STRESS INTENSITY FACTOR IN A CONTACT PROBLEM OF A COATED LAYER IN FIELD OF THERMODIFFUSION

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We present a model of a layered composite with thin layers bonded to the bounding plane or planes. The purpose of the paper is to investigate the contact shear stresses in the matrix, generated by fluxes of heat and mass diffusion. It was found that the singularity in the value of stresses at the circumference of the membrane depends on the ratio of the strata thickness to the membrane radius. The maximum value of the stress intensity factor takes place neither for $h \rightarrow \infty$ nor for small values of the ratio. Another phenomenon is the foundation effect, i.e. that for small ratio, depending on the geometry of the problem there is no singularity in the distribution of shear stress components at the vicinity of the membrane circumference. The results are presented also in the diagram form.

1. Introduction

We consider particular problems of thermodiffusion of solid elastic solids. The purpose of the paper is to find the influence of the fields of temperature and that of diffusion on the distribution of stresses in an elastic matrix in the form of a layer with a thin layer coating it. The problem can be generalized on the cases of layered composites. The main problem is to find the stress concentrations, in the theoretical model the regions of stress singularities. We introduce a model of a solid (compare the monograph by Nowacki and Olesiak, 1991) consisting of an elastic matrix in the form of a layer with a bonded thin, inextensible membrane of a very small thickness, bounding the matrix. It has been assumed that there are no heat and/or diffusive sources within the elastic layer, and that the heat and diffusive fluxes enter through the boundary of the layer. The idealization that the bounding membrane is inextensible makes it

possible to reduce the problem to that of the fracture mechanics and to make use of its typical methods. In order to find the criterion of debonding we have to find the distribution of stresses over the contact region. The problem we consider belongs also to those of the general theory of coupled fields, developed and discussed by Kupradze at al. (1979). See also a chapter devoted to the systems of equations of thermodiffusion in the monograph by Burchuladze and Gegelia (1985). The problems of uniqueness, existence and estimates of the solutions were discussed in a monograph by Fichera (1974). In the case of a semi-infinite elastic space and axially symmetric geometry we have found already (cf Olesiak, 1989) that there exist singular distributions of shear stress component at the circumference of circular membrane and at the lines of a jump in the distribution of temperature and/or of the mass diffusion on the boundary.

The assumption that the bonded layer is inextensible and very thin constitutes a model approximation. For an extensible boundary layer one can expect, the stress concentrations in the regions where in the theoretical model there exist the singular stress distributions. In the cases when the pertinent stress intensity factor (i.e. the coefficient at the singular term) has been exceeded we can expect the debonding of the layer from the elastic matrix. The coefficient at the singular value of shear stress component is dependent on the magnitude of thermal and diffusive fluxes and on the value of the coatings radii.

In this paper we derive the formulae and discuss the behavior of the corresponding contact stresses in the case of matrices in the form of thick layers. If we drop the assumption that the bounding membrane is inextensible and assume that its Young modulus is much higher as compared with that for the matrix we would obtain the problems of stress concentration. These problems are quite common in the so called engineering practice, in particular in the problems of solids with the surface layer hardened by heat treatment and/or diffusive processes. The numerous examples of solids with metal covered surfaces are encountered in electronic technology.

We consider the axially symmetric problems. The problems which do not possess the cylindrical symmetry, for Jordan contours, are basically similar.

2. Basic equations

As our point of departure we take the set of partial differential equation

of thermodiffusion of elastic solids in the form

$$(1 - 2\nu)\Delta u + \text{graddiv} u = 2(1 + \nu)(\alpha_\theta \text{grad} T + \alpha_c \text{grad} C) \tag{2.1}$$

$$\Delta T = 0 \tag{2.2}$$

$$\Delta C = 0 \tag{2.3}$$

Here

λ, μ	–	Lamé's constants
ν	–	Poisson ratio
α_θ, α_c	–	coefficients of thermal and diffusive expansions, respectively
T	–	temperature in kelvingrades with respect to the natural state
C	–	concentration of diffusing mass likewise with respect to the natural state.

