of Solution

Collocatiing at x_i in equation (11) gives

$$\begin{aligned}y(x_i)=&\sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x^k+\\&\frac{1}{\Gamma(\alpha)}\int_0^{x_i}(x_i-s)^{\alpha-1}P_2\left(\phi(s)''\right)ds\mathbf{A}\\&+\frac{1}{\Gamma(\alpha)}\int_0^{x_i}(x_i-s)^{\alpha-1}P_1\left(\phi(s)'\right)ds\mathbf{A}\\&+\frac{1}{\Gamma(\alpha)}\int_0^{x_i}(x_i-s)^{\alpha-1}P_0\left(\phi(s)\right)ds\mathbf{A}\\&-\frac{1}{\Gamma(\alpha)}\int_0^{x_i}(x_i-s)^{\alpha-1}\left(g(s)\right)ds-\\&\frac{1}{\Gamma(\alpha)}\int_0^{x_i}(x_i-s)^{\alpha-1}\left[\lambda_1\int_0^s k_1(s,t)\,\phi(t)\,dt\right.\\&\left.+\lambda_2\int_0^1 k_2(s,t)\,\phi(t)\,dt\right]ds\mathbf{A}\end{aligned}\tag{12}$$

Applying (4) on (12) gives

$$\begin{aligned}&\phi(x_i)\mathbf{A}=W(x_i)\\&+\left[\begin{array}{c}\frac{1}{\Gamma(\alpha)}\int_0^{x_i}(x_i-s)^{\alpha-1}P_2\left(\phi(s)''\right)ds+\\\frac{1}{\Gamma(\alpha)}\int_0^{x_i}(x_i-s)^{\alpha-1}P_1\left(\phi(s)'\right)ds+\\\frac{1}{\Gamma(\alpha)}\int_0^{x_i}(x_i-s)^{\alpha-1}P_0\left(\phi(s)\right)ds\\-\frac{1}{\Gamma(\alpha)}\int_0^{x_i}(x_i-s)^{\alpha-1}\left(\lambda_1\int_0^{x_i}k_1(x_i,t)\,\phi(t)\,dt\right.\\\left.+\lambda_2\int_0^1 k_2(x_i,t)\,\phi(t)\,dt\right)ds\end{array}\right]\mathbf{A},\end{aligned}\tag{13}$$

where

$$W(x_i)=\sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x^k-\frac{1}{\Gamma(\alpha)}\int_0^{x_i}(x_i-s)^{\alpha-1}g(s)ds.$$

Factorise the values of **A** from equation (13) gives

$$W(x_i)=\left[\begin{array}{c}\phi(x_i)-\frac{1}{\Gamma(\alpha)}\int_0^{x_i}(x_i-s)^{\alpha-1}P_2\left(\phi(s)''\right)ds-\\\frac{1}{\Gamma(\alpha)}\int_0^{x_i}(x_i-s)^{\alpha-1}P_1\left(\phi(s)'\right)ds-\\\frac{1}{\Gamma(\alpha)}\int_0^{x_i}(x_i-s)^{\alpha-1}P_0\left(\phi(s)\right)ds\\+\frac{1}{\Gamma(\alpha)}\int_0^{x_i}(x_i-s)^{\alpha-1}\\\left(\lambda_1\int_0^s k_1(s,t)\,\phi(t)\,dt\right.\\\left.+\lambda_2\int_0^1 k_2(s,t)\,\phi(t)\,dt\right)ds\end{array}\right]\mathbf{A}.\tag{14}$$

