MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
3, 32, 1994

ANALYSIS OF CHAOS IN SYSTEMS WITH GEARS

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The aim of this paper is investigation of a chaos phenomenon in complex models of gears that describe dynamics of toothing precisely (e.g. the Müller model). An analysis was carried out for the case of isolated gear system. A study of the results has shown, that chaos can be observed in system with gears, provided that the parameters of the model differ from those existing in real constructions. It can be seen that chaos should not appear in practice in systems with gears – particulary in structures designed according to the industrial standards (e.g. ISO or DIN).

1. Introduction

A great deal of research has been done recently to investigate a chaos phenomenon. Chaos has been observed in models describing various technical devices (cf Holmes and Moon (1983)). Chaos has also been investigated in systems with gears. The problem is extremely difficult due to great complexity of dynamic properties of such system (non-linear and parametrical functions, discontinuities, backlash, kinematical excitations arising from deviations, etc.) and important in practice because of common use of gears.

Küçükay and Pfeiffer (1986) present a gearbox model devoted to the "Generalized Impulsive Motion Theory". The solutions of equations describing model are explored by computer calculations. The authors conclude that the solutions of dynamic problems are of random character although the model is purely deterministic. The origin of this phenomenon is explained by sensitive dependence of solutions on the initial conditions. Clattering vibrations are presented in this paper and the gearbox model is used for determination of noise sources in gear transmissions.

Pfeiffer in the paper (1988) being a continuation of Küçükay and Pfeiffer (1986), considered the problem of clattering vibrations and noise in gear systems. This type of vibrations may be described as a sequential impact process. The equations of motion can be integrated between impacts, thus discrete and invertible mapping for the motions in phase-space may be obtained. Depending on the sets of parameters one may get either strange attractors or periodical vibrations. This is indicated by the Lyapunov exponents which are calculated here for one- and two-stage models of gears.

Hongler and Streit (1988) show the relationship of chaotic noise in gear transmissions to impact oscillators. These oscillators are characterized by the phenomenon of doubling subharmonic bifurcations and chaotic vibrations. It is visible that clattering vibrations may lead to the so-called Fermi map with dissipation. Originally, this map describes the motion of a model which consists of a small ball bouncing between two walls, one of which vibrates with time. Hongler and Streit (1988) presenting a theoretical study, provide however neither quantitative nor detailed information concerning dynamics of gears.

Moon and Broschart (1991) present the results of an experiment also confirmed by theoretical investigations of impact model which have been carried out to explain the deterministic sources of broadband noise in gears. A thin circular plate which modeled the gear housing has been impacted by chaotically oscillating mass excited by a harmonically moving base. Vibrations of the impacting mass show the classical period doubling phenomenon. Similarities between the Fermi oscillator and the experimental impacts of the plate suggest that simple models may be used for explaination of the deterministic origin of gear noise.

The mentioned above papers deal with the modelling of phenomena of the gear noise producing in terms of the impact theory. A model described by Sato et al. (1991) may be used for determination of vibrations and dynamic forces (dynamic overloads) in the one-stage gear transmission.

Calculation performed by Sato et al. (1991) uses a discrete, non-linear model of an isolated gear system. Teeth stiffness varies with time and depends on the tooth contact ratio, the backlash function is non-linear (see Sato et al. (1979)) and the external excitation is harmonic with its frequency equal to the meshing frequency.

Although Sato et al. (1991) claim that the assumption of a small amplitude sinusoidal wave form of transmission errors is not crucial in their paper, we can see, that this assumption allows to apply the Floquet theory for efficient investigation of bifurcation sets. The bifurcation sets shown by Sato et al. (1991) concern the parameters range in which nonlinearity of the system is manifested as a jump phenomenon. A similar phenomenon was confirmed by Sato et al. (1979) for the case of constant external load. A question arises how similar systems behave under the conditions described Sato et al. (1991).

Dynamic models of gears adopted by Sato et al. (1985) and (1991) are relativly simple. The aim of this paper is therefore investigation of a chaos phenomenon in more complex models of gears that describe dynamics of toothing more precisely (e.g. Müller model, cf Müller (1986)). In this paper the previous works (Dyk and Osiński (1991) and (1992)) have been adopted, where the Müller model is used in more complex systems. Model modifications allowing consideration of teeth meshing on the other side of the tooth (on the second line of action) was made. An analysis was carried out for the case of isolated gear system.

