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MOMENT LYAPUNOV EXPONENTS AND STOCHASTIC STABILITY OF A THIN-WALLED BEAM SUBJECTED TO ECCENTRIC AXIAL LOADS

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The Lyapunov exponent and moment Lyapunov exponents of two degrees-of-freedom linear systems subjected to white noise parametric excitation are investigated. The method of regular perturbation is used to determine the explicit asymptotic expressions for these exponents in the presence of small intensity noises. The Lyapunov exponent and moment Lyapunov exponents are important characteristics for determining the almost-sure and moment stability of a stochastic dynamic system. As an example, we study the almost-sure and moment stability of a thin-walled beam subjected to an eccentric stochastic axial load. The validity of the approximate results for moment Lyapunov exponents is checked by the numerical Monte Carlo simulation method for this stochastic system.

Key words: eigenvalues, perturbation, stochastic stability, thin-walled beam, mechanics of solids and structures

1. Introduction

In the recent years, there has been considerable interest in the study of the dynamic stability of non-gyroscopic conservative elastic systems whose parameters fluctuate in a stochastic manner. To have a complete picture of the dynamic stability of a dynamic system, it is important to study both the almost-sure and the moment stability, and to determine both the maximal Lyapunov exponent and the pth moment Lyapunov exponent. The maximal Lyapunov exponent, defined by

$$\lambda_q = \lim_{t\to\infty} \frac{1}{t} \log \|q(t;q_0)\| \tag{1.1}$$

where $q(t; q_0)$ is the solution process of a linear dynamic system. The almost-sure stability depends upon the sign of the maximal Lyapunov exponent which is an exponential growth rate of the solution of the randomly perturbed dynamic system. The negative sign of the maximal Lyapunov exponent implies the almost-sure stability, whereas a non-negative value indicates instability. The exponential growth rate $\mathrm{E}[\|q(t; q_0, \dot{q}_0)\|^p]$ is provided by the moment Lyapunov exponent defined as

$$\Lambda_q(p) = \lim_{t\to\infty} \frac{1}{t} \log \mathrm{E}[\|q(t; q_0)\|^p] \tag{1.2}$$

where $\mathrm{E}[\cdot]$ denotes the expectation. If $\Lambda_q(p) < 0$, then, by definition $\mathrm{E}[\|q(t; q_0, \dot{q}_0)\|^p] \to 0$ as $t \to \infty$ and this is referred to as the pth moment stability. Although the moment Lyapunov exponents are important in the study of the dynamic stability of stochastic systems, the actual evaluations of the moment Lyapunov exponents are very difficult.

Arnold *et al.* (1997) constructed an approximation for the moment Lyapunov exponents, the asymptotic growth rate of the moments of the response of a two-dimensional linear system driven by real or white noise. A perturbation approach was used to obtain explicit expressions for these exponents in the presence of small intensity noises. Khasminskii and Moshchuk (1998) obtained an asymptotic expansion of the moment Lyapunov exponents of a two-dimensional system under white noise parametric excitation in terms of the small fluctuation parameter ε, from which the stability index was obtained. Sri Namachchivaya *et al.* (1994) used a perturbation approach to calculate the asymptotic growth rate of a stochastically coupled two-degrees-of-freedom system. The noise was assumed to be white and of small intensity in order to calculate the explicit asymptotic formulas for the maximum Lyapunov exponent. Sri Namachchivaya and Van Roessel (2004) used a perturbation approach to obtain an approximation for the moment Lyapunov exponents of two coupled oscillators with commensurable frequencies driven by small intensity real noise with dissipation. The generator for the eigenvalue problem associated with the moment Lyapunov exponents was derived without any restriction on the size of pth moment. Kozić *et al.* (2009, 2010) investigated the Lyapunov exponent and moment Lyapunov exponents of two degrees-of-freedom linear systems subjected to a white noise parametric excitation. In the first, almost-sure and moment stability of the flexural-torsion stability of a thin elastic beam subjected to a stochastically fluctuating follower force were studied. In the second, moment Lyapunov exponents and stability boundary of the double-beam system under stochastic compressive axial loading were obtained. Pavlović *et al.* (2007) investigated the dynamic stability of thin-walled beams subjected to combined action of axial loads and end moments. By using the direct Lyapunov method, the authors obtained the almost-sure stochastic boundary and uniform stochastic stability boundary as the function of characteristics of the stochastic process and geometric and physical parameters.

The aim of this paper is to determine a weak noise expansion for the moment Lyapunov exponents of the four-dimensional stochastic system. The noise is assumed to be white noise of small intensity such that one can obtain an asymptotic growth rate. We apply the perturbation theoretical approach given in Khasminskii and Moshchuk (1998) to obtain second-order weak noise expansions of the moment Lyapunov exponents. The Lyapunov exponent is then obtained using the relationship between the moment Lyapunov exponents and the Lyapunov exponent. These results are applied to study the pth moment stability and almost-sure stability of a thin-walled beam subjected to eccentric stochastic axial loads. The motion of such an elastic system is governed by the partial differential equations in the paper by Pavlović *et al.* (2007). The approximate analytical results of the moment Lyapunov exponents are compared with the numerical values obtained by the Monte Carlo simulation approach for these exponents of a four-dimensional stochastic system.

2. Theoretical formulation

Consider linear oscillatory systems described by equations of motion of the form

$$\begin{aligned} \ddot{q}_1 + \omega_1^2 q_1 + 2\varepsilon\beta_1\dot{q}_1 - \sqrt{\varepsilon}\xi(t)(K_{11}q_1 + K_{12}q_2) = 0 \\ \ddot{q}_2 + \omega_2^2 q_2 + 2\varepsilon\beta_2\dot{q}_2 - \sqrt{\varepsilon}\xi(t)(K_{21}q_1 + K_{22}q_2) = 0 \end{aligned} \tag{2.1}$$

where q_1, q_2 are generalized coordinates, ω_1, ω_2 are natural frequencies and $2\varepsilon\beta_1, 2\varepsilon\beta_2$ represent small viscous damping coefficients. The stochastic term $\sqrt{\varepsilon}\xi(t)$ is the white-noise process with small intensity with zero mean and autocorrelation functions

$$R_{\xi\xi}(t_1, t_2) = \mathrm{E}[\xi(t_1)\xi(t_2)] = \sigma^2\delta(t_2 - t_1) \tag{2.2}$$

σ is the intensity of the random process $\xi(t)$, and $\delta(\cdot)$ is the Dirac delta.

