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FUNDAMENTAL SOLUTIONS RELATED TO THE STRESS INTENSITY FACTORS OF MODES *I*, *II* AND *III*. THE AXIALLY-SYMMETRIC PROBLEM

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Green's functions are obtained for the stress intensity factors of modes *I*, *II* and *III*. The Green's functions are defined as a solution to the problem of an elastic, transversely isotropic solid with a penny-shaped or an external crack under general axisymmetric loadings acting along a circle on the plane parallel to the crack plane. Exact solutions are presented in a closed form for the stress intensity factors under each type of axisymmetric ring forces assumed as fundamental solutions.

1. Introduction

It is clearly understood that the point force continuously distributed in radial (Fig.2) and axial (Fig.4) direction along a ring around the axis of symmetry gives the fundamental solutions to tension problems while those distributed in circumferential direction (Fig.3) give the fundamental solutions to a torsion problem. The problems of the crack treated in the present paper are solved by using three types of axisymmetric ring forces as fundamental solutions. The stress intensity factors of modes *I*, *II* and *III* are derived in this paper in terms of elementary functions and need no further elaboration. The results presented for general cases are new, but some of those relating to special cases of isotropic or transversely isotropic solids with crack surface tractions have been already known (cf Murakami, 1987; Rogowski, 1986).

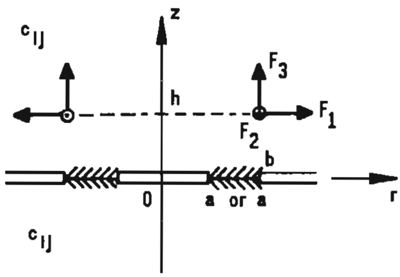

Fig. 1. A transversely isotropic elastic solid with a penny-shaped or external crack

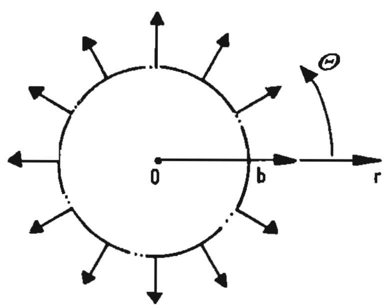

Fig. 2. A radial force acting along a circle

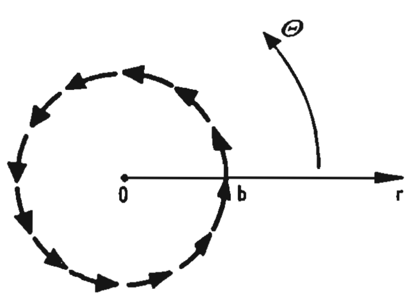

Fig. 3. A torsional force acting along a circle

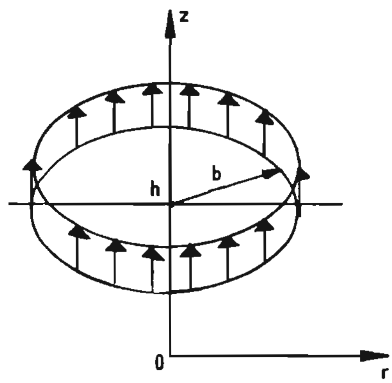

Fig. 4. An axial force acting along a circle

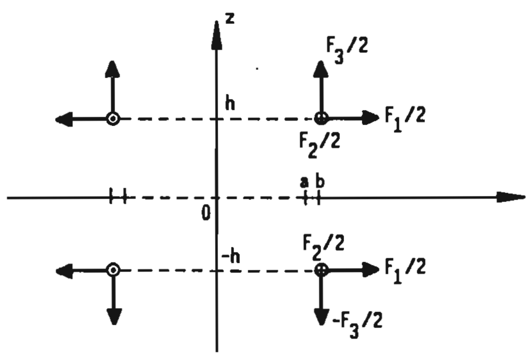

Fig. 5. Symmetric loadings

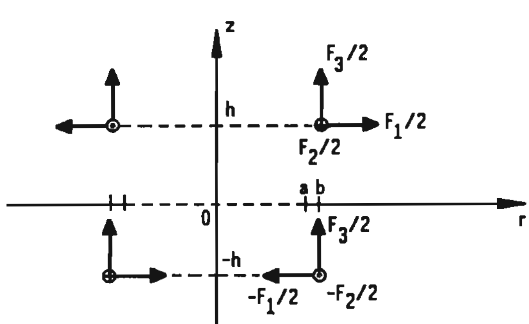

Fig. 6. Antisymmetric loadings

2. Basic equations

In this paper we use cylindrical coordinates and denote them by (r,θ,z) or $(i = 1,2,3)$. Let a penny-shaped crack or an external crack be located in the plane $z = 0$ of a homogeneous and transversely isotropic elastic solid.

The penny-shaped crack occupies the region $0 \leq r \leq a$ $(z = 0)$ and the external crack occupies the region $r \geq a$ $(z = 0)$. Both sides of the cracks are stress free.