Therefore, in the first part of the paper, the system described by Sato et al. (1991) with different tooth stiffness, damping and external load will be investigated to check if the method described by Sato et al. can be applied. In the second part the Müller model of isolated gear system with backlash subject to harmonic excitation will be studied.

2. Chaos in a model with backlash

The equation of motion for a gear system, according to Sato et al. (1991) is

$$\ddot{\psi}^* + 2\zeta\dot{\psi}^* + k(t^*)g(\psi^* + e_r^*(t^*), \eta) = \frac{T_1^*}{1+i^2} + \frac{i^3T_2^*}{1+i^2} \tag{2.1}$$

where

ψ^*	–	dimensionless relative angular displacement
ζ	–	dimensionless damping ratio
$k(t^*)$	–	dimensionless mesh stiffness
e_r^*	–	dimensionless transmission error
η	–	dimensionless backlash
i	–	gear ratio, $i = z_1/z_2$
z_1, z_2	–	teeth number
$g(\psi^* + e_r^*(t^*), \eta)$	–	nonlinear function representing a gear teeth, backlash model
T_1^*, T_2^*	–	dimensionless input and output torques, respectively.

Letting $x = \psi^* + e_\tau^*$ and assuming that external torque and transmission errors have the same frequency equal to the meshing frequency and introducing $\nu t^* = \tau$ Eq (2.1) becomes

$$\frac{d^2x}{d\tau^2} + 2\zeta\frac{dx}{d\tau} + k(\tau)g(x,\eta) = B\cos(\tau + \theta) + B_0 \tag{2.2}$$

where

- ν – dimensionless meshing frequency
- B – dimensionless amplitude of sinusoidal excitation
- B_0 – dimensionless static component of excitation
- θ – phase angle.

A nonlinear function describing gear teeth backlash is of the form

$$g(x,\eta) = \begin{cases} x & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad -\eta < x < 0 \\ x + \eta & \text{for} \quad x \leq -\eta \end{cases} \tag{2.3}$$

Mesh stiffness in terms of "real" time is assumed as

$$\begin{array}{lll} k_1 = 4c_1/t_{II} + 0.6 & \text{for} & 0 \leq c_1 < 0.1t_{II} \\ k_2 = 1 & \text{for} & 0.1t_{II} \leq c_1 < 0.9t_{II} \\ k_3 = -4c_1/t_{II} + 4.6 & \text{for} & 0.9t_{II} \leq c_1 < t_{II} \\ k_4 = 0.6 & \text{for} & t_{II} \leq c_1 < T \end{array} \tag{2.4}$$

where: $c_1 = t - \text{ent}(t/T)T$ and

- T – meshing period
- t_{II} – time of double-paired meshing.

A simplified version of meshing stiffness was also used

$$\begin{array}{lll} k_1 = 1 & \text{for} & 0 \leq c_1 < t_{II} \\ k_2 = 0.6 & \text{for} & t_{II} \leq c_1 < T \end{array} \tag{2.5}$$

and

$$k(t) = 0.8 + 0.2\cos(\nu t) \tag{2.6}$$

To obtain the results in a phase plane (x, v) a computer program was written. The gear method was used to solve numerically equations of motion, the calculations were performed with double precision and accuracy in the range $10^{-9} \div 10^{-12}$. Only analysis of numerical results will be considered in the sequel of this paper.