Using the transformation

$$q_1 = x_1 \qquad \dot{q}_1 = \omega_1 x_2 \qquad q_2 = x_3 \qquad \dot{q}_2 = \omega_2 x_4 \tag{2.3}$$

and denoting

$$p_{ij} = \frac{K_{ij}}{\omega_i}\sigma \qquad i,j = 1,2 \tag{2.4}$$

the above Eqs. (2.1) can be represented in the first-order form by a set of Stratonovich differential equations

$$d\boldsymbol{X} = \mathbf{A}_0\boldsymbol{X}\,dt + \varepsilon\mathbf{A}\boldsymbol{X}\,dt + \sqrt{\varepsilon}\mathbf{B}\boldsymbol{X}\,dw(t) \tag{2.5}$$

where $\boldsymbol{X} = [x_1, x_2, x_3, x_4]^\top$ is the state vector of the system, $w(t)$ is the standard Weiner process and $\mathbf{A}_0$, $\mathbf{A}$ and $\mathbf{B}$ are constant 4×4 matrices given by

$$\mathbf{A}_0 = \begin{bmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\beta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\beta_2 \end{bmatrix}$$
$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ p_{11} & 0 & p_{12} & 0 \\ 0 & 0 & 0 & 0 \\ p_{21} & 0 & p_{22} & 0 \end{bmatrix} \tag{2.6}$$

Applying the transformation

$$\begin{aligned} &x_1 = a\cos\varphi\cos\theta_1 \qquad && x_2 = -a\cos\varphi\sin\theta_1 \\ &x_3 = a\sin\varphi\cos\theta_2 && x_4 = -a\sin\varphi\sin\theta_2 \\ &P = \|a\|^p = \sqrt{(x_1^2+x_2^2+x_3^2+x_4^2)^p} \\ &0 \leqslant \theta_1 \leqslant 2\pi \qquad 0 \leqslant \theta_2 \leqslant 2\pi && 0 \leqslant \varphi \leqslant \frac{\pi}{2} \qquad -\infty < p < \infty \end{aligned} \tag{2.7}$$

yields the following set of Stratonovich equations for the pth power of the norm of the response and phase variables $(\varphi, \theta_1, \theta_2)$

$$\begin{aligned} &d\|a\|^p = \varepsilon\alpha_1^*\,dt + \sqrt{\varepsilon}\gamma_1^*\,dw(t) \qquad && d\varphi = \varepsilon\alpha_2^*\,dt + \sqrt{\varepsilon}\gamma_2^*\,dw(t) \\ &d\theta_1 = (\omega_1 + \varepsilon\alpha_3^*)\,dt + \sqrt{\varepsilon}\gamma_3^*\,dw(t) && d\theta_2 = (\omega_2 + \varepsilon\alpha_4^*)\,dt + \sqrt{\varepsilon}\gamma_4^*\,dw(t) \end{aligned} \tag{2.8}$$

In the above transformations, a represents the norm of the response, θ_1 and θ_2 are the angles of the first and second oscillators, respectively, and φ describes the coupling or exchange of energy between the first and second oscillator. In the previous equation, we introduced the following marking

$$
\begin{aligned}
\alpha_1^* &= -2pP(\beta_1 \sin^2\theta_1 \cos^2\varphi + \beta_2 \sin^2\theta_2 \sin^2\varphi) \\
\alpha_2^* &= \beta_1 \sin^2\theta_1 \sin 2\varphi - \beta_2 \sin^2\theta_2 \sin 2\varphi \\
\alpha_3^* &= -\beta_1 \sin 2\theta_1 \qquad\qquad \alpha_4^* = -\beta_2 \sin 2\theta_2 \\
\gamma_1^* &= -\frac{pP}{2}(p_{11} \sin 2\theta_1 \cos^2\varphi + p_{22} \sin 2\theta_2 \sin^2\varphi \\
&\quad + p_{12} \sin\theta_1 \cos\theta_2 \sin 2\varphi + p_{21} \cos\theta_1 \sin\theta_2 \sin 2\varphi) \\
\gamma_2^* &= \frac{p_{11}}{4} \sin 2\theta_1 \sin 2\varphi - \frac{p_{22}}{4} \sin 2\theta_2 \sin 2\varphi + \frac{p_{12}}{2} \sin\theta_1 \cos\theta_2 \sin^2\varphi \\
&\quad - \frac{p_{21}}{2} \cos\theta_1 \sin\theta_2 \cos^2\varphi \\
\gamma_3^* &= -p_{11} \cos^2\theta_1 - p_{12} \cos\theta_1 \cos\theta_2 \tan\varphi \\
\gamma_4^* &= -p_{22} \cos^2\theta_2 - p_{21} \cos\theta_1 \cos\theta_2 \cot\varphi
\end{aligned}
\tag{2.9}
$$

The Itô versions of Eqs. (2.8) have the following form

$$
\begin{aligned}
d\|a\|^p &= \varepsilon\alpha_1\,dt + \sqrt{\varepsilon}\gamma_1\,dw_1(t) \qquad & d\varphi &= \varepsilon\alpha_2\,dt + \sqrt{\varepsilon}\gamma_2\,dw(t) \\
d\theta_1 &= (\omega_1 + \varepsilon\alpha_3)\,dt + \sqrt{\varepsilon}\gamma_3\,dw(t) \qquad & d\theta_2 &= (\omega_2 + \varepsilon\alpha_4)\,dt + \sqrt{\varepsilon}\gamma_4\,dw(t)
\end{aligned}
\tag{2.10}
$$

where α_i are given in Appendix 1 and $\gamma_i = \gamma_i^*$, $(i = 1, 2, 3, 4)$.

Following Wedig (1998), we perform the linear stochastic transformation

$$
S = T(\varphi, \theta_1, \theta_2)P \qquad\qquad P = T^{-1}(\varphi, \theta_1, \theta_2)S \tag{2.11}
$$

introducing the new norm process S by means of the scalar function $T(\varphi, \theta_1, \theta_2)$ which is defined in the stationary phase processes θ_1, θ_2 and φ

$$
\begin{aligned}
dS &= P(\omega_1 T'_{\theta_1} + \omega_2 T'_{\theta_2})\,dt + \varepsilon P\Big(\alpha_1 T + m_0 T'_\varphi + m_1 T'_{\theta_1} + m_2 T'_{\theta_2} + \frac{1}{2}\gamma_2^2 T''_{\varphi\varphi} \\
&\quad + \gamma_2\gamma_3 T''_{\varphi\theta_1} + \gamma_2\gamma_4 T''_{\varphi\theta_2} + \frac{1}{2}\gamma_3^2 T''_{\theta_1\theta_1} + \gamma_3\gamma_4 T''_{\theta_1\theta_2} + \frac{1}{2}\gamma_4^2 T''_{\theta_2\theta_2}\Big)\,dt \\
&\quad + \sqrt{\varepsilon}P(T\gamma_1 + T'_\varphi\gamma_2 + T'_{\theta_1}\gamma_3 + T'_{\theta_2}\gamma_4)\,dw(t)
\end{aligned}
\tag{2.12}
$$

where

$$
m_0 = \alpha_2 + \gamma_1\gamma_2 \qquad\qquad m_1 = \alpha_3 + \gamma_1\gamma_3 \qquad\qquad m_2 = \alpha_4 + \gamma_1\gamma_4 \tag{2.13}
$$