In a cylindrical coordinate system (r, ϑ, z), and for the displacement vector $\boldsymbol{u} = (u, 0, w)$ we obtain the following system of equations

$$2(1 - \nu)\mathcal{B}_1 u + (1 - 2\nu)D^2 u + Dw_{,r} = 2(1 + \nu)(\alpha_\theta T_{,r} + \alpha_c C_{,r}) \tag{2.4}$$

$$(1 - 2\nu)\mathcal{B}_0 w + 2(1 - \nu)D^2 w + D(ru)_{,r} = 2(1 + \nu)D(\alpha_\theta T + \alpha_c C) \tag{2.5}$$

$$(\mathcal{B}_0 + D^2)T = 0 \tag{2.6}$$

$$(\mathcal{B}_0 + D^2)C = 0 \tag{2.7}$$

where we have introduced the following differential operators

$$\mathcal{B}_k \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - kr^2 \qquad k = 0, 1 \qquad D \equiv \frac{\partial}{\partial z} \tag{2.8}$$

It is sufficient to assume that the required functions multiplied by $\sqrt{r}$ are integrable (i.e. $\in L^1$). Sommerfeld radiation (or regularity) conditions are necessary for $\sqrt{r}f(r) \in L^1$. We apply the first order Hankel transform with respect to Eq (2.4) and zero order Hankel transform with respect to Eqs (2.5) $\div$ (2.7), namely we assume

$$\hat{u}(\xi, z) = \mathcal{H}_1[u(r, z); \; r \to \xi] \equiv \int_0^\infty ru(r, z)J_1(r\xi)\,dr \tag{2.9}$$

$$\bar{w}(\xi, z), \bar{T}(\xi, z), \bar{C}(\xi, z) = \mathcal{H}_0[w(r, z), T(r, z), C(r, z); \; r \to \xi] \equiv \tag{2.10}$$

$$\equiv \int_0^\infty u(r, z), T(r, z), C(r, z)rJ_0(r\xi)\,dr$$

The solution of the system of equations (2.4) ÷ (2.7), expressed in terms of the Hankel transforms of the corresponding functions takes the following form

$$u(r, z) = \mathcal{H}_1[\hat{u}(\xi, z); \; \xi \to r] \tag{2.11}$$

$$w(r, z), T(r, z), C(r, z) = \mathcal{H}_0[\bar{w}(\xi, z), \bar{T}(\xi, z), \bar{C}(\xi, z); \; \xi \to r] \tag{2.12}$$

where, taking into account the general solutions of the system of ordinary differential equations in the space of transforms, obtained from the system of partial differential equations (2.4) ÷ (2.7), and some algebra, we find that

$$\hat{u}(\xi, z) = \left[-\gamma_3 - (3 - 4\nu)\beta_3(\xi) + \frac{2(1 + \nu)}{\xi}A(\xi) - \xi z \delta_3(\xi)\right]\cosh(\xi z) + \tag{2.13}$$

$$+ \left[-\alpha_3(\xi) - (3 - 4\nu)\delta_3(\xi) + \frac{2(1 + \nu)}{\xi}B(\xi) - \xi z \beta_3(\xi)\right]\sinh(\xi z)$$

$$\bar{w}(\xi, z) = [\alpha_3(\xi) + \xi z \beta_3(\xi)]\cosh(\xi z) + [(\gamma_3(\xi) + \xi z \delta_3(\xi)]\sinh(\xi z) \tag{2.14}$$

$$\bar{T}(\xi, z) = A_\theta(\xi)\cosh(\xi z) + B_\theta(\xi)\sinh(\xi z) \tag{2.15}$$

$$\bar{C}(\xi, z) = A_c(\xi)\cosh(\xi z) + B_c(\xi)\sinh(\xi z) \tag{2.16}$$

Here $\alpha_3(\xi)$, $\beta_3(\xi)$, $\gamma_3(\xi)$, $\delta_3(\xi)$, $A_\theta(\xi)$, $B_\theta(\xi)$, $A_c(\xi)$ and $B_c(\xi)$ denote the functions of ξ to be determined from the pertinent Hankel transforms of the boundary conditions.

Similarly we obtain from the generalized Duhamel-Neumann constitutive relations

$$\sigma_{zz}(r, z) = \mathcal{H}_0[\bar{\sigma}_{zz}(\xi, z); \; \xi \to r] \tag{2.17}$$

$$\sigma_{rz}(r, z) = \mathcal{H}_1[\hat{\sigma}_{rz}(\xi, z); \; \xi \to r] \tag{2.18}$$
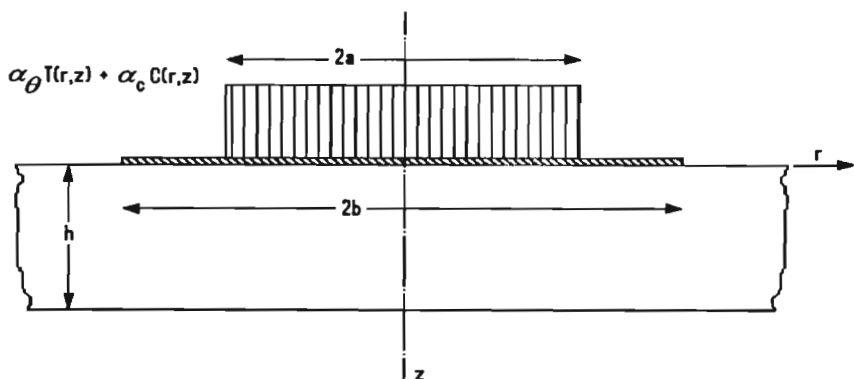
$$\sigma_{rr}(r, z) + \sigma_{\theta\theta}(r, z) = \mathcal{H}_0[\bar{\sigma}_{rr}(\xi, z) + \bar{\sigma}_{\theta\theta}(\xi, z); \; \xi \to r] \tag{2.19}$$