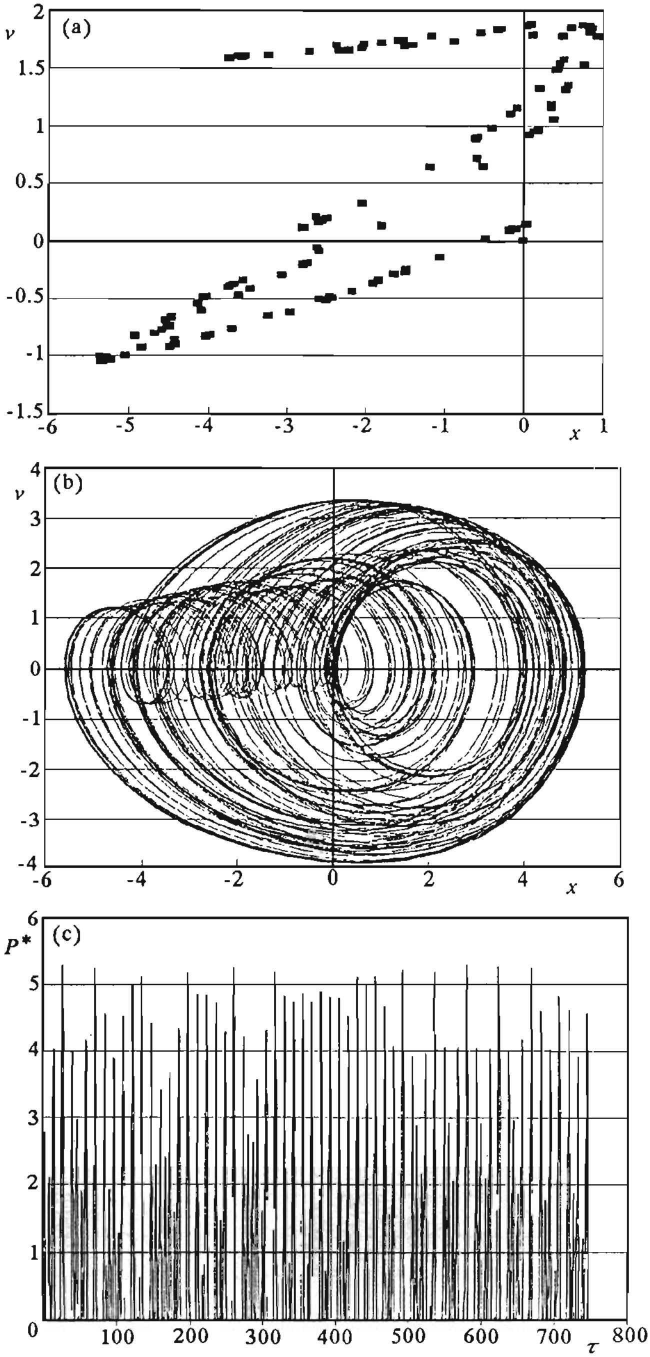

Fig. 1. (a) – point sequences by the Poincaré map; (b) – phase plane; (c) – dynamic load; ($\nu = 1.5$, $\zeta = 0.08$, $B = 3.5$, $B_0 = 1$, $\eta = 7$)