If the transformation function $T(\theta_1, \theta_2, \varphi)$ is bounded and non-singular, both processes P and S possess the same stability behavior. Therefore, the transformation function $T(\theta_1, \theta_2, \varphi)$ is chosen so that the drift term, of Itô differential Eq. (2.13), does not depend on the phase processes θ_1, θ_2 and φ, so that

$$
dS = \Lambda(p)S\,dt + ST^{-1}(T\gamma_1 + T'_\varphi\gamma_2 + T'_{\theta_1}\gamma_3 + T'_{\theta_2}\gamma_4)\,dw(t) \tag{2.14}
$$

By comparing Eqs. (2.12) and (2.14), it can be seen that such a transformation function $T(\varphi, \theta_1, \theta_2)$ is given by the following equation

$$[L_0 + \varepsilon L_1]T(\varphi, \theta_1, \theta_2) = \Lambda(p)T(\varphi, \theta_1, \theta_2) \tag{2.15}$$

Here L_0 and L_1 are the following first and second-order differential operators

$$\begin{aligned} L_0 &= \omega_1 \frac{\partial}{\partial \theta_1} + \omega_2 \frac{\partial}{\partial \theta_2} \\ L_1 &= a_1 \frac{\partial^2}{\partial \varphi^2} + a_2 \frac{\partial^2}{\partial \theta_1^2} + a_3 \frac{\partial^2}{\partial \theta_2^2} + a_4 \frac{\partial^2}{\partial \varphi \partial \theta_1} + a_5 \frac{\partial^2}{\partial \varphi \partial \theta_2} \\ &\quad + a_6 \frac{\partial^2}{\partial \theta_1 \partial \theta_2} + b_1 \frac{\partial}{\partial \varphi} + b_2 \frac{\partial}{\partial \theta_1} + b_3 \frac{\partial}{\partial \theta_2} + c \end{aligned} \tag{2.16}$$

where $a_i = a_i(\varphi, \theta_1, \theta_2)$, $(i = 1, 2, \ldots, 6)$, $b_j = b_j(\varphi, \theta_1, \theta_2)$, $(j = 1, 2, 3)$, and $c = c(\varphi, \theta_1, \theta_2)$ are given in Appendix 2.

Equation (2.15) defines an eigenvalue problem for a second-order differential operator of three independent variables, in which $\Lambda(p)$ is the eigenvalue and $T(\varphi, \theta_1, \theta_2)$ the associated eigenfunction. From Eq. (2.14), the eigenvalue $\Lambda(p)$ is seen to be the Lyapunov exponent of the pth moment of system (2.5), i.e., $\Lambda(p) = \Lambda_{x(t)}(p)$. This approach was first applied by Wedig (1998) to derive the eigenvalue problem for the moment Lyapunov exponent of a two-dimensional linear Itô stochastic system. In the following section, the method of regular perturbation is applied to eigenvalue problem (2.15) to obtain a weak noise expansion of the moment Lyapunov exponent of a four-dimensional stochastic linear system.

3. Weak noise expansion of the moment Lyapunov exponent

Applying the method of regular perturbation, both the moment Lyapunov exponent $\Lambda(p)$ and the eigenfunction $T(\varphi, \theta_1, \theta_2)$ are expanded in the power series of ε as

$$\begin{aligned} \Lambda(p) &= \Lambda_0(p) + \varepsilon \Lambda_1(p) + \varepsilon^2 \Lambda_2(p) + \ldots + \varepsilon^n \Lambda_n(p) + \ldots \\ (\varphi, \theta_1, \theta_2) &= T_0(\varphi, \theta_1, \theta_2) + \varepsilon T_1(\varphi, \theta_1, \theta_2) + \varepsilon^2 T_2(\varphi, \theta_1, \theta_2) \\ &\quad + \ldots + \varepsilon^n T_n(\varphi, \theta_1, \theta_2) + \ldots \end{aligned} \tag{3.1}$$

Substituting perturbation series (3.1) into eigenvalue problem (2.15) and equating the terms of equal powers of ε leads to the following equations

$$\begin{aligned}
&\varepsilon^0 \rightarrow L_0 T_0 = \Lambda_0(p) T_0\\
&\varepsilon^1 \rightarrow L_0 T_1 + L_1 T_0 = \Lambda_0(p) T_1 + \Lambda_1(p) T_0\\
&\varepsilon^2 \rightarrow L_0 T_2 + L_1 T_1 = \Lambda_0(p) T_2 + \Lambda_1(p) T_1 + \Lambda_2(p) T_0\\
&\varepsilon^3 \rightarrow L_0 T_3 + L_1 T_2 = \Lambda_0(p) T_3 + \Lambda_1(p) T_2 + \Lambda_2(p) T_1 + \Lambda_3(p) T_0\\
&\vdots\\
&\varepsilon^n \rightarrow L_0 T_n + L_1 T_{n-1} = \Lambda_0(p) T_n + \Lambda_1(p) T_{n-1} + \Lambda_2(p) T_{n-2}\\
&\qquad + \ldots + \Lambda_{n-1}(p) T_1 + \Lambda_n(p) T_0
\end{aligned} \tag{3.2}$$

where each function $T_i = T_i(\varphi, \theta_1, \theta_2)$, $(i = 0, 1, 2, \ldots)$ must be positive and periodic in the range $0 \leqslant \varphi \leqslant \pi/2$, $0 \leqslant \theta_1 \leqslant 2\pi$ and $0 \leqslant \theta_2 \leqslant 2\pi$.

3.1. Zeroth order perturbation

The zeroth order perturbation equation is $L_0 T_0 = \Lambda_0(p) T_0$ or

$$\omega_1 \frac{\partial T_0}{\partial \theta_1} + \omega_2 \frac{\partial T_0}{\partial \theta_2} = \Lambda_0(p) T_0 \tag{3.3}$$

From the property of the moment Lyapunov exponent, it is known that

$$\Lambda(0) = \Lambda_0(0) + \varepsilon \Lambda_1(0) + \varepsilon^2 \Lambda_2(0) + \ldots + \varepsilon^n \Lambda_n(0) = 0 \tag{3.4}$$

which results in $\Lambda_n(0) = 0$ for $n = 0, 1, 2, \ldots$. Since eigenvalue problem (3.3) does not contain p, the eigenvalue $\Lambda_0(p)$ is independent of p. Hence, $\Lambda_0(0) = 0$ leads to

$$\Lambda_0(p) = 0 \tag{3.5}$$

Now, partial differential Eqs. (3.3) have the form

$$\omega_1 \frac{\partial T_0}{\partial \theta_1} + \omega_2 \frac{\partial T_0}{\partial \theta_2} = 0 \tag{3.6}$$

The solution to Eq. (3.6) may be taken as

$$T_0(\varphi, \theta_1, \theta_2) = \psi_0(\varphi) \tag{3.7}$$

where $\psi_0(\varphi)$ is an unknown function of φ which has yet to be determined.