The half-space $z \geq 0$ is subjected to axisymmetric body forces

$$
\begin{aligned}
F_i &= \frac{1_i}{2\pi r}\delta(r-b)\delta(z-h) \qquad\qquad i = 1,3 \\
F_2 &= \frac{1_2}{2\pi r^2}\delta(r-b)\delta(z-h)
\end{aligned}
\tag{2.1}
$$

distributed along a circle $(r = b, z = h)$ in the interior of the solid, where $\delta(\cdot)$ is the Dirac delta function and F_1, F_2, F_3 are a radial, a torsional and an axial forces, respectively, as shown in Fig.2, Fig.3 and Fig.4.

We consider axisymmetric deformations of an elastic transversely isotropic solid. That is, the displacements and stresses treated here are independent of angle θ in cylindrical coordinates (r,θ,z). We restrict our attention to the determination of the singular stresses at the crack tip, since these are the quantities of greatest physical interest. Due to the symmetry (Fig.5) or antisymmetry (Fig.6) of the problem, it can be reduced to a mixed boundary – value problem for half – space with the following mixed boundary conditions

— for a penny-shaped crack, $0 \leq r \leq a$

$$
\begin{array}{llllllr}
u_z = 0 & r > a & z = 0 & \sigma_z = 0 & r < a & z = 0 & (2.2.a) \\
u_r = 0 & r > a & z = 0 & \sigma_{zr} = 0 & r < a & z = 0 & (2.2.b) \\
u_\theta = 0 & r > a & z = 0 & \sigma_{z\theta} = 0 & r < a & z = 0 & (2.2.c)
\end{array}
$$

— for an external crack, $r \geq a$

$$
\begin{array}{llllllr}
u_z = 0 & r < a & z = 0 & \sigma_z = 0 & r > a & z = 0 & (2.3.a) \\
u_r = 0 & r < a & z = 0 & \sigma_{zr} = 0 & r > a & z = 0 & (2.3.b) \\
u_\theta = 0 & r < a & z = 0 & \sigma_{z\theta} = 0 & r > a & z = 0 & (2.3.c)
\end{array}
$$

under symmetric, antisymmetric and antisymmetric torsional loadings, respectively. The symmetric torsional loading yields $\sigma_{z\theta} = 0$ for $r \geq 0$, $z = 0$.

Suitable elasticity solutions for cracked solid that represent unit ring loading are obtained using the theory of Hankel transforms (cf Sneddon, 1972).

Avoiding details of calculations it can be shown that the displacement and stress fields associated with the action of concentrated axisymmetric ring forces meeting the mixed boundary conditions (2.2a,b,c) or (2.3a,b,c) on the plane where the crack appears are:

(i) For axial and radial symmetric forces as shown in Fig.4, Fig.2 and Fig.5

$$u_z(r,0) = \frac{1}{4\pi G_z C}\Big[1_3 \int\limits_0^\infty J_0(\xi r)J_0(\xi b)H_0(\xi s_i h)\, d\xi + -v_0 1_1 \int\limits_0^\infty J_0(\xi r)J_1(\xi b)H_1(\xi s_i h)\, d\xi + \int\limits_0^\infty A(\xi)J_0(\xi r)\, d\xi\Big] \tag{2.4}$$

$$\sigma_z(r,0) = -\frac{1}{4\pi}\int\limits_0^\infty \xi A(\xi)J_0(\xi r)\, d\xi$$

(ii) For axial and radial antisymmetric forces as shown in Fig.4, Fig.2 and Fig.6

$$u_r(r,0) = \frac{1}{4\pi G_z C s_1 s_2}\Big[-v_1 1_3 \int\limits_0^\infty J_1(\xi r)J_0(\xi b)H_2(\xi s_i h)\, d\xi + +1_1 \int\limits_0^\infty J_1(\xi r)J_1(\xi b)H_3(\xi s_i h)\, d\xi + \int\limits_0^\infty B(\xi)J_1(\xi r)\, d\xi\Big] \tag{2.5}$$

$$\sigma_z(r,0) = -\frac{1}{4\pi}\int\limits_0^\infty \xi B(\xi)J_1(\xi r)\, d\xi$$

(iii) For antisymmetric torsional force as shown in Fig.2 and Fig.6

$$u_\theta(r,0) = \frac{1}{4\pi G_z s_3}\Big[\frac{1_2}{b}\int\limits_0^\infty e^{-\xi s_3 h}J_1(\xi r)J_1(\xi b)\, d\xi + \int\limits_0^\infty C(\xi)J_1(\xi b)\, d\xi\Big] \tag{2.6}$$

$$\sigma_{z\theta}(r,0) = -\frac{1}{4\pi}\int\limits_0^\infty \xi C(\xi)J_1(\xi r)\, d\xi$$

where

J_ν – Bessel function of the first kind of order ν

H_i – known functions in the form of exponentials, $i = 0,1,2,3$ (cf Appendix Eqs(A.11)

G_z – shear modulus of material in the z-direction and four material parameters: s_i, C, v_0 and v_1 see Appendix Eqs (A.9) and (A.10).