where

$$\bar{\sigma}_{zz}(\xi, z) = 2\mu\xi\Big\{[\alpha_3(\xi) + (1 - 2\nu)\delta_3(\xi) + \xi z \beta_3(\xi)]\sinh(\xi z) +$$

$$+ [\gamma_3 + (1 - 2\nu)\beta_3(\xi) + \xi z \delta_3(\xi)]\cosh(\xi z) + \tag{2.20}$$

$$- \big[(1 + \nu)A(\xi)\cosh(\xi z) + B(\xi)\sinh(\xi z)\big]\Big\}$$

$$\hat{\sigma}_{rz}(\xi, z) = -2\mu\xi\Big\{\left[\gamma_3(\xi) + 2(1 - \nu)\beta_3(\xi) - \frac{1 + \nu}{\xi}A(\xi) + \xi z \delta_3(\xi)\right]\sinh(\xi z) + \tag{2.21}$$

$$+ \left[\alpha_3(\xi) + 2(1 - \nu)\delta_3(\xi) - \frac{1 + \nu}{\xi}B(\xi) + \xi z \beta_3(\xi)\right]\cosh(\xi z)\Big\}$$

3. Boundary conditions

Fig. 1. An elastic stratum with bonded membrane of radius a

We solve the thermodiffusive problems for an elastic layer of width h and bonded circular membrane presented in Fig.1. The stresses in the matrix are exerted by a flow of heat and/or diffusive mass from the sources on the upper and lower planes bounding the body. We assume that on the upper plane there is a known distribution of temperature and diffusion over a circle of diameter a. Outside the region $r < a$ the solid is being kept in the natural state of temperature and diffusion. On the lower plane we assume either known distributions of temperature and diffusion or their fluxes. For the mechanical boundary conditions we assume that the upper layer is free from tractions over the region outside $r < a$ and free from normal components of tractions for $r < a$. For the bottom plane of the layer $z = h$ we assume that the shear stress tensor component and the normal component of the displacement vector vanish. First we take into account only three boundary conditions, namely

$$\left.\begin{array}{l} \sigma_{zz}(r,0) = 0 \\ \sigma_{rz}(r,h) = 0 \\ w(r,h) = 0 \end{array}\right\} \qquad \text{for} \quad r \in < 0, \infty) \tag{3.1}$$

By making use of the above boundary conditions we can eliminate three parameters, say: $\alpha_3(\xi)$, $\delta_3(\xi)$ and $\gamma_3(\xi)$ and express all quantities in terms of $\beta_3(\xi)$ and the parameters $A(\xi)$ and $B(\xi)$. In this way we obtain three linear algebraic equations $\gamma_3(\xi)$, $\delta_3(\xi)$, $\alpha_3(\xi)$. We obtain the equations for

the components of displacement vector (compare Olesiak, 1993)

$$
\begin{aligned}
\hat{u}(\xi, z) &= \Big\{[-2(1-\nu) + \xi z \tanh(\xi h)]\beta_3(\xi) + \frac{1+\nu}{\xi} A(\xi) + \\
&- \frac{1+\nu}{2(1-\nu)} z[B(\xi) + A(\xi)\tanh(\xi h)]\Big\}\cosh(\xi z) + \Big\{\{[2(1-\nu) + \\
&- \xi h \tanh(\xi h)]\tanh(\xi h) + \xi(h-z)\}\beta_3(\xi) + \frac{1+\nu}{\xi}B(\xi) + \\
&- \frac{1+\nu}{2(1-\nu)\xi}[1 - 2\nu - \xi h \tanh(\xi h)][B(\xi) + A(\xi)\tanh(\xi h)]\Big\}\sinh(\xi z)
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
\bar{w}(\xi, z) &= \Big\{\{[1 - 2\nu + \xi h \tanh(\xi h)]\tanh(\xi h) - \xi(h-z)\}\beta_3(\xi) + \\
&- \frac{1+\nu}{\xi}A(\xi)\tanh(\xi h) - \frac{1+\nu}{2(1-\nu)}h\tanh(\xi h)[B(\xi) + \\
&+ A(\xi)\tanh(\xi h)\Big\}\cosh(\xi x) + \Big\{-[1 - 2\nu + \xi z \tanh(\xi h)]\beta_3(\xi) + \\
&+ \frac{1+\nu}{\xi}A(\xi) - \frac{1+\nu}{2(1-\nu)}z[B(\xi) + A(\xi)\tanh(\xi h)]\Big\}\sinh(\xi z)
\end{aligned}
\tag{3.3}
$$