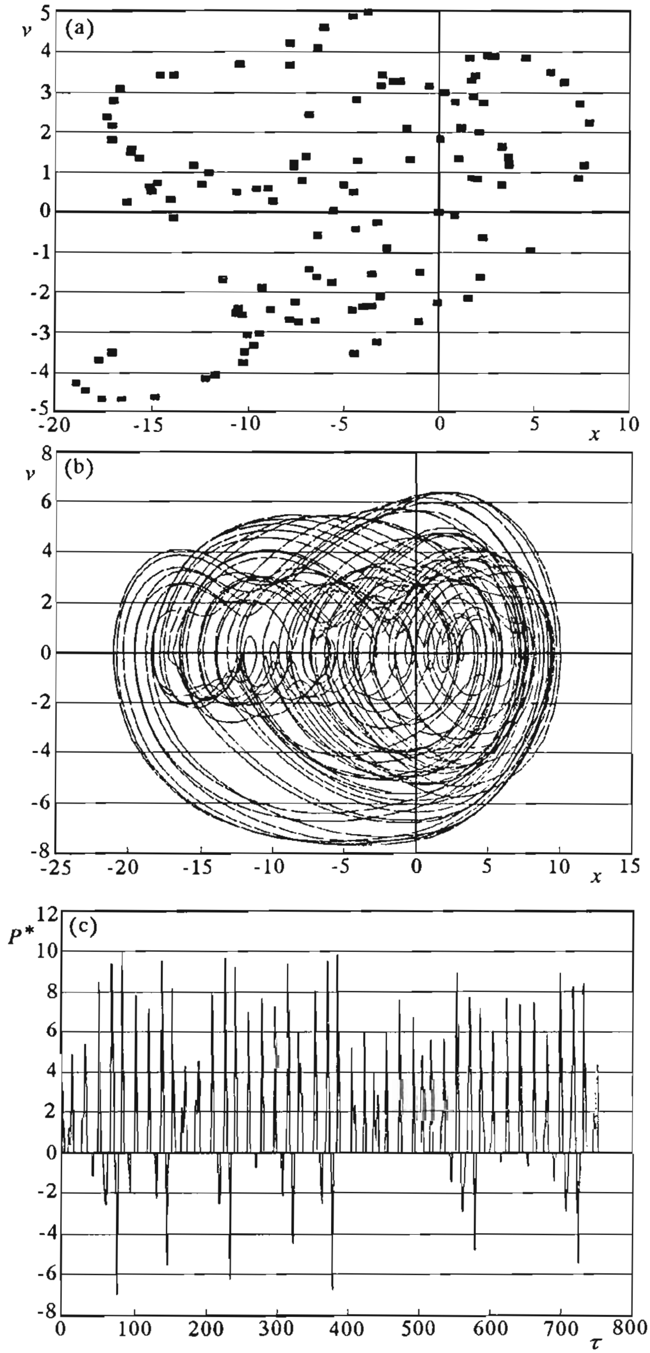

Fig. 2. (a) – point sequences by the Poincaré map; (b) – phase plane; (c) – dynamic load; ($\nu = 1.5$, $\zeta = 0.02$, $B = 3.5$, $B_0 = 1$, $\eta = 14$)

At the beginning a Poincaré map was obtained for the data shown in Fig.6a,b,c by Sato et al. (1991) but for zero initial conditions. It can be observed, that in the first figure there are four distinct curves corresponding to period four solutions. In the next figure they merge to form one curve, and in 6c there is a complicated trajectory as a stroboscopic portrait of the solution. Next, the Poincaré map and a phase portrait was obtained for the data from 6c by Sato et al. (1991) with dimensionless backlash $\eta = 1$ and dimensionless amplitude $B = 0$. These values represent the case of constant external load. The steady-state response is periodical. Results shown in Fig.1a,b,c and Fig.2a,b,c were calculated for the data from 6c in Sato et al. (1991) with dimensionless damping four times greater and backlash two times bigger then earlier, respectively. Fig.1c and Fig.2c show the dynamic load $P^* = k(\tau)g(x,\eta)$. The effect of separation of tooth meshing can be observed as well as mating of opposite sides of teeth (only in Fig.2 c) in spite of a relatively big value of backlash. Both phenomena have a great influence on the values of dynamic load. In Fig.3 and Fig.4 the Poincaré maps under the conditions used in Fig.6c (Sato et al. (1991)) and stiffnesses from Eqs (2.5) and (2.6), respectively, are shown. Thus it can be stated that the chaos in gears arise from the separation of tooth meshing due to the backlash, independently of the influence of the stiffness on the Poincaré map.

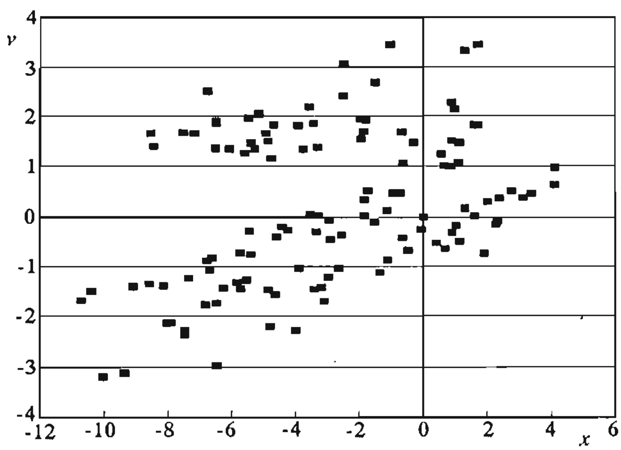

Fig. 3. Point sequences by the Poincaré map ($\nu = 1.5$, $\zeta = 0.02$, $B = 3.5$, $B_0 = 1$, $\eta = 7$), stiffness according to Eq (2.5)