Within the context of linear elastic fracture mechanics, the stress intensity factors are defined as

$$\left.\begin{array}{c} K_I \\ K_{II} \\ K_{III} \end{array}\right\} = \lim_{r\to a+} \sqrt{2(r-a)} \left\{\begin{array}{c} \sigma_z(r,0) \\ \sigma_{zr}(r,0) \\ \sigma_{z\theta}(r,0) \end{array}\right. \tag{2.7}$$

$$\left.\begin{array}{c} K_I \\ K_{II} \\ K_{III} \end{array}\right\} = \lim_{r\to a-} \sqrt{2(a-r)} \left\{\begin{array}{c} \sigma_z(r,0) \\ \sigma_{zr}(r,0) \\ \sigma_{z\theta}(r,0) \end{array}\right. \tag{2.8}$$

for a penny-shaped crack and an external crack, respectively.

K_I, K_{II}, K_{III} are the mode I, II, III stress intensity factors (Kanninen and Popelar, 1985), respectively, corresponding to the cases (i), (ii) and (iii) of loading, respecively.

3. Mode I loading

3.1. A penny-shaped crack

The boundary conditions (2.2.a) and the solutions (2.4) yield

$$\begin{aligned} \int_0^\infty A(\xi)J_0(\xi r)\,d\xi &= -\mathrm{l}_3 \int_0^\infty J_0(\xi r)J_0(\xi b)H_0(\xi s_i h)\,d\xi + \\ &+ v_0 \mathrm{l}_1 \int_0^\infty J_0(\xi r)J_1(\xi b)H_1(\xi s_i h)\,d\xi \qquad r > a \end{aligned} \tag{3.1}$$

$$\int_0^\infty \xi A(\xi)J_0(\xi r)\,d\xi = 0 \qquad r < a \tag{3.2}$$

The dual integral equations (3.1), (3.2) are converted to the Abel integral equation by means of the following integral representation of $A(\xi)$

$$A(\xi) = \sqrt{\frac{2}{\pi}} \int_0^a g(x)\sin(\xi x)\,dx - \mathrm{l}_3 J_0(\xi b)H_0(\xi s_i h) + v_0 \mathrm{l}_1 J_1(\xi b)H_1(\xi s_i h) \tag{3.3}$$

on the assumption that $g(x) \to 0$ as $x \to 0^+$.

This representation of $A(\xi)$ identically satisfies Eq (3.1). Substituting for $A(\xi)$ into Eq (3.2) leads to the following Abel integral equation in an auxiliary function $g(x)$

$$\sqrt{\frac{2}{\pi}} \int_0^r \Big(\frac{dg(x)}{dx} \frac{1}{\sqrt{r^2 - x^2}}\Big)\, dx = 1_3 \int_0^\infty \xi J_0(\xi r) J_0(\xi b) H_0(\xi s_i h)\, d\xi + \\ -v_0 1_1 \int_0^\infty \xi J_0(\xi r) J_1(\xi b) H_1(\xi s_i h)\, d\xi \tag{3.4}$$

Applying the Abel solution method to invert the left-hand side of Eq (3.4), give the formula for $g(x)$

$$g(x) = \sqrt{\frac{2}{\pi}} \Big[1_3 \int_0^\infty J_0(\xi b) \sin(\xi x) H_0(\xi s_i h)\, d\xi + \\ - v_0 1_1 \int_0^\infty J_1(\xi b) \sin(\xi x) H_1(\xi s_i h)\, d\xi \Big] \tag{3.5}$$

The improper integrals appearing in Eq (3.5) are calculated analytically (see Appendix, Eqs (A.1) and (A.2)). Consequently, the auxiliary function $g(x)$ is obtained explicitly in terms of the oblate spheroidal coordinates ζ_i and η_i (see Appendix) as

$$g(x) = \sqrt{\frac{2}{\pi}} \Big[\frac{1_3}{(k-1)x} \Big(\frac{k\eta_2}{\zeta_2^2 + \eta_2^2} - \frac{\eta_1}{\zeta_1^2 + \eta_1^2} \Big) + \\ - \frac{v_0 1_1 b}{(ks_2 - s_1)x^2} \Big(\frac{ks_2 \zeta_1}{(\zeta_1^2 + \eta_1^2)(1 + \zeta_1^2)} - \frac{s_1 \zeta_2}{(\zeta_2^2 + \eta_2^2)(1 + \zeta_2^2)} \Big) \Big] \tag{3.6}$$

where the material parameters s_1, s_2, k and v_0 are given in the Appendix.