and the stress components

$$
\begin{aligned}
\bar{\sigma}_{zz}(\xi, z) &= -2\mu\xi\Big\{\xi z\{\tanh(\xi h)\beta_3(\xi) - \frac{1+\nu}{2(1-\nu)\xi}[B(\xi) + \\
&+ A(\xi)\tanh(\xi h)]\}\cosh(\xi z) + \{\xi[(h-z) - \xi h \tanh^2(\xi h)]\beta_3(\xi) + \\
&+ \frac{1+\nu}{2(1-\nu)\xi}(1 + \xi h \tanh(\xi h)[B(\xi) + A(\xi)\tanh(\xi h)])\sinh(\xi z)\Big\}
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
\hat{\sigma}_{rz}(\xi, z) &= 2\mu\xi\Big\{\{[(1 - \xi h \tanh(\xi h))\tanh(\xi h) + \xi(h-z)]\beta_3(\xi) + \\
&+ \frac{1+\nu}{2(1-\nu)}h[B(\xi) + A(\xi)\tanh(\xi h)]\}\cosh(\xi z) + \\
&- \{[1 - \xi z \tanh(\xi h)]\beta_3(\xi) + \\
&- \frac{1+\nu}{2(1-\nu)}z[B(\xi) + A(\xi)\tanh(\xi h)]\}\sinh(\xi z)\Big\}
\end{aligned}
\tag{3.5}
$$

Parameters $A(\xi)$ and $B(\xi)$ can be determined from the thermal and diffusive boundary conditions. First let us assume the boundary conditions on the

lower plane, i.e. for $z = h$. We shall discuss three basic cases, namely: case 1. when over $z = h$ both the temperature and the diffusion are being kept in the natural state, i.e. $T(r,h) = 0$, and $C(r,h) = 0$. In case 2. we assume the thermal and diffusive perfect isolation while in case 3. free heat and mass exchange. For case 2., i.e. for perfect isolation over $z = -h$

$$\frac{\partial}{\partial z}\bar{T}(\xi, z)\Big|_{z=h} = 0 \qquad \frac{\partial}{\partial z}\bar{C}(\xi, z)\Big|_{z=h} = 0$$

we obtain the following relations between the parameters $A_\theta(\xi)$, $B_\theta(\xi)$, $A_c(\xi)$ and $B_c(\xi)$

$$\bar{T}(\xi, z) = A_\theta(\xi)[\cosh(\xi z) - \tanh(\xi h)\sinh(\xi z)] \tag{3.6}$$

$$\bar{C}(\xi, z) = A_c(\xi)[\cosh(\xi z) - \tanh(\xi h)\sinh(\xi z)] \tag{3.7}$$

$$B(\xi) + A(\xi)\tanh(\xi h) = 0 \tag{3.8}$$

4. Reduction to a system of dual integral equations and Fredholm integral equation

In order to find the state of stresses and the contact shear stresses between the membrane of diameter $2b$ and the elastic layer we have to solve the system of equations, resulting from the boundary conditions, for $\beta_3(\xi)$. The mechanical boundary conditions corresponding to Fig.1 are as follows

$$\begin{aligned} u(r,0) &= 0 \quad &\text{for} \quad r \in< 0, b) \\ \sigma_{rz}(r,0) &= 0 \quad &\text{for} \quad r \in< b, \infty) \end{aligned} \tag{4.1}$$

We have the following expressions for the transforms of the radial displacement component and of the shear stress tensor component, expressed by a single parameter β_3, respectively

$$\hat{u}(\xi, 0) = -2(1 - \nu)\beta_3(\xi) + \frac{1+\nu}{\xi}A(\xi) \tag{4.2}$$

$$\hat{\sigma}_{rz}(\xi, 0) = 2\mu\xi\{[(1 - \xi h\tanh(\xi h))\tanh(\xi h) + \xi h]\beta_3(\xi) + $$

$$+ \frac{1+\nu}{2(1-\nu)}h\tanh(\xi h)[B(\xi) + A(\xi)\tanh(\xi h)]\} \tag{4.3}$$

We arrive at the following system of dual integral equations

$$\int_0^\infty \psi(\xi)[1 - k(\xi h)]J_1(\xi r)\,d\xi = f(r) \qquad r \in <0, b) \qquad (4.4)$$

$$\int_0^\infty \xi\psi(\xi)J_1(\xi r)\,d\xi = 0 \qquad r \in (b, \infty) \qquad (4.5)$$