Equation 14 can be in the form

$$\tau_n(x_i)\mathbf{A}=W(x_i),\tag{15}$$

where

$$\tau_n(x_i)=\left[\begin{array}{c}\phi(x_i)-\frac{1}{\Gamma(\alpha)}\int_0^{x_i}(x_i-s)^{\alpha-1}P_2\left(\phi(s)''\right)ds-\\\frac{1}{\Gamma(\alpha)}\int_0^{x_i}(x_i-s)^{\alpha-1}P_1\left(\phi(s)'\right)ds-\\\frac{1}{\Gamma(\alpha)}\int_0^{x_i}(x_i-s)^{\alpha-1}P_0\left(\phi(s)\right)ds+\frac{1}{\Gamma(\alpha)}\int_0^{x_i}(x_i-s)^{\alpha-1}\\\left(\lambda_1\int_0^s k_1(s,t)\,\phi(t)\,dt+\lambda_2\int_0^1 k_2(s,t)\,\phi(t)\,dt\right)ds,\end{array}\right]$$

and

$$\mathbf{A}=\left[a_0\quad a_1\quad\cdots\quad a_N\right]^T.$$

Multiply both sides of equation(15) by$\tau_n^{-1}(x_i)$ gives

$$\mathbf{A}=\tau_n^{-1}(x_i)W(x_i).\tag{16}$$

Substituting **A** into the approximate solution (4) gives

$$y(x)=\phi(x_i)\tau_n^{-1}(x_i)\,W(x_i)\tag{17}$$

4. Uniqueness of the Method

In order to establish the uniqueness of the method, we introduce the following hypothesis

$$H_1 \quad : \quad k_1^* = \max_{x\in[0,1]} \int_0^x \lambda_1 \left|k_1(x,t)\right| dt$$
$$H_2 \quad : \quad k_2^* = \max_{x\in[0,1]} \int_0^1 \lambda_2 \left|k_2(x,t)\right| dt$$
$$H_3 \quad : \quad \left|y_1^{(m)} - y_2^{(m)}\right| \leq L_m \left|y_1 - y_2\right|.$$

Lemma (4.1) (q-contraction) Let $T : X \longrightarrow X$ be a mapping defined by theorem (3.0) for $y_1,\ y_2 \in X$. T is q-contraction if and only if

$$\left[\frac{\sum_{n=0}^{2} L_n - \left(k_1^* + k_2^*\right)}{\Gamma(\alpha+1)}\right] < 1,$$

then there exist a unique solution of T.

$$\begin{aligned}(Ty_1)(x) = {} & W(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(P_2 y_1''(s)\right) ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(P_1 y_1'(s)\right) ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(P_0 y_1(s)\right) ds \\ & - \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\lambda_1 \int_0^s k_1(s,t)\, y_1(t)\, dt \right. \\ & \left. + \lambda_2 \int_0^1 k_2(s,t)\, y_1(t)\, dt\right] ds\end{aligned}$$

and

$$\begin{aligned}(Ty_2)(x) = {} & W(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(P_2 y_2''(s)\right) ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(P_1 y_2'(s)\right) ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(P_0 y_2(s)\right) ds \\ & - \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\lambda_1 \int_0^s k_1(s,t)\, y_2(t)\, dt \right. \\ & \left. + \lambda_2 \int_0^1 k_2(s,t)\, y_2(t)\, dt\right] ds.\end{aligned}$$

Thus,

$$\begin{aligned}& \left|(Ty_1)(x) - (Ty_2)(x)\right| \\ & = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} P_2 \left|y_1''(s) - y_2''(s)\right| ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} P_1 \left|y_1'(s) - y_2'(s)\right| ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} P_0 \left|y_1(s) - y_2(s)\right| ds \\ & - \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\begin{array}{c}\int_0^s \lambda_1 \left|k_1(s,t)\right| \left|y_1(t) - y_2(t)\right| dt \\ + \int_0^1 \lambda_1 \left|k_2(s,t)\right| \left|(y_1(t) - y_2(t))\right| dt\end{array}\right] ds.\end{aligned}$$

Taking maximum of both sides and using H_1 *to* H_3 gives

$$d\left(Ty_1(x), Ty_2(x)\right) \leq \left[\frac{L_2 + L_1 + L_0 - \left(k_1^* + k_2^*\right)}{\Gamma(\alpha+1)}\right] d(y_1, y_2)$$

Since T is a contraction

$$\left[\frac{\sum_{n=0}^{2} L_n - \left(k_1^* + k_2^*\right)}{\Gamma(\alpha+1)}\right] < 1.$$

We can conclude that T has a unique solution.