The singular part of the axial stress is given by a formula

$$\sigma_z(r, 0) = \sqrt{\frac{2}{\pi}} \frac{g(a)}{4\pi \sqrt{r^2 - a^2}} \qquad \text{as} \qquad r \to a^+ \tag{3.7}$$

Consequently, from Eqs $(2.7)_1$, (3.6) and (3.7), the stress intensity factor at the crack tip is obtained explicitly in terms of the oblate spheroidal coordinates

$\bar{\zeta}_i$ and $\bar{\eta}_i$ (for the values of ζ_i and η_i for $x = a$, see Appendix) as

$$
\begin{aligned}
K_I &= \frac{1}{2\pi^2\sqrt{a^3}}\Big[\frac{l_3}{k-1}\Big(\frac{k\bar{\eta}_2}{\bar{\zeta}_2^2+\bar{\eta}_2^2} - \frac{\bar{\eta}_1}{\bar{\zeta}_1^2+\bar{\eta}_1^2}\Big) + \\
&- \frac{v_0 l_1 b}{(ks_2 - s_1)a}\Big(\frac{ks_2\bar{\zeta}_1}{(1+\bar{\zeta}_1^2)(\bar{\zeta}_1^2+\bar{\eta}_1^2)} - \frac{s_1\bar{\zeta}_2}{(1+\bar{\zeta}_2^2)(\bar{\zeta}_2^2+\bar{\eta}_2^2)}\Big)\Big]
\end{aligned}
\tag{3.8}
$$

3.2. An external crack

The boundary conditions (2.3a) with the use of Eq (2.4) yield

$$
\begin{aligned}
\int\limits_0^\infty A(\xi)J_0(\xi r)\,d\xi &= -l_3\int\limits_0^\infty J_0(\xi r)J_0(\xi b)H_0(\xi s_i h)\,d\xi + \\
&+ v_0 l_1\int\limits_0^\infty J_0(\xi r)J_1(\xi b)H_1(\xi s_i h)\,d\xi \qquad r < a
\end{aligned}
\tag{3.9}
$$

$$
\int\limits_0^\infty \xi A(\xi)J_0(\xi r)\,d\xi = 0 \qquad r > a \tag{3.10}
$$

The dual integral equations (3.9), (3.10) are converted to the Abel integral equation, by means of the following integral representation of $A(\xi)$

$$
A(\xi) = \sqrt{\frac{2}{\pi}}\int\limits_0^a f(x)\cos(\xi x)\,dx \tag{3.11}
$$

In this representation the auxiliary function $f(x)$ is assumed to be continuous over the interval $[0,a]$. This representation of $A(\xi)$ identically satisfies Eq (3.10). Substituting for $A(\xi)$ into Eq (3.9) lead to the following Abel integral equation

$$
\begin{aligned}
\sqrt{\frac{2}{\pi}}\int\limits_0^r \frac{f(x)}{\sqrt{r^2-x^2}}\,dx &= -l_3\int\limits_0^\infty J_0(\xi r)J_0(\xi b)H_0(\xi s_i h)\,d\xi + \\
&+ v_0 l_1\int\limits_0^\infty J_0(\xi r)J_1(\xi b)H_1(\xi s_i h)\,d\xi
\end{aligned}
\tag{3.12}
$$

Applying the Abel solution method, give the formula for $f(x)$

$$
\begin{aligned}
f(x) &= \sqrt{\frac{2}{\pi}}\Big[-1_3\int\limits_0^\infty \cos(\xi x)J_0(\xi b)H_0(\xi s_i h)\,d\xi + \\
&+ v_0 1_1\int\limits_0^\infty \cos(\xi x)J_1(\xi b)H_1(\xi s_i h)\,d\xi\Big]
\end{aligned}
\tag{3.13}
$$

Substituting for the integrals (A.3) and (A.4) (see Appendix) give the final formula for $f(x)$

$$
\begin{aligned}
f(x) &= \sqrt{\frac{2}{\pi}}\Big\{-\frac{1_3}{(k-1)x}\Big(\frac{k\zeta_2}{\zeta_2^2+\eta_2^2} - \frac{\zeta_1}{\zeta_1^2+\eta_1^2}\Big) + \\
&+ \frac{v_0 1_1}{b}\Big[1 - \frac{1}{ks_2 - s_1}\Big(\frac{ks_2\eta_1(1+\zeta_1^2)}{\zeta_1^2+\eta_1^2} - \frac{s_1\eta_2(1+\zeta_2^2)}{\zeta_2^2+\eta_2^2}\Big)\Big]\Big\}
\end{aligned}
\tag{3.14}
$$

where the oblate spheroidal coordinates ζ_i, η_i are defined in the Appendix.