3.2. First order perturbation

The first order perturbation equation is

$$L_0 T_1 = \Lambda_1(p) T_0 - L_1 T_0 \tag{3.8}$$

Since homogeneous Eq. (3.6) has a non-trivial solution as given by Eq. (3.7), for Eq. (3.8) to have a solution, it is required that, from the Fredholm alternative

$$(L_0 T_1, T_0^*) = (\Lambda_1(p)T_0 - L_1 T_0, T_0^*) = 0 \tag{3.9}$$

In the previous equation, $T_0^* = \psi_0(\varphi)$ is the unknown solution to the associated adjoint differential equation of (3.6), and (f, g) denotes the inner product of functions $f(\varphi, \theta_1, \theta_2)$ and $g(\varphi, \theta_1, \theta_2)$ defined by

$$(f, g) = \int\limits_0^{\pi/2} \int\limits_0^{2\pi} \int\limits_0^{2\pi} f(\varphi, \theta_1, \theta_2) g(\varphi, \theta_1, \theta_2)\, d\theta_1\, d\theta_2\, d\varphi \tag{3.10}$$

Considering (3.7), (3.8) and (3.10), expression (3.9) now has the form

$$\int\limits_0^{\pi/2} \int\limits_0^{2\pi} \int\limits_0^{2\pi} (\Lambda_1(p)\psi_0 - L_1\psi_0)\psi\psi_0(\varphi)\, d\theta_1\, d\theta_2\, d\varphi = 0 \tag{3.11}$$

and will be satisfied if and only if

$$\int\limits_0^{2\pi} \int\limits_0^{2\pi} (\Lambda_1(p)\psi_0 - L_1\psi_0)\, d\theta_1\, d\theta_2 = 0 \tag{3.12}$$

After the integration of the previous expression, we have that

$$\overline{L}(\psi_0) = A_1(\varphi)\frac{d^2\psi_0}{d\varphi^2} + B_1(\varphi)\frac{d\psi_0}{d\varphi} + C_1(\varphi)\psi_0 - \Lambda_1(p)\psi_0 = 0 \tag{3.13}$$

where

$$\begin{aligned} A_1(\varphi) &= \int\limits_0^{2\pi} \int\limits_0^{2\pi} a_1(\varphi, \theta_1, \theta_2)\, d\theta_1\, d\theta_2 \qquad B_1(\varphi) = \int\limits_0^{2\pi} \int\limits_0^{2\pi} b_1(\varphi, \theta_1, \theta_2)\, d\theta_1\, d\theta_2 \\ C_1(\varphi) &= \int\limits_0^{2\pi} \int\limits_0^{2\pi} c(\varphi, \theta_1, \theta_2)\, d\theta_1\, d\theta_2 \end{aligned} \tag{3.14}$$

Finally, there

$$\begin{aligned} A_1(\varphi) = &-\frac{1}{128}[p_{11}^2 + p_{22}^2 - 2(p_{12}^2 + p_{21}^2)]\cos 4\varphi - \frac{p_{12}^2 - p_{21}^2}{16}(\omega_1^2 - \omega_2^2)\cos 2\varphi \\ &+\frac{1}{128}[p_{11}^2 + p_{22}^2 + 6(p_{12}^2 + p_{21}^2)] \end{aligned}$$

$$\begin{aligned}B_1(\varphi) =& -\frac{1}{64}(p-1)[p_{11}^2+p_{22}^2-2(p_{12}^2+p_{21}^2)]\sin 4\varphi \\ &-\frac{1}{8}(p_{12}^2\sin^2\varphi\tan\varphi - p_{21}^2\cos^2\varphi\cot\varphi) \\ &+\frac{1}{32}\{16\beta_1-16\beta_2-[(p+2)(p_{11}^2-p_{22}^2)+2(p-1)(p_{12}^2-p_{21}^2)]\}\sin 2\varphi \\ C_1(\varphi) =& \frac{1}{128}p(p-2)[p_{11}^2+p_{22}^2-2(p_{12}^2+p_{21}^2)]\cos 4\varphi \\ &-\frac{1}{32}\{16\beta_1-16\beta_2-[(p+2)(p_{11}^2-p_{22}^2)-4(p_{12}^2-p_{21}^2)]\}\cos 2\varphi \\ &+\frac{1}{128}p\{-64\beta_1-64\beta_2+[(10+3p)(p_{11}^2+p_{22}^2)+2(6+p)(p_{12}^2+p_{21}^2)]\}\end{aligned} \tag{3.15}$$

Since coefficients (3.15) of Eq.(3.13) are periodic functions of φ, a series expansion of the function $\psi_0(\varphi)$ may be taken in the form

$$\psi_0(\varphi) = \sum_{k=0}^{N} K_k \cos 2k\varphi \tag{3.16}$$

Substituting (3.16) in (3.13), multiplying the resulting equation by $\cos 2k\varphi$ ($k = 0, 1, 2, \ldots$) and integrating with respect to φ from 0 to $\pi/2$ leads to a set of $2N+1$ homogeneous linear equations for the unknown coefficients $K_0, K_1, K_2, \ldots$

$$\sum_{j=0}^{N} A_{jk}K_j = \Lambda_1(p)K_k \tag{3.17}$$

where

$$A_{jk} = \int_0^{\pi/2} \overline{L}(\cos 2j\varphi)\cos 2k\varphi\, d\varphi \qquad k = 0, 1, 2, \ldots, N \tag{3.18}$$

When N tends to infinity, solution (3.16) to equations tends to the exact solution. The condition for system homogeneous linear equation (3.17) to have nontrivial solutions is that the determinant of system homogeneous linear equations (3.17) is equal to zero. The coefficients A_{jk} to order $N = 3$ are presented in Appendix 3.

In the case when $N = 0$, we assume solution (3.16) in the form $\psi_0(\varphi) = K_0$, from conditions that $A_{00} = 0$, the moment Lyapunov exponent in the first perturbation is defined as

$$\Lambda_1(p) = -\frac{p}{2}(\beta_1+\beta_2) + \frac{p(10+3p)}{128}(p_{11}^2+p_{22}^2) + \frac{p(6+p)}{64}(p_{12}^2+p_{21}^2) \tag{3.19}$$

In the case when $N = 1$, solution (3.16) has the form $\psi_0(\varphi) = K_0 + K_1 \cos 2\varphi$, the moment Lyapunov exponent in the first perturbation is the solution to the equation $\Lambda_1^2 + d_1^{(1)} \Lambda_1 + d_0^{(1)} = 0$ where coefficients $d_0^{(1)}$ and $d_1^{(1)}$ are presented in Appendix 4. In the case when $N = 2$, solution (3.16) has the form $\psi_0(\varphi) = K_0 + K_1 \cos 2\varphi + K_2 \cos 4\varphi$, the moment Lyapunov exponent in the first perturbation is the solution to the equation $\Lambda_1^3 + d_2^{(2)} \Lambda_1^2 + d_1^{(2)} \Lambda_1 + d_0^{(2)} = 0$ where coefficients $d_0^{(2)}$, $d_1^{(2)}$ and $d_2^{(2)}$ are presented in Appendix 5. However, for $N = 3$, it is impossible to obtain explicit expressions of $\Lambda_1(p)$ and numerical results must be given.