On the right hand side of the first of the dual integral equation we have

$$f(r) \equiv \frac{1+\nu}{2(1-\nu)}\int_0^\infty A(\xi)J_1(\xi r)\,d\xi + \int_0^\infty \xi\phi_A[1 - k(\xi h)]J_1(\xi r)\,d\xi \qquad (4.6)$$

where

$$k(\xi h) = \frac{\xi h - \exp(-\xi h)\cosh(\xi h)}{\xi h + \sinh(\xi h)\cosh(\xi h)} \qquad (4.7)$$

and

$$\phi_A = \frac{1+\nu)}{2(1-\nu)}h\tanh(\xi h)[B + A\tanh(\xi h)] \qquad (4.8)$$

The system of dual integral equations can be reduced, by application of Abel transforms and the use of certain properties of Bessel functions and of elliptic integrals to a single integral equation of the second kind

$$g(t) = \frac{1}{\pi}\int_0^b g(x)K(t, x)\,dx + F(t) \qquad (4.9)$$

The kernel is given in the form of the sum of the Fourier cosine transform integrals

$$K(t, x) = \frac{2}{h}\int_0^\infty k\eta\sin\left(\eta\frac{t}{h}\right)\sin\left(\eta\frac{x}{h}\right)d\eta \equiv$$

$$\qquad (4.10)$$

$$\equiv \frac{1}{h}\left\{\mathcal{F}_c\left[k(\eta); \eta \to \frac{t-x}{h}\right] - \mathcal{F}_c\left[k(\eta); \eta \to \frac{t+x}{h}\right]\right\}$$

where

$$\mathcal{F}_c[k(\xi); \xi \to y] \equiv \int_0^\infty k(\xi)\cos(\xi y)\,d\xi \qquad (4.11)$$

If the solution of the integral equation (4.9) is known we are able to find function $\psi(\xi)$ appearing in the system of dual integral equations (4.4) and (4.5), and in turn the contact shear stress component, from the formula

$$\sigma(r,0) = 2\mu \int_0^\infty \psi(\xi) J_1(r\xi)\, d\xi \qquad (4.12)$$

The stresses can be expressed directly by the solution of the Fredholm integral equation $g(t,h)$

$$\sigma_{rz}(r,0) = 2\mu \sqrt{\frac{2}{\pi}} \left\{ \frac{r}{b\sqrt{b^2 - r^2}} g(b) - \int_r^b \frac{s^2 - r^2}{s^2} - \frac{1}{2}[sg'(s) - g(s)]\, ds \right\} \quad (4.13)$$

In order to obtain the distribution of stresses we have to find the kernel (4.10), then solve the Fredholm integral equation, and finally the stress tensor components. The kernel is an improper integral and can be evaluated numerically, consequently the solution of the Fredholm integral equation, and the contact shear stresses from formula (4.13) also has to be computed numerically with decreasing accuracy. The accuracy of the entire procedure is not high. To increase it we replace function $k(\xi h)$ by such a function which would behave at infinity and zero precisely as $k(\xi h)$ and the kernel integral could be calculated analytically. There exist a number of such functions, for example

$$\varphi_1 = \left(4\eta - \frac{1}{2\eta}\right) \exp(-2\eta) \qquad\qquad \eta \equiv \xi h$$

$$\varphi_2 = \left(4\eta^2 - \frac{1}{2\eta}\right) \exp(-2\eta)$$

(4.14)

In Fig.2a we show three curves: $k(\eta)$, $\varphi_1(\eta)$ and the difference $\varphi_1(\eta) - k(\eta)$. We see that the difference is a smooth function, vanishing at zero and tending rapidly to zero for values of $\eta \geq 3$. In Fig.2b curve called $\text{hump}(\eta)$ is the corrected curve $\varphi_1(\eta)$. We see that the difference $\text{hump}(\eta) - k(\eta)$ has been reduced to a small region, the segment of η between 1 and 2 and the integral limits reduce to this segment. In calculations we have assumed that $\text{hump}(\eta) = \varphi_1(\eta) - \text{corr}(\eta)$ where $\text{corr}(\eta) = 0.55 \sin(\eta\pi 5/6) H(1.2 - \eta)$, here $H(x)$ denotes Heaviside step distribution.