The stress $\sigma_z(r,0)$ for $r < a$ is given by

$$
\sigma_z(r,0) = -\frac{1}{4\pi}\sqrt{\frac{2}{\pi}}\Big[\frac{f(a)}{\sqrt{a^2-r^2}} - \int\limits_r^a \Big(\frac{df(x)}{dx}\frac{1}{\sqrt{x^2-r^2}}\Big)\,dx\Big] \qquad r < a \tag{3.15}
$$

Conseqently, from Eqs (3.14), (3.15) and $(2.8)_1$, the stress intensity factor of mode I can be obtained in terms of the coordinates $\bar{\zeta}_i$, $\bar{\eta}_i$ such that

$$
\begin{aligned}
K_I &= \frac{1}{2\pi^2\sqrt{a^3}}\Big\{\frac{1_3}{k-1}\Big(\frac{k\bar{\zeta}_2}{\bar{\zeta}_2^2+\bar{\eta}_2^2} - \frac{\bar{\zeta}_1}{\bar{\zeta}_1^2+\bar{\eta}_1^2}\Big) + \\
&- \frac{v_0 1_1 a}{b}\Big[1 - \frac{1}{ks_2 - s_1}\Big(\frac{ks_2\bar{\eta}_1(1+\bar{\zeta}_1^2)}{\bar{\zeta}_1^2+\bar{\eta}_1^2} - \frac{s_1\bar{\eta}_2(1+\bar{\zeta}_2^2)}{\bar{\zeta}_2^2+\bar{\eta}_2^2}\Big)\Big]\Big\}
\end{aligned}
\tag{3.16}
$$

where $\bar{\zeta}_i$, $\bar{\eta}_i$ are obtained from ζ_i, η_i for $x = a$ (see Appendix).

4. Mode *II* loading

4.1. A penny-shaped crack

Substituting Eq (2.5) into the boundary conditions (2.2b) the following dual integral equations for antisymmetric loading cases are obtained

$$\int_0^\infty B(\xi)J_1(\xi r)\,d\xi = v_1 1_3 \int_0^\infty J_1(\xi r)J_0(\xi b)H_2(\xi s_i h)\,d\xi + \\ -1_1 \int_0^\infty J_1(\xi r)J_1(\xi b)H_3(\xi s_i h)\,d\xi \qquad r > a \tag{4.1}$$

$$\int_0^\infty \xi B(\xi)J_1(\xi r)\,d\xi = 0 \qquad r < a \tag{4.2}$$

The integral representation of $B(\xi)$

$$B(\xi) = \sqrt{\xi}\int_0^a \sqrt{x}h(x)J_{3/2}(x\xi)\,dx + \\ + v_1 1_3 J_0(\xi b)H_2(\xi s_i h) - 1_1 J_1(\xi b)H_3(\xi s_i h) \tag{4.3}$$

on the assumption that $\sqrt{x}h(x) \to 0$ as $x \to 0^+$, satisfies identically Eq (4.1), while Eq (4.2) is converted to the Abel integral equation

$$\sqrt{\frac{2}{\pi}}\int_0^r \Big(\frac{d[xh(x)]}{dx}\frac{1}{\sqrt{r^2-x^2}}\Big)\,dx = -v_1 1_3 r\int_0^\infty \xi J_1(\xi r)J_0(\xi b)H_2(\xi s_i h)\,d\xi + \\ +1_1 r\int_0^\infty J_1(\xi r)J_1(\xi b)H_3(\xi s_i h)\,d\xi \tag{4.4}$$

The solution to this equation is

$$h(x) = \sqrt{\frac{2}{\pi}}\Big[-v_1 1_3\int_0^\infty J_0(\xi b)\Big(\frac{\sin\xi x}{\xi x} - \cos\xi x\Big)H_2(\xi s_i h)\,d\xi + \\ + 1_1\int_0^\infty J_1(\xi b)\Big(\frac{\sin\xi x}{\xi x} - \cos\xi x\Big)H_3(\xi s_i h)\,d\xi\Big] \tag{4.5}$$

Using the integrals (A.3) $\div$ (A.6) (see Appendix) give the final formula for the auxiliary function $h(x)$

$$
h(x) = \sqrt{\frac{2}{\pi}}\Big\{-\frac{v_1 1_3}{x(ks_2 - s_1)}\Big[ks_2\Big(\frac{\pi}{2} - \arctan\zeta_2 - \frac{\zeta_2}{\zeta_2^2+\eta_2^2}\Big) + \\
-s_1\Big(\frac{\pi}{2} - \arctan\zeta_1 - \frac{\zeta_1}{\zeta_1^2+\eta_1^2}\Big)\Big] + \frac{1_1}{b(k-1)}\Big[\frac{k\eta_1(1-\eta_1^2)}{\zeta_1^2+\eta_1^2} - \frac{\eta_2(1-\eta_2^2)}{\zeta_2^2+\eta_2^2}\Big]\Big\} \tag{4.6}
$$

The singular part of the shear stress is given by

$$
\sigma_{zr}(r,0) = \sqrt{\frac{2}{\pi}}\frac{ah(a)}{4\pi r\sqrt{r^2-a^2}} \qquad \text{as} \quad r \to a^+ \tag{4.7}
$$