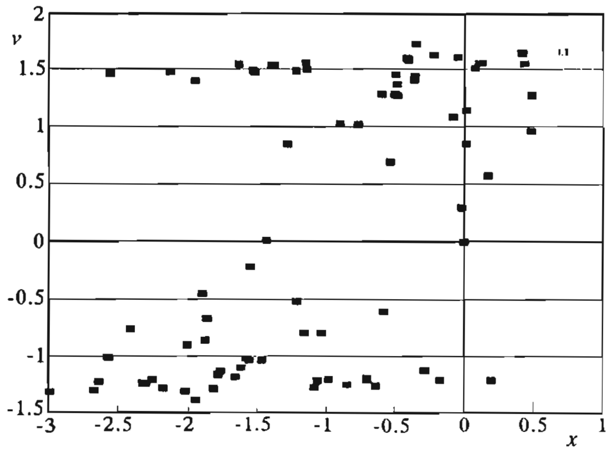

Fig. 4. Point sequences by the Poincaré map ($\nu = 1.5$, $\zeta = 0.08$, $B = 3.5$, $B_0 = 1$, $\eta = 7$), stiffness according to Eq (2.6)

3. Chaos in the Müller model

In the Müller model (cf Müller (1979) and (1986); Dyk and Osiński (1991); Holmes and Moon (1983)) the line motion has been substituted for the revolving motion of gear wheels. This model makes it possible to represent mesh stiffness changes, technological deviations, premature or delayed coming in mesh and changes of tooth profile (modification). The model can be described by the equation in the general form

$$\ddot{y} + 2\zeta\omega_0\dot{y} + \omega_n^2 F(t,y) = A_D \tag{3.1}$$

where the mesh stiffness function is of the form

$$\begin{aligned} F(t,y) = {} & F_1[y + y_{p3} - a(t_z - vC_1)^2] + F_2[y + y_{p2} - bvC_1] + \\ & + F_3[y + y_{p1} - b(vC_1 + t_z)] + F_4\{y + y_{p1} - bvT - a[(\varepsilon - 1)t_z - vC_1]^2\} \end{aligned} \tag{3.2}$$

In formulae (3.1) and (3.2) the following denotations are employed

ω_n – natural frequency

ε – tooth contact ratio

a – parameter of the parabola

b – coefficient describing the indication of the bottom shape of an equivalent solid-body

t_z	–	base pitch
v	–	velocity of the spring palisade
T	–	cycle of tooth contact
y_{p1}, y_{p2}, y_{p3}	–	errors of base pitches
C_1	–	time measured from the beginning of a given cycle of tooth contact
F_i	–	1 – when the expression in paranthesis is positive; 0 – when the expression in paranthesis is negative.

More details about the way of determination of all above-mentioned parameters can be found in papers by Dyk and Osiński (1991), Müller (1986), Müller (1979).

Eq (2.6) can be written using dimensionless coordinates and dimensionless time. By adding the upper spring palisade an additional condition for backlash may be introduced. The right-hand side of this equation can be written as it was in Eq (2.1). Taking into account the denotations from Sato et al. (1991) the equation (3.1) may be written in form of Eq (1.1). E.g. taking

$$T_D = Pr_1 \qquad\qquad T_{D\,max} = Pr_1$$
$$T_L = -Pr_2 - \Delta P r_2 \cos \nu t \tag{3.3}$$

the constants in Eq (2.2) are

$$B_0 = 1 \qquad\qquad B = \frac{\Delta P}{P\left(\frac{i^2}{i^2+1}\right)} \qquad\qquad \theta = 0 \tag{3.4}$$

where

T_D	–	input torque
T_L	–	output torque
P	–	force along the line of action
ΔP	–	additional load
r_1, r_2	–	base circles of gears
$k_{tr\,max}$	–	maximum variable stiffness
θ_{st}	–	angular displacement of the meshing gear teeth, $\theta_{st} = T_{D\,max}/k_{tr\,max}$
I_1, I_2	–	moments of inertia of the gears
I_c	–	equivalent mass moment of inertia of the two mating gears, $I_c = \left(\frac{1}{I_1} + \frac{1}{i^2 I_2}\right)$

Thus Eq (3.1) can be written in the form

$$\ddot{\psi}^* + 2\zeta\dot{\psi}^* + k(t^*)F(\psi^*, y_r^*(t^*), \eta) = B_0 + B\cos(\nu t^* + \theta) \qquad (3.5)$$

In the equation above $F(\psi^*, y_r^*(t^*), \eta)$ is a function $F(t,y)$ written in terms of dimensionless time and transformed to dimensionless coordinates; the backlash is included there. Function $y_r^*(t^*)$ represents manufacturing errors. Taking $B_0 = 1$, $B = 0$ and $\eta \to \infty$ Eq (3.5) becomes a dimensionless equation describing the Müller model (cf Müller (1979) and (1986)).