5. Convergence Analysis

We establish the convergence of the method

$$\begin{aligned}y_N(x) = {} & W(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(P_2 y_N''(s)\right) ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(P_1 y_N'(s)\right) ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(P_0 y_N(s)\right) ds \\ & - \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left[\lambda_1 \int_0^s k_1(s,t)\, y_N(t)\, dt \right. \\ & \left. + \lambda_2 \int_0^1 k_2(s,t)\, y_N(t)\, dt\right] ds.\end{aligned} \qquad (18)$$

Subtract (8) from (18) gives

$$E_N(x) = y_N(x) - y(x).$$

Hence

$$\begin{aligned}\left|E_N(x)\right| = {} & \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left|E_N(s)\left(P_2 + P_1\right)\right| ds \\ & - \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} \\ & \left[\left|\lambda_1\right| \left|\int_0^s k_1(s,t)\, E_N(t) dt\right| \right. \\ & \left. + \left|\lambda_2\right| \left|\int_0^1 k_2(s,t)\, E_N(t) dt\right| ds.\right.\end{aligned}$$

(20)

Therefore

$$\frac{\|E_N(x)\|_\infty}{\|E_N(t)\|_\infty} \leq \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} \times \left[\begin{array}{c}\left|E_n(s)\left(P_2 + P_1\right)\right| \\ - \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} \\ \left(\left|\lambda_1\right| \left|\int_0^s k_1(s,t)\, E_n(t) dt\right|\right. \\ \left. + \left|\lambda_2\right| \left|\int_0^1 k_2(s,t)\, E_n(t) dt\right|\right.\end{array}\right] ds.$$

The method converges

6. Numerical Examples

In this section, two numerical problems with boundary conditions are presented to text the efficiency and simplicity of the method. All computations are done with the aid of Maple 18. Let $y_n(x)$ and $y(x)$ be the approximate and exact solution respectively. $\text{Error}_N = |y_n(x) - y(x)|$

Example 1:

Considering fractional integro-differential equation [17]

$$y''(x) + \frac{1}{x} D_x^{0.5} y(x) + \frac{1}{x^2} y(x) = f(x) + \int_0^x \sin(x-t)\, y(t)dt + \int_0^1 \cos(x-t)\, y(t)\, dt,$$

subject to the boundary conditions

$$y(0) = 0,\ y(1) = 0,$$

where

$$f(x) = 5 + 1.50451x^{0.5} - 13x - 1.80541x^{1.5} - x^2 + x^3 - 2.067x\cos(x) + 5.95385\sin(x).$$

Exact solution $y(x) = x^2(1-x)$

Solution 1

We solve this problem at $N = 3$ *and* 5 but we use $N = 3$ for demonstration. Integral form of example 1 is

$$y''(x) + \frac{1}{x}\left(\frac{1}{\Gamma(1-0.5)}\int_0^x (x-t)^{-0.5} y^{(1)}(t)dt\right) + \frac{1}{x^2} y(x) = f(x) + \int_0^x \sin(x-t)\, y(t)dt + \int_0^1 \cos(x-t)\, y(t)\, dt \quad (21)$$

Using approximate solution (4) on (14) gives

$$\left[\begin{array}{c} \phi''(x) + \frac{1}{x}\left(\frac{1}{\Gamma(1-0.5)}\int_0^x (x-t)^{-0.5} \frac{d}{dx}(\phi(t))dt\right) \\ + \frac{1}{x^2}\phi''(x) - \int_0^x \sin(x-t)\,\phi(t)dt \\ - \int_0^1 \cos(x-t)\,\phi(t)\,dt \end{array}\right] \mathbf{A} = f(x) \quad (22)$$

Equation (15) gives

$$\tau(x)\mathbf{A} = f(x), \quad (23)$$

where

$$\tau(x) = \begin{array}{c} \phi''(x) + \frac{1}{x}\left(\frac{1}{\Gamma(1-0.5)}\int_0^x (x-t)^{-0.5} \frac{d}{dx}(\phi(t))dt\right) \\ + \frac{1}{x^2}\phi''(x) - \int_0^x \sin(x-t)\,\phi(t)dt - \int_0^1 \cos(x-t)\,\phi(t)\,dt. \end{array}$$