Defining the stress intensity factor of mode II as in Eq $(2.8)_2$, and substituting for $h(a)$ we obtain

$$
K_{II} = \frac{1}{2\pi^2\sqrt{a^3}}\Big\{-\frac{v_1 1_3}{ks_2 - s_1}\Big[ks_2\Big(\frac{\pi}{2} - \arctan\bar{\zeta}_2 - \frac{\bar{\zeta}_2}{\bar{\zeta}_2^2+\bar{\eta}_2^2}\Big) + \\
-s_1\Big(\frac{\pi}{2} - \arctan\bar{\zeta}_1 - \frac{\bar{\zeta}_1}{\bar{\zeta}_1^2+\bar{\eta}_1^2}\Big)\Big] + \frac{1_1 a}{(k-1)b}\Big[\frac{k\bar{\eta}_1(1-\bar{\eta}_1^2)}{\bar{\zeta}_1^2+\bar{\eta}_1^2} - \frac{\bar{\eta}_2(1-\bar{\eta}_2^2)}{\bar{\zeta}_2^2+\bar{\eta}_2^2}\Big]\Big\} \tag{4.8}
$$

where $\bar{\zeta}_i$, $\bar{\eta}_i$ are the values of ζ_i, η_i for $x = a$ (see Appendix).

4.2. An external crack

Substituting Eq (2.5) into the boundary conditions (2.3b) the following dual integral equations for antisymmetric loading cases are obtained

$$
\int\limits_0^\infty B(\xi)J_1(\xi r)\,d\xi = v_1 1_3 \int\limits_0^\infty J_1(\xi r)J_0(\xi b)H_2(\xi s_i h)\,d\xi + \\
-1_1 \int\limits_0^\infty J_1(\xi r)J_1(\xi b)H_3(\xi s_i h)\,d\xi \qquad r < a \tag{4.9}
$$

$$
\int\limits_0^\infty \xi B(\xi)J_1(\xi r)\,d\xi = 0 \qquad r > a \tag{4.10}
$$

The integral representation of $B(\xi)$

$$
B(\xi) = \sqrt{\frac{2}{\pi}}\int\limits_0^a t(x)\sin(\xi x)\,dx \tag{4.11}
$$

satisfies identically Eq (4.10), while Eq (4.9) is converted to the Abel integral equation

$$\sqrt{\frac{2}{\pi}}\int\limits_0^r \frac{xt(x)}{\sqrt{r^2-x^2}}\,dx = v_1 1_3 r\int\limits_0^\infty J_1(\xi r)J_0(\xi b)H_2(\xi s_i h)\,d\xi + -1_1 r\int\limits_0^\infty J_1(\xi r)J_1(\xi b)H_3(\xi s_i h)\,d\xi \tag{4.12}$$

The solution to this equation is

$$t(x) = \sqrt{\frac{2}{\pi}}\Big[v_1 1_3\int\limits_0^\infty J_0(\xi b)\sin(\xi x)H_2(\xi s_i h)\,d\xi + - 1_1\int\limits_0^\infty J_1(\xi b)\sin(\xi x)H_3(\xi s_i h)\,d\xi\Big] \tag{4.13}$$

Substituting the analytical formulae for the improper integrals (Eqs (A.1) and (A.2) in the Appendix) we get

$$t(x) = \sqrt{\frac{2}{\pi}}\Big[\frac{v_1 1_3}{x(ks_2-s_1)}\Big(\frac{ks_2\eta_2}{\zeta_2^2+\eta_2^2}-\frac{s_1\eta_1}{\zeta_1^2+\eta_1^2}\Big) + - \frac{1_1 b}{x^2(k-1)}\Big(\frac{k\zeta_1}{(1+\zeta_1^2)(\zeta_1^2+\eta_1^2)}-\frac{\zeta_2}{(1+\zeta_2^2)(\zeta_2^2+\eta_2^2)}\Big)\Big] \tag{4.14}$$

The stress $\sigma_{zr}(r,0)$ is

$$\sigma_{zr}(r,0) = \sqrt{\frac{2}{\pi}}\frac{1}{4\pi}\Big[-\frac{rt(a)}{a\sqrt{a^2-r^2}} + r\int\limits_r^a \frac{d}{dx}\Big(\frac{t(x)}{x}\Big)\frac{dx}{\sqrt{x^2-r^2}}\Big] \qquad r<a \tag{4.15}$$

The mode II stress intensity factor of the external crack is obtained as

$$K_{II} = \frac{1}{2\pi^2\sqrt{a^3}}\Big[-\frac{v_1 1_3}{ks_2-s_1}\Big(\frac{ks_2\bar{\eta}_2}{\bar{\zeta}_2^2+\bar{\eta}_2^2}-\frac{s_1\bar{\eta}_1}{\bar{\zeta}_1^2+\bar{\eta}_1^2}\Big) + + \frac{1_1 b}{(k-1)a}\Big(\frac{k\bar{\zeta}_1}{(1+\bar{\zeta}_1^2)(\bar{\zeta}_1^2+\bar{\eta}_1^2)}-\frac{\bar{\zeta}_2}{(1+\bar{\zeta}_2^2)(\bar{\zeta}_2^2+\bar{\eta}_2^2)}\Big)\Big] \tag{4.16}$$