4. Application to a thin-walled beam subjected to an eccentric stochastic axial load

The purpose of this section is to present the general results of the above sections in the context of real engineering applications and show how these results can be applied to physical problems. To this end, we consider the flexural-torsinal vibration stability of a homogeneous, isotropic, thin-walled beam with two planes of summetry which is subjected to an eccentric axial load (Fig. 1a), where R is the eccentricity. By transferring the eccentric load to the plane of symmetry of the cross-section of the beam, an axial load and a couple are obtained, which are shown in Fig. 1b.

The governing differential equations for the coupled flexural and torsional motion of the beam can be written as (Pavlović *et al.*, 2007)

$$\begin{aligned}
&\rho A \frac{\partial^2 U}{\partial T^2} + \alpha_u \frac{\partial U}{\partial t} + EI_y \frac{\partial^4 U}{\partial Z^4} + \overline{M}(T) \frac{\partial^2 \phi}{\partial Z^2} + \overline{F}(T) \frac{\partial^2 U}{\partial Z^2} = 0 \\
&\rho I_p \frac{\partial^2 \phi}{\partial T^2} + \alpha_\phi \frac{\partial \phi}{\partial T} - \Big(GJ - \overline{F}(t) \frac{I_p}{A}\Big) \frac{\partial^2 \phi}{\partial Z^2} + \overline{M}(T) \frac{\partial^2 U}{\partial Z^2} + EI_s \frac{\partial^4 \phi}{\partial Z^4} = 0
\end{aligned} \tag{4.1}$$

where U is the flexural displacement in the x-direction, ϕ is the torsional displacement, ρ is the mass density, A is the area of the cross-section of beam, I_y, I_p, I_S are the axial, polar and sectorial moment of inertia, J is Saint-Venant's torsional constant, E is Young's modulus of elasticity, G is the shear modulus, α_U, α_ϕ are viscous damping coefficients, T is time and Z is the axial coordinate. Using the following transformations

$$U = u\sqrt{\frac{I_p}{A}} \qquad\qquad Z = zl \qquad\qquad R = rl$$

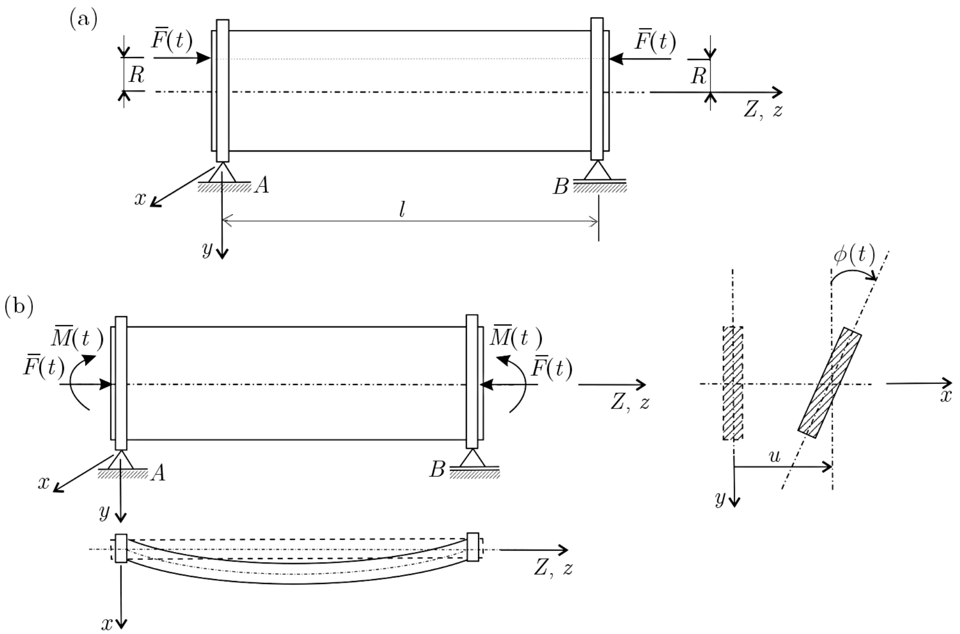

Fig. 1. Geometry of the thin-walled beam system

$$
\begin{array}{lll}
T = k_t t & \overline{M}(T) = R\overline{F}(T) & \overline{F}(T) = F_{cr}F(t) \\[2mm]
F_{cr} = \dfrac{\pi^2 EI_y}{l^2} & k_t^2 = \dfrac{\rho A l^4}{EI_y} & e = \dfrac{AI_s}{I_y I_p} \\[2mm]
\varepsilon\beta_1 = \dfrac{1}{2}\alpha_U \dfrac{l^2}{\sqrt{\rho AEI_y}} & \varepsilon\beta_2 = \dfrac{1}{2}\alpha_\phi l^2 \sqrt{\dfrac{A}{\rho EI_y I_p^2}} & S_1 = \dfrac{l^2 r^2 A}{I_p} \\[2mm]
S_2 = \dfrac{GJAl^2}{\pi^2 EI_y I_p} & &
\end{array}
\tag{4.2}
$$

where l is the length of the beam, F_{cr} is the Euler critical force for the simply supported narrow rectangular beam, S_1 and S_2 are slenderness parameters, β_1 and β_2 are reduced viscous damping coefficients, we get governing equations as

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} + 2\varepsilon\beta_1 \frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial z^4} + \pi^2\sqrt{S_1}F(t)\frac{\partial^2 \phi}{\partial z^2} + \pi^2 F(T)\frac{\partial^2 u}{\partial z^2} = 0 \\
&\frac{\partial^2 \phi}{\partial t^2} + 2\varepsilon\beta_2 \frac{\partial \phi}{\partial T} - \pi^2[S_2 - F(t)]\frac{\partial^2 \phi}{\partial z^2} + \pi^2\sqrt{S_1}F(t)\frac{\partial^2 u}{\partial z^2} + e\frac{\partial^4 \phi}{\partial z^4} = 0
\end{aligned}
\tag{4.3}
$$

Taking free warping displacement and zero angular displacements into account, the boundary conditions for the simply supported beam are

$$\begin{aligned} u(t,0) = u(t,1) = \frac{\partial^2 u}{\partial z^2}\Big|_{(t,0)} = \frac{\partial^2 u}{\partial z^2}\Big|_{(t,1)} = 0 \\ \phi(t,0) = \phi(t,1) = \frac{\partial^2 \phi}{\partial z^2}\Big|_{(t,0)} = \frac{\partial^2 \phi}{\partial z^2}\Big|_{(t,1)} = 0 \end{aligned} \tag{4.4}$$