Collocating at $x_3 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}$ and substituting the boundary conditions gives

$$\tau(x)^* \mathbf{A} = f(x)^*, \quad (24)$$

where

$$\tau_i(x) = \begin{bmatrix} 8.1837497830 & 4.5298367530 & 3.5647166670 & 2.4321976680 \\ 1.0664602830 & 2.3357941530 & 3.8888778550 & 5.4288519160 \\ -0.3011686789 & 1.5101524580 & 4.1068429140 & 8.5147669310 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Table 1. Exact and approximate values, Example 1

x	Exact	$N = 3$	$N = 5$
0.2	0.032000000000	0.031999821270	0.031999851830
0.4	0.096000000000	0.095999891010	0.095999961860
0.6	0.144000000000	0.143999795600	0.143999892900
0.8	0.128000000000	0.127999546700	0.127999648400
1.0	0.000000000000	-8.444000000000e-7	-7.382000000000e-7

Table 2. Absolute Error for Example 1

x	error_3	error_5	error [17]=32
0.2	-1.787300000000e-7	-1.481700000000e-7	2.048e-5
0.4	-1.089900000000e-7	-3.814000000000e-8	2.503e-5
0.6	-2.044000000000e-7	-1.071000000000e-8	1.789e-5
0.8	-1.071000000000e-7	-3.516000000000e-8	7.682e-6
1	-8.444000000000e-7	-7.382000000000e-8	3.034e-6

$$f(x) = \begin{bmatrix} 1.1325152410 & -1.5399784720 & -4.4079289640 & 0 & 0 \end{bmatrix}.$$

We now solve for the unknown values $\mathbf{A}$ (17) making use of matrix inversion results in;

$$y_3 = \left(\begin{array}{c} -4.2497275388 \times 10^{-7} + 0.16917e - 5x + \\ 0.9999976497x^2 - 0.9999997608x^3 \end{array}\right).$$

Applying the same procedure for $N = 5$ gives

$$y_5 = \left(\begin{array}{c} -3.4773620430 \times 10^{-7} + 8.9350760390 \times 10^7 x + 1.0000017198x^2 \\ -1.0000070682x^3 + 0.57092e - 5x^4 - 0.16454e - 5x^5 \end{array}\right)$$

Example 2

Considering fractional integro-differential equation [17]

$$y''(x) + D_x^{\alpha} y(x) = f(x) + 2\int_0^x (x-t)\, y(t)dt + \int_0^1 \left(x^2 - t\right) y(t)\, dt$$

with the given boundary conditions

$$y(0) = 0,\ y(1) = 0$$

where

$$f(x) = -\frac{1}{30} - 6x + \frac{181x^2}{20} + 4x^3 - \frac{x^5}{10} + \frac{x^6}{15} \text{ and } \alpha = 1$$

Exact solution $y(x) = x^3(x-1)$

Solution 2

We solve this problem at $N = 4$ and 5, however, make use of $N = 4$ for demonstration.
Integral form of example 2 is

$$y''(x) + \left(\frac{1}{\Gamma(2-1)}\int_0^x (x-t)^0 y^{(2)}(t)dt\right) = f(x) + 2\int_0^x (x-t)\, y(t)dt + \int_0^1 \left(x^2 - t\right) y(t)\, dt \quad (25)$$

Table 3. Exact and approximate values, Example 2

x	Exact	$N = 4$	$N = 5$
0.2	-0.6400000000e-2	-0.6400000000e-2	-0.6400000000e-2
0.4	-0.3840000000e-1	-0.3840000000e-1	-0.3840000000e-1
0.6	-0.864000000e-1	-0.8640000000e-1	-0.8640000000e-1
0.8	-0.1024000000	-0.1024000000	-0.1024000000
1.0	-0.9000000000e-3	-0.900000000e-3	-0.900000000e-3