5. Mode *III* loading

5.1. A penny-shaped crack

The boundary conditions (2.2c) with the use of Eq (2.6) yields the following dual integral equations of axisymmetric torsion of the penny-shaped crack

$$\int_0^\infty C(\xi)J_1(\xi r)\,d\xi = -\frac{1_2}{b}\int_0^\infty J_1(\xi r)J_1(\xi b)\mathrm{e}^{-\xi s_3 h}\,d\xi \qquad r > a \tag{5.1}$$

$$\int_0^\infty \xi C(\xi)J_1(\xi r)\,d\xi = 0 \qquad r < a \tag{5.2}$$

The integral representation of $C(\xi)$

$$C(\xi) = \sqrt{\xi}\int_0^a \sqrt{x}\varphi(x)J_{3/2}(x\xi)\,d\xi - \frac{1_2}{b}J_1(\xi b)\mathrm{e}^{-\xi s_3 h} \tag{5.3}$$

under assumption that $\sqrt{x}\varphi(x) \to 0$ as $x \to 0^+$, satisfies identically Eq (5.1) and gives the Abel integral equation

$$\sqrt{\frac{2}{\pi}}\int_0^r \Big(\frac{d[x\varphi(x)]}{dx}\frac{1}{\sqrt{r^2-x^2}}\Big)\,dx = \frac{r1_2}{b}\int_0^\infty \xi J_1(\xi r)J_1(\xi b)\mathrm{e}^{-\xi s_3 h}\,d\xi \tag{5.4}$$

Applying the Abel solution method to invert the left-hand side of Eq (5.4) and then substituting for integrals Eqs (A.4) and (A.6) from the Appendix one obtains the final formula for $\varphi(x)$ where

$$\varphi(x) = \sqrt{\frac{2}{\pi}}\frac{1_2\eta_3}{x^2(1+\zeta_3^2)(\zeta_3^2+\eta_3^2)} \tag{5.5}$$

where ζ_3 and η_3 are defined in the Appendix.

The singular part of the stress $\sigma_{z\theta}$ is given by

$$\sigma_{z\theta}(r,0) = \sqrt{\frac{2}{\pi}}\frac{a\varphi(a)}{4\pi r\sqrt{r^2-a^2}} \qquad \text{as} \quad r \to a^+ \tag{5.6}$$

The solution (5.5) and Eq (5.6) give

$$K_{III} = \frac{1}{2\pi^2\sqrt{a^5}}\frac{\bar{\eta}_3}{(1+\bar{\zeta}_3^2)(\bar{\zeta}_3^2+\bar{\eta}_3^2)} \tag{5.7}$$

where $\bar{\zeta}_3$, $\bar{\eta}_3$ are the values of ζ_3, η_3 for $x = a$ (see Appendix).

5.2. An external crack

The boundary conditions (2.3c) and the solutions (2.6) yield

$$\int_0^\infty C(\xi)J_1(\xi r)\,d\xi = -\frac{1_2}{b}\int_0^\infty J_1(\xi r)J_1(\xi b)e^{-\xi s_3 h}\,d\xi \qquad r < a \qquad (5.8)$$

$$\int_0^\infty \xi C(\xi)J_1(\xi r)\,d\xi = 0 \qquad r > a \qquad (5.9)$$

These equations have a form similar to that of Eqs (4.9), (4.10).

Thus, the integral representation for $C(\xi)$

$$C(\xi) = \sqrt{\frac{2}{\pi}}\int_0^a \psi(x)\sin(\xi x)\,dx \qquad (5.10)$$

gives the final formula for an auxiliary function $\psi(x)$

$$\psi(x) = -\sqrt{\frac{2}{\pi}}\frac{1_2\zeta_3}{x^2(1+\zeta_3^2)(\zeta_3^2+\eta_3^2)} \qquad (5.11)$$

where ζ_3, η_3 are the oblate spheroidal coordinates associated with the material parameter s_3 (see Appendix).

The stress $\sigma_{z\theta}(r,0)$ is

$$\sigma_{z\theta}(r,0) = \sqrt{\frac{2}{\pi}}\frac{1}{4\pi}\Big[-\frac{r\psi(a)}{a\sqrt{a^2-r^2}} + r\int_r^a \frac{d}{dx}\Big(\frac{\psi(x)}{x}\Big)\frac{dx}{\sqrt{x^2-r^2}}\Big] \qquad r < a \qquad (5.12)$$

The stress intensity factor of mode III for an external crack reads

$$K_{III} = \frac{1_2}{2\pi^2\sqrt{a^5}}\frac{\bar{\zeta}_3}{(1+\bar{\zeta}_3^2)(\bar{\zeta}_3^2+\bar{\eta}_3^2)} \qquad (5.13)$$

where $\bar{\zeta}_3$, $\bar{\eta}_3$ are obtained from ζ_3, η_3 for $x = a$.