The research has been conducted under the condition used by Sato et al. (1991). The dynamic load factor was also investigated. Assuming dimensionless backlash $\eta = 100$ the separation of tooth meshing can be observed and mating of opposite sides of teeth is avoided (cf Eq (3.2)). In other cases the value of $\eta = 7$ was used, after Sato et al. (1991). For the value of dimensionless meshing frequency $\nu = 1.5$ (overcritical gear speed) dimansionless damping ratio values $\zeta = 0.02$ and $\zeta = 0.08$ were used. The dimensionless static component of excitation $B_0 = 1$ and dimensionless amplitude of sinusoidal excitation $B = 3.5$ or $B = 0$ were used. The values of parameter of the parabola and coefficient describing the indication of the bottom shape of an equivalent solid-body were $a = 0$ or $a = 20$, and $b = 0$ or $b = 2$, respectively. The meshing stiffness was assumed under the conditions imposed in Eq (2.5).

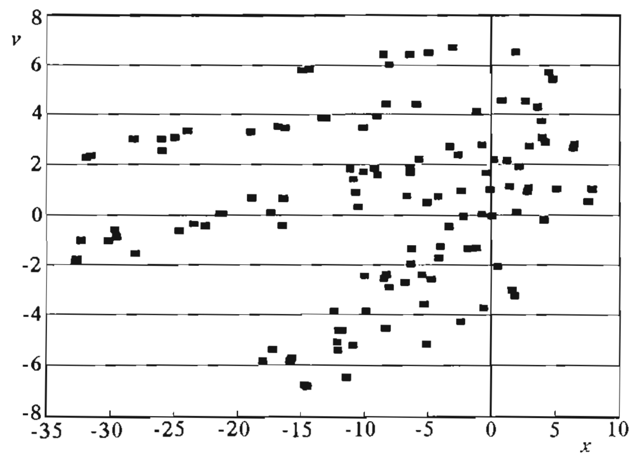

Fig. 5. Point sequences by the Poincaré map ($\nu = 1.5$, $\zeta = 0.02$, $B = 3.5$, $B_0 = 1$, $\eta = 100$, $a = 0$, $b = 0$)

Comparing Fig.5 and Fig.6 it can be seen that introducing greater degrees of damping apart from obvious changes in extremal values causes big

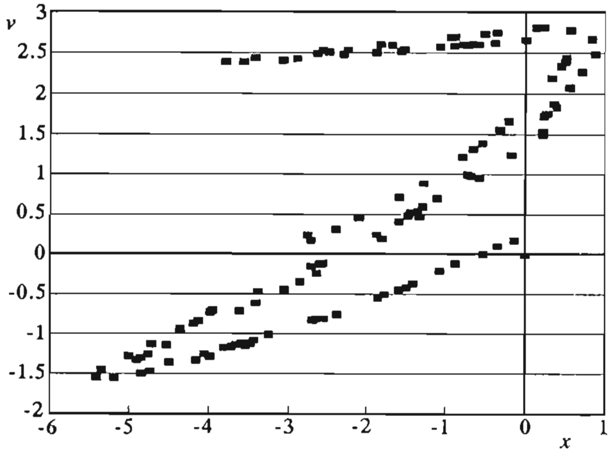

Fig. 6. Point sequences by the Poincaré map ($\nu = 1.5$, $\zeta = 0.08$, $B = 3.5$, $B_0 = 1$, $\eta = 100$, $a = 0$, $b = 0$)

differences of Poincaré maps, but the charateristic features of chaos remain unchanged.

In the picture of dynamic load it can be observed that in case of impossibility of mating on the second line of action the assumption of small values of the damping ratio results in unreal (too big) values of dynamic load. For the same set of data and a constant external load in Eq (3.5) a stroboscopic portrait of the solution is characteristic for semi-periodical or periodical vibrations (the latter in the case of greater values of damping ratio).

Introduction of additional perturbation in the form of the parameter of the parabola that describes premature or delayed coming in mesh doesn't cause significant changes to the Poincaré map. In the case of $\eta = 7$ the dynamic load may be of negative value as a result of meshing on the second line of action, this phenomenon vanishes for higher values of damping ratio. Simultaneous appearance of coefficients a and b, that describe errors of base pitch is shown in Fig.7 and Fig.8. A moderating influence of the positive value of teeth profile error on the dynamic load values can be observed. Such a value of teeth profile error makes it "easier" coming in mesh of the successive teeth. In the end all the above mentioned kinds of errors were introduced in the calculations as well as random distribution of error of base pitch $f_{pi} = \pm 2$. The shapes of the graphs obtained were similar to those in Fig.5 and Fig.8. It can't be clearly explained whether the random nature of the graphs is the result of chaos in the system or whether it arises from random kinematic excitation with uniform distribution.