Consider the shape function $\sin \pi z$ which satisfies the boundary conditions for the first mode vibration, the displacement $u(t,z)$ and twist angle $\phi(t,z)$ can be described by

$$u(t,z) = q_1(t) \sin \pi z \qquad \phi(t,z) = \psi_1(t) \sin \pi z \tag{4.5}$$

Substituting $u(t,z)$ and $\phi(t,z)$ from (4.5) into equations of motion (4.3) and employing Galerkin's method, the unknown time functions can be expressed as

$$\begin{aligned} \ddot{u}_1 + \omega_1^2 u_1 + 2\beta_1 \varepsilon \dot{u}_1 - \sqrt{\varepsilon}(K_{11}u_1 + K_{12}\psi_1)F(t) = 0 \\ \ddot{\psi}_1 + \omega_2^2 \psi_1 + 2\beta_2 \varepsilon \dot{\psi}_1 - \sqrt{\varepsilon}(K_{21}u_1 + K_{22}\psi_1)F(t) = 0 \end{aligned} \tag{4.6}$$

If we define the expressions

$$\begin{aligned} \omega_1^2 &= \pi^4 & \omega_2^2 &= \pi^4(S_2 + e) \\ K_{11} = K_{22} &= \pi^4 & K_{12} = K_{21} &= \pi^4\sqrt{S_1} \end{aligned} \tag{4.7}$$

and assume that the compressive axial force is stochastic white-noise process (2.2) with small intensity

$$F(t) = \sqrt{\varepsilon}\xi(t) \tag{4.8}$$

then Eq. (4.4) is reduced to Eq. (2.1).

Using the above result for the moment Lyapunov exponent

$$\Lambda(p) = \varepsilon\Lambda_1(p) + O(\varepsilon^2) \tag{4.9}$$

with the definition of the moment stability $\Lambda(p) < 0$, we determine analytically (the case where $N = 0$, $\Lambda_1(p)$ is shown with Eq. (3.19)) the pth moment stability boundary of the oscillatory system in the first-order perturbation

$$\beta_1 + \beta_2 > \sigma^2\pi^4\Big(1 + \frac{1}{S_2 + e}\Big)\Big(\frac{10 + 3p}{64} + \frac{6 + p}{32}S_1\Big) \tag{4.10}$$

It is known that oscillatory system (4.3) is asymptotically stable only if the Lyapunov exponent $\lambda < 0$. Then expression

$$\lambda = \varepsilon\lambda_1 + O(\varepsilon^2) \tag{4.11}$$

is employed to determine the almost-sure stability boundary of the oscillatory system in the first-order perturbation

$$\beta_1 + \beta_2 > \sigma^2\pi^4\Big(1 + \frac{1}{S_2 + e}\Big)\Big(\frac{5}{32} + \frac{3}{16}S_1\Big) \tag{4.12}$$

For the sake of simplicity, we assume, in what follows, that two viscous damping coefficients are equal

$$\beta_1 = \beta_2 = \beta \tag{4.13}$$

For this case, we determine the almost-sure stability boundary of the oscillatory system in the first-order perturbation

$$\beta > \frac{3\pi^4\sigma^2}{32}\Big(1 + \frac{1}{S_2 + e}\Big)\Big(\frac{5}{6} + S_1\Big) \tag{4.14}$$

and the pth moment stability boundary is

$$\beta > \frac{\sigma^2\pi^4}{64}\Big(1 + \frac{1}{S_2 + e}\Big)\Big[5 + \frac{3}{2}p + (6 + p)S_1\Big] \tag{4.15}$$

With respect to standard I-section, we can approximatiely take $h/ \approx 2$, $b/\delta_1 \approx 11$, $\delta/\delta_1 \approx 1.5$, where h is depth, b is width, δ is thickness of the flanges and δ_1 is thickness of the rib of I-section. These ratios yield $S_1 \approx 6(R/h)^2$, $S_2 \approx 0.01928(l/h)^2$ and $e \equiv 1.276$.

Figure 2 shows the almost-sure stability boundaries with respect to the damping coefficient β and intensity of random process σ. The stability regions are given in space for a constant geometrical ratio $(l/h = 10)$ of length of the beam and depth of the standard I-profil. They are enlarged when the axial force is closer to the axis of symmetry, with the greatest enlargement in the case when the force acts towards the main axis of symmetry. With the increase of ratio R/h, the stability regions are reduced.

Figure 3 shows the almost-sure and pth moment stability boundaries with respect to the damping coefficient β and intensity of the random process σ. Note that the moment stability boundaries are more conservative than the almost-sure boundary. These boundaries become increasingly more conservative as p increases, as shown in Fig. 3.

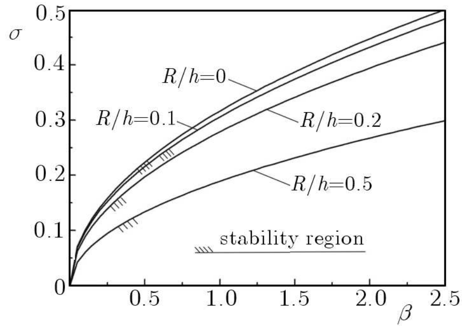

Fig. 2. Stability regions for almost-sure (a-s) stability for $\varepsilon = 0.1$ and $l/h = 10$

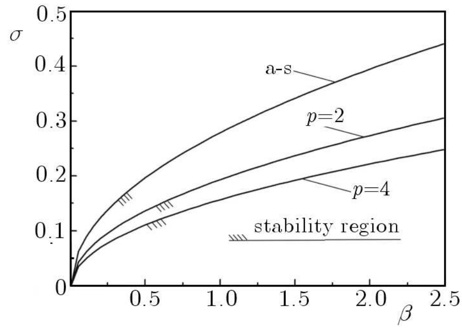

Fig. 3. Stability regions for almost-sure (a-s) and pth moment stability for $\varepsilon = 0.1$, $l/h = 10$ and $R/h = 0.2$

5. Numerical determination of the pth moment Lyapunov exponent

Numerical determination of the pth moment Lyapunov exponents is important in assessing the validity and ranges of applicability of the approximate analytical results. In many engineering applications, amplitudes of noise excitations are not small, and the approximate analytical methods, such as the method of perturbation of the method of stochastic averaging, cannot be applied. Therefore, numerical approaches have to be employed to evaluate the moment Lyapunov exponents. The numerical approach is based on expanding the exact solution to the system of Ito stochastic differential equations in powers of the time increment h and the small parameter ε as proposed in Milstein and Tret'Yakov (1997). The state vector of system (2.5) is to be rewritten as a system of Ito stochastic differential equations with small noise in the form

$$
\begin{aligned}
dx_1 &= \omega_1 x_2\,dt & dx_2 &= (-\omega_1 x_1 - \varepsilon 2\beta_1 x_2)\,dt + \sqrt{\varepsilon}(p_{11}x_1 + p_{12}x_3)\,dw(t)\\
dx_3 &= \omega_2 x_4\,dt & dx_4 &= (-\omega_2 x_3 - \varepsilon 2\beta_2 x_4)\,dt + \sqrt{\varepsilon}(p_{21}x_1 + p_{22}x_3)\,dw(t)
\end{aligned}
\tag{5.1}
$$