The approximate values of kernel $K1(x,t)$ have been presented in the diagram form in Fig.3. Kernel $K1(x,t)$ is symmetric and can be computed
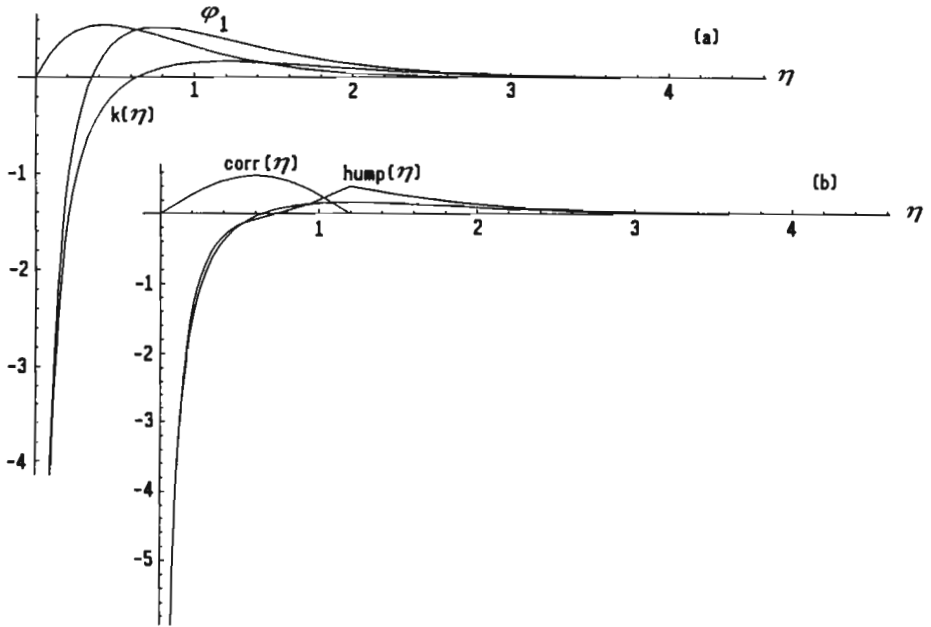
Fig. 2. (a) Functions $k(\eta)$, $\varphi_1(\eta)$ and their difference, (b) Functions $k(\eta)$, corr(η) and hump(η)
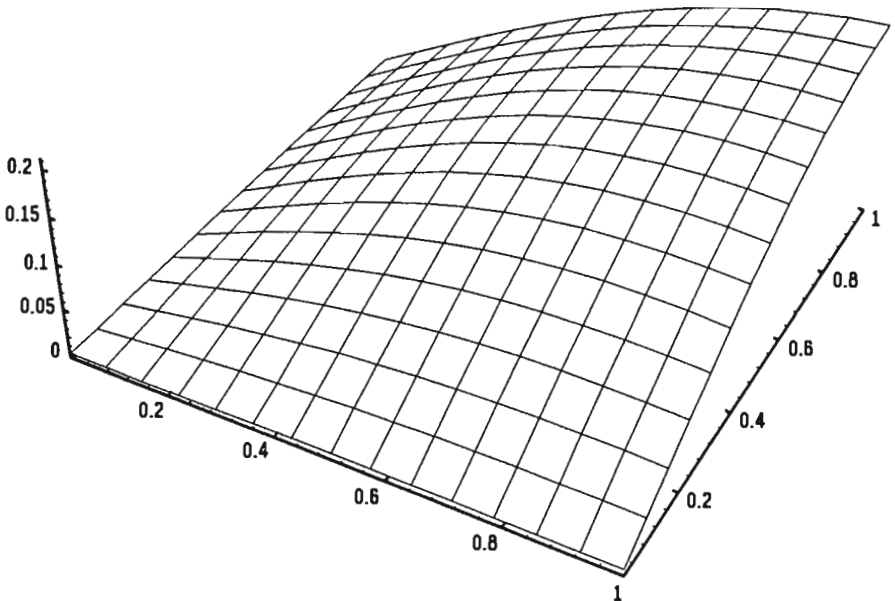

Fig. 3. Approximate kernel $K1(x,t)$ of the Fredholm integral equation (4.9)

analytically. We obtain

$$K1(t,x) = \frac{1}{h} \int\limits_{0}^{\infty} \left(4\eta - \frac{1}{2\eta}\right) \exp(-2\eta) \sin\left(\eta\frac{x}{h}\right) \sin\left(\eta\frac{t}{h}\right) d\eta =$$

(4.15)

$$= \frac{8h^2 tx[8h^2(6h^2 + t^2 + x^2) - (t^2 - x^2)^2]}{[4h^2 + (t-x)^2]^2[4h^2 + (t+x)^2]^2} - \frac{1}{8h} \ln \frac{4h^2 + (t+x)^2}{4h^2 + (t-x)^2}$$