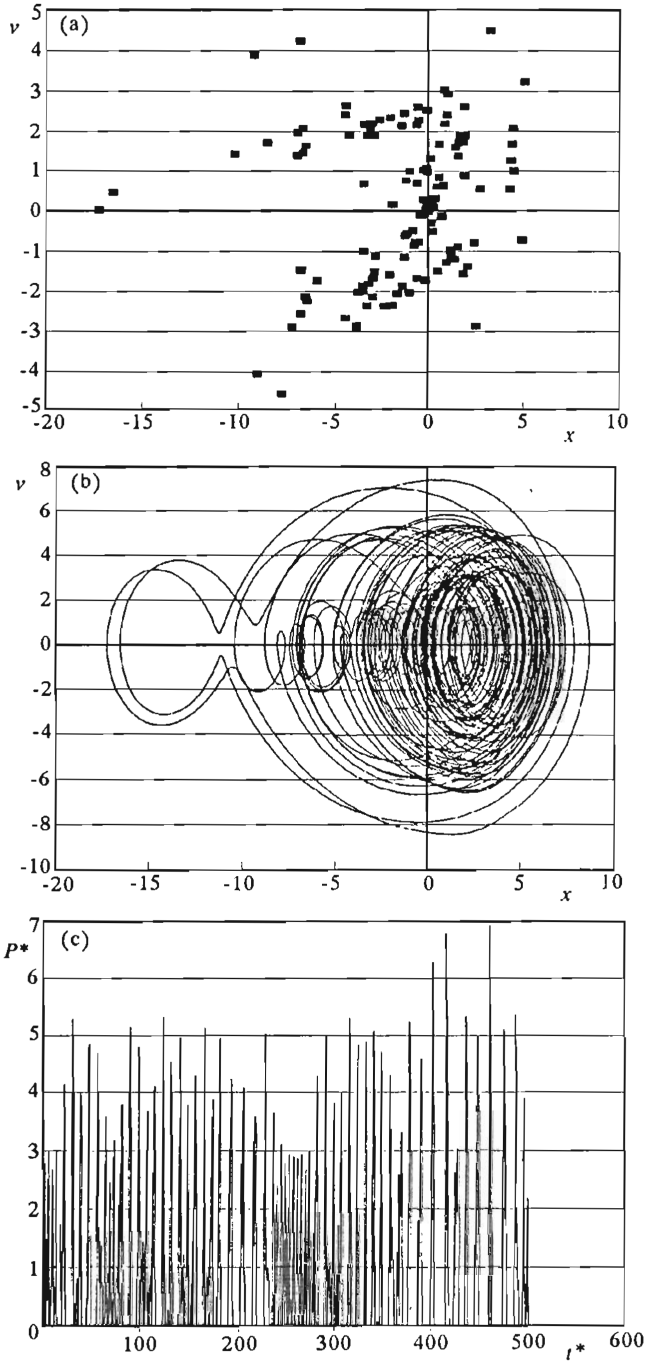

Fig. 7. (a) – point sequences by the Poincaré map; (b) – phase plane; (c) – dynamic load; ($\nu = 1.5$, $\zeta = 0.02$, $B = 3.5$, $B_0 = 1$, $\eta = 100$, $a = 20$, $b = 2$)