Table 4. Absolute Error for Example 2

x	error_4	error_5	error $_{[17]=64}$
0.2	0.0	0.0	2.931e-7
0.4	0.0	0.0	3.915e-7
0.6	0.0	0.0	2.696e-7
0.8	0.0	0.0	1.011e-7
1	0.0	0.0	3.596e-8

Using approximate solution (4) on (18) gives

$$\begin{bmatrix} \phi''(x) + \left(\frac{1}{\Gamma(2-1)} \int_0^x (x-t)^0 \, \frac{d^2}{dx^2}(\phi(t))dt\right) \\ -2\int_0^x (x-t)\,\phi(t)dt - \int_0^1 \left(x^2 - t\right)\phi(t)\,dt \end{bmatrix} \mathbf{A} = f(x). \tag{26}$$

Equation (6.6) gives

$$\tau(x)\mathbf{A} = f(x) \tag{27}$$

where

$$\tau(x) = \begin{matrix} \phi''(x) + \left(\frac{1}{\Gamma(2-1)} \int_0^x (x-t)^0 \, \frac{d^2}{dx^2}(\phi(t))dt\right) \\ -2\int_0^x (x-t)\,\phi(t)dt - \int_0^1 \left(x^2 - t\right)\phi(t)\,dt. \end{matrix}$$

Collocating at $x_4 = \begin{bmatrix} 0 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 \end{bmatrix}$ and substituting the boundary conditions gives

$$\tau(x)^*\mathbf{A} = f(x)^* \tag{28}$$

where

$$\tau_i(x) = \begin{bmatrix} \frac{1}{2} & 0.3333333333 & 2.250000000 & 0.2000000000 & 0.1666666667 \\ \frac{1}{2} & \frac{59}{192} & \frac{4193}{1536} & \frac{19169}{10240} & \frac{59393}{61440} \\ \frac{1}{2} & \frac{1}{4} & \frac{305}{96} & \frac{249}{64} & \frac{3473}{960} \\ \frac{1}{2} & \frac{37}{192} & \frac{1851}{512} & \frac{64211}{10240} & \frac{522457}{61440} \\ \frac{1}{2} & \frac{1}{6} & \frac{49}{12} & \frac{181}{20} & \frac{481}{30} \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$f(x) = \begin{bmatrix} -\frac{1}{30} & -\frac{55621}{61440} & -\frac{131}{480} & \frac{137191}{61440} & \frac{419}{60} & 0 & 0 \end{bmatrix}$$

We solve for the unknown values $\mathbf{A}$ using matrix inversion, results;

$$y_4 = \begin{pmatrix} -2.2211399390 \times 10^{-13} \\ +3.0783153800 \times 10^{-12}x \\ -1.4125589590 \times 10^{-11}x^2 \\ -1.0000000000x^3 + 1.0000000000x^4 \end{pmatrix}$$

Applying the same procedure for $N = 5$ gives

$$y_5 = \begin{pmatrix} 5.329070518000 \times 10^{-15} \\ +1.781685910000 \times 10^{-12}x \\ -1.580957587000 \times 10^{-11}x^2 \\ -1.000000000000x^3 \\ +1.000000000000x^4 \\ +1.062971933000 \times 10^{-11}x^5 \end{pmatrix}$$

7. Conclusion

Collocation approach is utilized to solving the fractional order Fredholm-Volterra integro-differential problems in this paper. The applied method is consistent, efficient and easy to compute. The results of the numerical example 1 as shown in table 1 shows that the approximate solution at $N = 3$ gives $y_3(x) = -4.2497275388 \times 10^{-7} + 0.16917e - 5x + 0.9999976497x^2 - 0.9999997608x^3$ and solving at $N = 5$ indicates that as the value of N increases, the error becomes smaller. We also compare our absolute errors with Alkan *et al* (2017) as shown in table 2, this clearly shows that our method performs better. The results of the numerical example 2 as shown in table 3 shows that the approximate solution obtained at $N = 4$ and 5 converges to the exact solution. Table 4 compares the errors in the method proposed by Alkan *et al* (2017) at $N = 64$ and errors in our new method which converges to the exact solution at $N = 4$.

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