For the numerical solutions of the stochastic differential equations, the Runge-Kutta approximation may be applied, both with the error $R = O(h^4 + \varepsilon^4 h)$. The interval discretization is $[t_0, T]$: $\{t_k\ :\ k = 0,1,2,\ldots,M;\ t_0 < t_1 < t_2 < \ldots < t_M = T\}$ and the time increment is $h = t_{j+1} - t_j$. The following Runge-Kutta method is obtained for the $(k+1)$th iteration of the state vector $\boldsymbol{X} = [x_1, x_2, x_3, x_4]$

$$
\boldsymbol{X}^{(k+1)} = \left\{ \begin{bmatrix} \mathbf{N}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_2 \end{bmatrix} + \varepsilon \begin{bmatrix} \beta_1 \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \beta_2 \mathbf{M}_2 \end{bmatrix} + \sqrt{\varepsilon} \begin{bmatrix} p_{11}\mathbf{P}_{11} & p_{12}\mathbf{P}_1 \\ p_{21}\mathbf{P}_2 & p_{22}\mathbf{P}_{22} \end{bmatrix} \right\} \boldsymbol{X}^{(k)}
\tag{5.2}
$$

where $\mathbf{N}_k$, $\mathbf{M}_k$, $\mathbf{P}_k$ and $\mathbf{P}_{kk}$ $(k = 1,2)$ are 2×2 matrices

$$
\begin{aligned}
\mathbf{N}_k &= \begin{bmatrix} N_{1k} & N_{2k} \\ -N_{2k} & N_{1k} \end{bmatrix} & \mathbf{M}_k &= \begin{bmatrix} M_{1k} & M_{3k} \\ M_{4k} & M_{2k} \end{bmatrix}\\
\mathbf{P}_{kk} &= \begin{bmatrix} W_{1k} & W_{3k} \\ W_{4k} & W_{2k} \end{bmatrix} & \mathbf{P}_k &= \begin{bmatrix} W_{1k} & W_{3k} \\ W_5 & W_{k+5} \end{bmatrix}
\end{aligned}
\tag{5.3}
$$

and the members of previous matrices can be evaluated as follows

$$
\begin{aligned}
N_{1k} &= 1 - \frac{h^2\omega_k^2}{2} + \frac{h^4\omega_k^4}{24} & N_{2k} &= h\omega_k\Big(1 - \frac{h^2\omega_k^2}{6}\Big)\\
M_{1k} &= \frac{h^3}{3}\omega_k^2 & M_{2k} &= -2h\Big(1 - \frac{h^2\omega_k^2}{3} + \frac{h^4\omega_k^4}{36}\Big)\\
M_{3k} &= -h^2\omega_k\Big(1 - \frac{h^2\omega_k^2}{9}\Big) & M_{4k} &= h^2\omega_k\Big(1 - \frac{h^2\omega_k^2}{6}\Big)\\
W_{1k} &= \frac{h^{3/2}}{2}(\xi + 2\eta)\omega_k & W_{2k} &= \frac{h^{3/2}}{2}(\xi - 2\eta)\omega_k\\
W_{3k} &= \frac{h^{5/2}}{6}\xi\omega_k^2 & W_{4k} &= \sqrt{h}\Big(1 - \frac{h^2\omega_k^2}{3}\Big)\xi\\
W_5 &= \sqrt{h}\Big[1 - \frac{h^2}{6}(\omega_1^2 + \omega_2^2)\Big]\xi & W_6 &= \frac{h^{3/2}}{2}(\xi - 2\eta)\omega_2\\
W_7 &= \frac{h^{3/2}}{2}(\xi - 2\eta)\omega_1
\end{aligned}
\tag{5.4}
$$

Random variables ξ_i and η_i $(i = 1,2)$ are simulated as

$$
P(\xi_i = -1) = P(\xi_i = 1) = \frac{1}{2} \qquad P\Big(\eta_i = \frac{-1}{\sqrt{12}}\Big) = P\Big(\eta_i = \frac{1}{\sqrt{12}}\Big) = \frac{1}{2}
\tag{5.5}
$$

Having obtained L samples of the solutions to stochastic differential equations (5.1), the pth moment can be determined as follows

$$\begin{aligned}\mathrm{E}[\|\boldsymbol{X}(t_{k+1})\|^p] &= \frac{1}{L}\sum_{j=1}^{L}\|\boldsymbol{X}_j(t_{k+1})\|^p \\ \|\boldsymbol{X}_j(t_{k+1})\| &= \sqrt{[\boldsymbol{X}_j^\top(t_{k+1})\boldsymbol{X}_j(t_{k+1}}\end{aligned} \tag{5.6}$$

By the Monte-Carlo technique (Xie, 2005), we numerically calculate the pth moment Lyapunov exponent for all values of p of interest defined as

$$\Lambda(p) = \frac{1}{T}\log\mathrm{E}[\|\boldsymbol{X}(T)\|^p] \tag{5.7}$$

6. Numerical results and conclusions

In this paper, the moment Lyapunov exponents of a thin-walled beam subjected to eccentric stochastic axial loads are studied. The method of regular perturbation is applied to obtain a weak noise expansion of the moment Lyapunov exponent in terms of the small fluctuation parameter. The weak noise expansion of the Lyapunov exponent is also obtained. The slope of the moment Lyapunov exponent curve at $p = 0$ is the Lyapunov exponent. When the Lyapunov exponent is negative, system (4.6) is stable with probability 1, otherwise it is unstable. For the purpose of illustration, in the numerical study we considere the set of system parameters $\beta_1 = \beta_2 = \beta = 1$, $\varepsilon = 0.1$, $L = 4000$, $h = 0.0005$, $M = 10\,000$ and $x_1(0) = x_2(0) = x_3(0) = x_4(0) = 1/2$.

Typical results of the moment Lyapunov exponents $\Lambda(p)$ for system (4.6) given by Eq. (4.9) in the first perturbation are shown in Fig. 4 for I-section, $\varepsilon = 0.1$, $l/h = 10$ and $R/h = 0.2$, the noise intensity $\sigma = 0.2$ and damping coefficient $\beta = 1$. The accuracy of the approximate analytical results is validated and assessed by comparing them to the numerical results. The Monte Carlo simulation approach is usually more versatile, especially when the noise excitations cannot be described in such a form that can be treated easily using analytical tools. From the Central Limit Theorem, it is well known that the estimated pth moment Lyapunov exponent is a random number, with the mean being the true value of the pth moment Lyapunov exponent and standard deviation equal to $n_p/\sqrt{L}$, where n_p is the sample standard deviation determined from L samples.