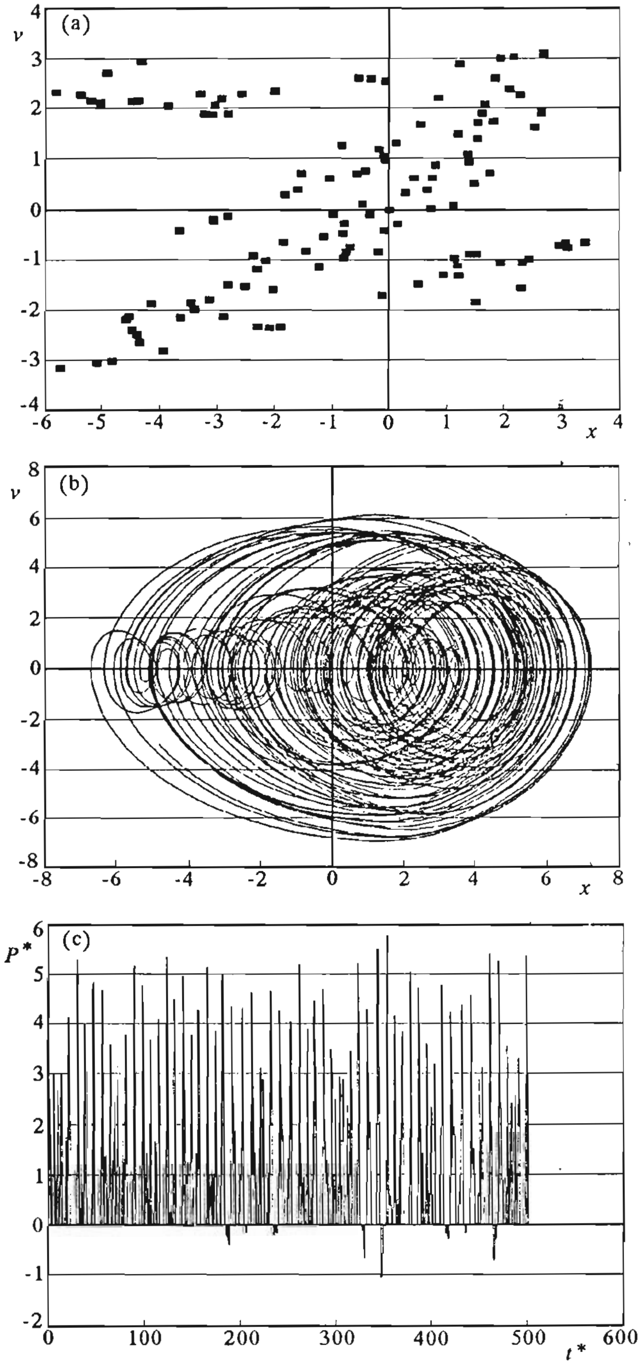

Fig. 8. (a) – point sequences by the Poincaré map; (b) – phase plane; (c) – dynamic load; ($\nu = 1.5$, $\zeta = 0.02$, $B = 3.5$, $B_0 = 1$, $\eta = 7$, $a = 20$, $b = 2$)

4. Conclusions

In this report several cases of chaos in systems with gears taken after Sato et al. (1991) have been investigated. A new program was applied for obtaining solution of equations of motion presented there. A good agreement of results was observed, which implies corectness of the used method and algorythm. The equation presented by Sato et al. (1991) is relatively simple, and does not fully describe the dynamic features of toothing. Therefore, in the new version of the program, the Müller model, which describes dynamics of toothing more precisely, was employed. The series of calculations was made and a possibility of chaos existence was confirmed. The premature and delayed coming in mesh was considered. A deep study of the results has shown, that chaos can be observed in systems with gears, provided that the parameters of the model differ from those existing in real constructions. Particularly, it concerns values of damping ratio, manufacturing errors, backlashes, rotational speeds, respectively, not relevant for such gears and harmonical external excitations with frequencies equal to the frequency of meshing, great amplitudes and negative coefficients of cycle assymetry. It can be seen that chaos shouldn't appear in practice in systems with gears – particularly in structures designed according to the industrial standards (e.g. ISO or DIN). It concerns systems with spur gears and relatively small tangential velocities.

5. References

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Analiza ruchu chaotycznego w układach z przekładnią zębatą

Streszczenie

W pracy zbadano ruch chaotyczny występujący w przekładniack zębatych z uwzględnieniem dokładnego opisu dynamiki uzębień (model Müllera). Analizę wykonano dla przykładu przekładni izolowanej. Stwierdzono, że ruch chaotyczny występuje w układzie z przekładnią zębatą w przypadku łącznego przyjęcia parametrów modelu odbiegających od rzeczywiście występujących. Zjawisko ruchu chaotycznego nie powinno mieć praktycznego znaczenia w przekładniach, w konstrukcji których przestrzegane są dotychczas obowiązujące zalecenia (normy ISO, DIN).

Manuscript received July 9, 1993; accepted for print January 4, 1994