The difference of kernel $K1(x,t)$ and the exact kernel $K(t,x)$ Eq (4.10), computed numerically, is presented in Fig.4.
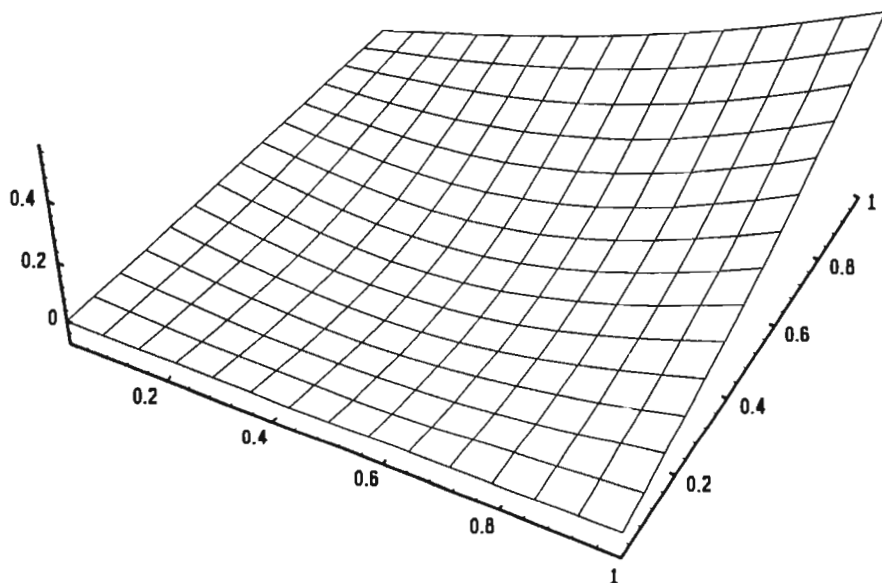

Fig. 4. Difference of two kernels $K1(t,x) - K(t,x)$

If, as a function under the improper integral sign, we take not $\varphi_1(\eta)$ but $\text{hump}(\eta) = \varphi_1(\eta) - \text{corr}(\eta)$ then the difference of the exact kernel $K(t,x)$ and the corrected kernel

$$K_{\text{corr}}(x,t) = K1(x,t) - \int\limits_{0}^{\frac{6}{5}} \text{corr}(\eta) \sin\left(\eta\frac{x}{h}\right) \sin\left(\eta\frac{t}{h}\right) d\eta$$

(4.16)

does not exceed 2%. Consequently it is justified to replace, in our calculations, kernel $K(t,x)$ by $K_{\text{corr}}(t,x)$. Since now the kernel of the Fredholm integral

equation can be written down analytically, we are able to solve it by the method of successive approximations. The Neumann series for the resolvent kernel takes the following form

$$g(t) = F(t) + \frac{1}{\pi}\psi_1(t) + \frac{1}{\pi^2}\psi_2(t) + \ldots \tag{4.17}$$

where

$$\psi_1(t) = \int\limits_0^b K(x,t)F(x)\,dx \tag{4.18}$$

and similarly for the next approximations. The solution depends on the free term $F(t)$. For the case of the thermal and diffusive boundary conditions discussed here, i.e. for the perfect isolation for $z = h$, function $F(t)$ corresponds to the pertinent function encountered in the solution of the analogous problem for an elastic semi-space. We have, (see Olesiak et al., 1992)

$$F(t) = \frac{1+\nu}{\sqrt{2\pi}(1-\nu)}\alpha\left(t - \sqrt{t^2 - a^2}H(t-a)\right) \tag{4.19}$$

where a is radius of the region in Fig.1, α is a constant in thermodiffusive boundary conditions, ν denotes Poisson ratio. The function $F(t)$ is continuous in the range $< 0, b)$ and has one point for which its derivative has a finite jump corresponding to the line of jump in the value of temperature and/or diffusion on the boundary. We assume that $b = 1$ and $a = 1/2$. For these values we obtain an approximate solution of the Fredholm integral equation from the truncated Neumann series. The solution depends on the parameter h, the width of the layer.

We see from the solution presented in Fig.5 that for smaller values of the thickness h of strata it assumes the negative values. In this problem it is important to know the behavior of function $g(1,h) = g(t)$ i.e. for $x = 1$ on the circumference of the membrane.

5.   Conclusions

The shear stress component is given by Eq (4.13). The coefficient at the singular term is the stress intensity factor K_{II}. Its magnitude is equal to