Appendix

The following integrals are used to evaluate the auxiliary functions appearing in this paper

$$\int_0^\infty J_0(\xi b)\sin(\xi x)e^{-\xi s_i h}\,d\xi = \frac{\eta_i}{x(\zeta_i^2+\eta_i^2)} \qquad (A.1)$$

$$\int_0^\infty J_1(\xi b)\sin(\xi x)e^{-\xi s_i h}\,d\xi = \frac{b\zeta_i}{x^2(1+\zeta_i^2)(\zeta_i^2+\eta_i^2)} \tag{A.2}$$

$$\int_0^\infty J_0(\xi b)\cos(\xi x)e^{-\xi s_i h}\,d\xi = \frac{\zeta_i}{x(\zeta_i^2+\eta_i^2)} \tag{A.3}$$

$$\int_0^\infty J_1(\xi b)\cos(\xi x)e^{-\xi s_i h}\,d\xi = \frac{1}{b}\Big[1-\frac{\eta_i(1+\zeta_i^2)}{\zeta_i^2+\eta_i^2}\Big] \tag{A.4}$$

$$\int_0^\infty \frac{1}{\xi}J_0(\xi b)\sin(\xi x)e^{-\xi s_i h}\,d\xi = \frac{\pi}{2}-\arctan\zeta_i \tag{A.5}$$

$$\int_0^\infty \frac{1}{\xi}J_1(\xi b)\sin(\xi x)e^{-\xi s_i h}\,d\xi = \frac{x}{b}(1-\eta_i) \tag{A.6}$$

The oblate spheroidal coordinates ζ_i, η_i are related to b, s_ih, x by the equations

$$b^2 = x^2(1+\zeta_i^2)(1-\eta_i^2) \qquad s_ih = x\zeta_i\eta_i \tag{A.7}$$

where $-1 \le \eta_i \le 1$ and $\zeta_i \ge 0$. The surfaces $\zeta_i = 0$ and $\eta_i = 0$ are the interior and the exterior of circle $b = x$, $h = 0$, respectively; here therefore

$$\begin{array}{llll} h=0 & b<x & \zeta_i = 0 & \eta_i = \sqrt{1-\frac{b^2}{x^2}} \\ h=0 & b>x & \zeta_i = \sqrt{\frac{b^2}{x^2}-1} & \eta_i = 0 \\ & b=0 & \zeta_i = \frac{s_ih}{x} & \eta_i = 1 \end{array} \tag{A.8}$$

The coordinates ζ_i, η_i for $x = a$ are denoted by $\bar\zeta_i$, $\bar\eta_i$.

Three sets of oblate spheroidal coordinates ζ_i, η_i $(i = 1,2,3)$ are associated with three material parameters s_i $(i = 1,2,3)$ which are given by equations

$$s_i: \quad c_{33}c_{44}s^4 - [c_{11}c_{33} - c_{13}(c_{13}+2c_{44})]s^2 + c_{11}c_{44} = 0 \qquad i = 1,2 \tag{A.9}$$

$$s_2 = \frac{c_{11}-c_{12}}{2c_{44}} = \frac{G_r}{G_z}$$

where c_{ij} are five elastic constants of a transversely isotropic solid and G_r, and G_z are the shear moduli along r- and z-axes, respectively; z being the axis of elastic symmetry of material.

Other material parameters are given as

$$k = \frac{c_{33}s_1^2 - c_{44}}{c_{13} + c_{44}} \qquad C = \frac{(k+1)(s_1 - s_2)}{(k-1)s_1 s_2}$$
$$v_0 = \frac{ks_2 - s_1}{(k-1)s_1 s_2} \qquad v_1 = \frac{ks_2 - s_1}{k-1} \tag{A.10}$$

Expressions for functions $H_j(\xi s_i h)$ that appear in analysis are as follows

$$H_0(\xi s_i h) = \frac{1}{k-1}\Big(ke^{-\xi s_2 h} - e^{-\xi s_1 h}\Big)$$
$$H_1(\xi s_i h) = \frac{1}{ks_2 - s_1}\Big(ks_2 e^{-\xi s_1 h} - s_1 e^{-\xi s_2 h}\Big)$$
$$H_2(\xi s_i h) = \frac{1}{ks_2 - s_1}\Big(ks_2 e^{-\xi s_2 h} - s_1 e^{-\xi s_1 h}\Big) \tag{A.11}$$
$$H_3(\xi s_i h) = \frac{1}{k-1}\Big(ke^{-\xi s_1 h} - e^{-\xi s_2 h}\Big)$$

Each of these functions tend to unity as h tends to zero.

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Rozwiązania podstawowe dla współczynników intensywności naprężenia typów *I*, *II* i *III*. Zagadnienie osiowo symetryczne

Streszczenie

Otrzymano funkcje Greena dla współczynników intensywności naprężenia typów *I*, *II* i *III*. Funkcje Greena zdefiniowano jako rozwiązanie zagadnienia sprężystego, poprzecznie izotropowego ciała z kołową lub zewnętrzną szczeliną, gdy na płaszczyźnie równoległej do płaszczyzny szczeliny działają dowolne osiowo symetryczne obciążenia rozłożone na okręgu. Przedstawiono rozwiązania ścisłe, analityczne w postaci zamkniętej, dla każdego typu obciążeń, jako rozwiązania podstawowe.

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