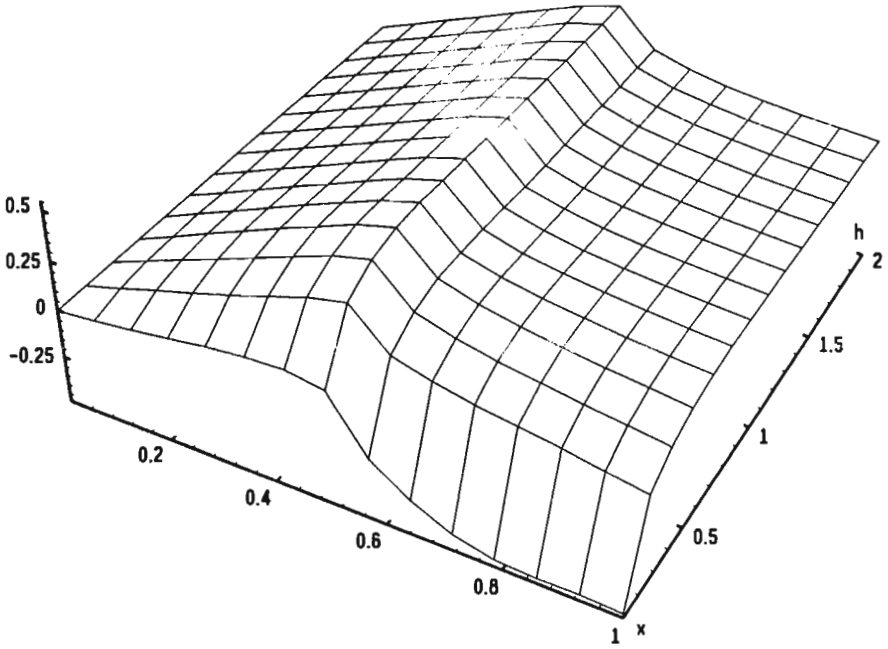$$K_{II} = 2\frac{1+\nu}{\sqrt{2\pi}(1-\nu)}\alpha\mu\sqrt{\frac{2}{\pi}}g(1,h) \tag{5.1}$$

Fig. 5. Solution $g(x, h)$ of the Fredholm integral equation (4.9)

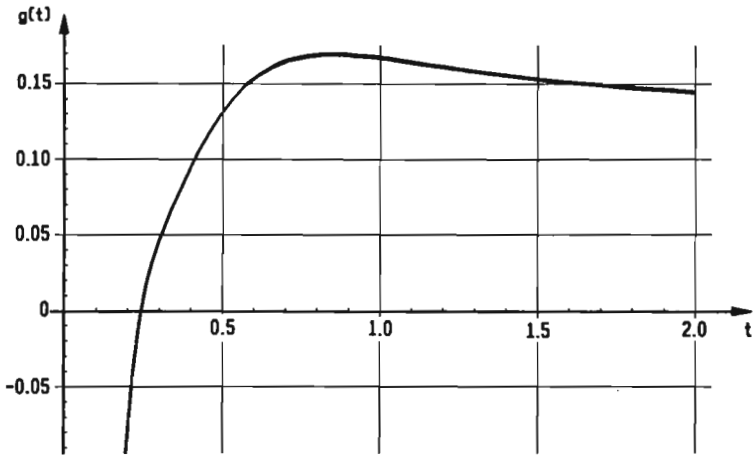Fig. 6. Function $g(t) \equiv g(1, h)$ versus h for $h \in\, <0.1, 2.0>$

From Fig.6 and Fig.5 it is evident that function $g(1, h)$ has maximum at $h = 0.8$ and is negative for $h \leq 0.24$. The behavior of $F(t)$ is crucial since for a corresponding value at which it has a jump in the value of its derivative the magnitude of the contact shear stress tensor commponent exhibits the singularity of the logarithmic type. The singularity of one over square root type in the magnitude of the shear stresses appears at all points for which $g(1, h) \geq 0$. This fact implies that there might be a region of delamination or debonding of the membrane from the elastic matrix and that it depends on the values of the stress intensity factor K_{II}. The existence of the regions of the negative values of the solution $g(t, h)$ implies that in the cases for smaller values of the ratios h/b and a/b the problem has to be treated as one with uknown regions of contact shear stresses. The negative values of $g(t, h)$ mean that the membrane coating the elastic strata is under compression.

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Współczynnik intensywności naprężeń w zagadnieniu warstwy z cienkim pokryciem w polu termodyfuzji

Streszczenie

Rozpatrujemy zagadnienie warstwy sprężystej pokrytej cienką nierozciągliwą membraną o promieniu b. Na skutek działania pola termodyfuzji powstają styczne naprężenia kontaktowe, które w pobliżu obwodu membrany wzrastają nieograniczenie. W pracy pokazaliśmy, dość nieoczekiwane zjawiska, a mianowicie, że wartość współczynnika intensywności naprężenia wzrasta do wartości maksymalnej dla stosunku grubości do promienia membrany 0.8, przy ustalonym promieniu działania źródeł termodyfuzji. Drugim zjawiskiem jest efekt podłoża, to znaczy, że dla małych wartości stosunku grubości do promienia osobliwość w wartości naprężeń przestaje występować, a w zagadnieniu należy uwzględnić warunki brzegowe ze zmiennymi granicami.