It is evident that analytical results agree very well with the numerical results, except for $N = 0$ when the function $\psi_0(\varphi)$ does not depend on φ and

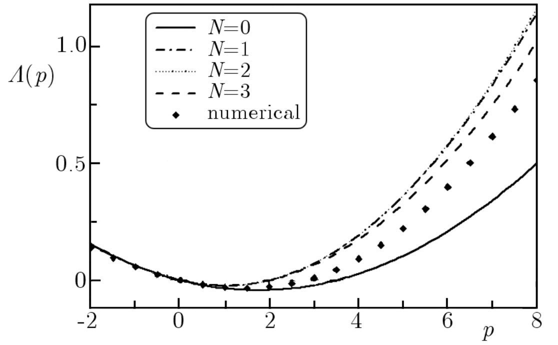

Fig. 4. Moment Lyapunov exponent $\Lambda(p)$ for $\sigma = 0.2$ and $\beta = 1$

assumes the form $\psi_0(\varphi) = K_0 =$ const. It is observed that the discrepancies between the approximate analytical and numerical results decrease for a larger number N of series (3.16). Further increase of the N number of members does not make sense, because the curves merge into one. Further increase in the number of members in the supposed solution does not make sense also because the approximation of the exact solutions is worse. On the other side, the equation from which we can determine the value of the exponent of the moment Lyapunov exponent is of a higher order and the coefficients in them are of a more complex form.

Acknowledgments

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Appendix 1

$$
\begin{aligned}
\alpha_1 = & -2pP(\beta_1 \sin^2\theta_1 \cos^2\varphi + \beta_2 \sin^2\theta_2 \sin^2\varphi) \\
& + \frac{p(p-2)P}{16}(p_{11}p_{22} + p_{12}p_{21}) \sin 2\theta_1 \sin 2\theta_2 \sin^2 2\varphi \\
& + \frac{pP}{2}(p_{11}^2 \cos^2\theta_1 \cos^2\varphi + p_{12}^2 \cos^2\theta_2 \sin^2\varphi)[\cos^2\theta_1 + (p\cos^2\varphi - \cos 2\varphi)] \\
& + \frac{pP}{2}(p_{22}^2 \cos^2\theta_2 \sin^2\varphi + p_{21}^2 \cos^2\theta_1 \cos^2\varphi)[\cos^2\theta_2 + (p\sin^2\varphi + \cos 2\varphi)] \\
& + \frac{pP}{8}\cos\theta_1 \cos\theta_2 \sin 2\varphi\{p_{22}p_{21}[p+2-(p-2)(\cos 2\theta_2 + 2\sin^2\theta_2 \cos 2\varphi)] \\
& + p_{11}p_{12}[p+2-(p-2)(\cos 2\theta_1 - 2\sin^2\theta_1 \cos 2\varphi)]\} \\
& + \frac{p(p-2)P}{2}\sin 2\varphi \sin\theta_1 \sin\theta_2 (p_{11}p_{21}\cos^2\theta_1 \cos^2\varphi + p_{22}p_{12}\cos^2\theta_2 \sin^2\varphi)
\end{aligned}
$$

$$
\begin{aligned}
\alpha_2 &= (\beta_1 \sin^2\theta_1 - \beta_2 \sin^2\theta_2)\sin 2\varphi - \frac{1}{16}(p_{11}p_{22} + p_{12}p_{21})\sin 2\theta_1 \sin 2\theta_2 \sin 4\varphi \\
&\quad -(p_{11}p_{21}\cos^2\theta_1\cos^2\varphi + p_{22}p_{12}\cos^2\theta_2\sin^2\varphi)\sin\theta_1\sin\theta_2\cos 2\varphi \\
&\quad -\frac{1}{4}p_{11}^2\cos^2\theta_1\sin 2\varphi(\cos 2\theta_1 - \cos 2\varphi\sin^2\theta_1) \\
&\quad +\frac{1}{4}p_{22}^2\cos^2\theta_2\sin 2\varphi(\cos 2\theta_2 + \cos 2\varphi\sin^2\theta_2) \\
&\quad +\frac{1}{2}p_{12}^2\cos^2\theta_2\sin^2\varphi(\sin^2\theta_1\sin 2\varphi - \cos^2\theta_1\tan\varphi) \\
&\quad -\frac{1}{2}p_{21}^2\cos^2\theta_1\cos^2\varphi(\sin^2\theta_2\sin 2\varphi - \cos^2\theta_2\cot\varphi) \\
&\quad -p_{11}p_{12}\cos\theta_1\cos\theta_2\sin^2\varphi(\cos 2\theta_1 - \cos 2\varphi\sin^2\theta_1) \\
&\quad +p_{22}p_{21}\cos\theta_1\cos\theta_2\cos^2\varphi(\cos 2\theta_2 + \cos 2\varphi\sin^2\theta_2) \\
\alpha_3 &= -\beta_1\sin 2\theta_1 - \frac{1}{2}(p_{11}\cos\theta_1 + p_{12}\cos\theta_2\tan\varphi)^2\sin 2\theta_1 \\
\alpha_4 &= -\beta_2\sin 2\theta_2 - \frac{1}{2}(p_{22}\cos\theta_2 + p_{21}\cos\theta_1\cot\varphi)^2\sin 2\theta_2
\end{aligned}
$$

Appendix 2

$$
\begin{aligned}
a_1 &= \frac{1}{32}(p_{11}\sin 2\theta_1 - p_{22}\sin 2\theta_2)^2\sin^2 2\varphi \\
&\quad +\frac{1}{2}(p_{12}\cos\theta_1\sin\theta_2\cos^2\varphi - p_{21}\sin\theta_1\cos\theta_2\sin^2\varphi)^2 \\
&\quad -\frac{1}{4}(p_{11}\sin 2\theta_1 - p_{22}\sin 2\theta_2)(p_{12}\sin\theta_1\cos\theta_2\sin^2\varphi - p_{21}\cos\theta_1\sin\theta_2\cos^2\varphi)\sin 2\varphi \\
a_2 &= \frac{1}{2}\cos^2\theta_1(p_{11}\cos\theta_1 + p_{12}\cos\theta_2\tan\varphi)^2 \\
a_3 &= \frac{1}{2}\cos^2\theta_2(p_{22}\cos\theta_2 + p_{21}\cos\theta_1\cot\varphi)^2 \\
a_4 &= -\frac{1}{4}\cos^2\theta_1\sin 2\varphi[p_{11}^2\sin 2\theta_1 - (p_{11}p_{22} - p_{12}p_{21})\sin 2\theta_2] \\
&\quad +p_{11}p_{21}\cos^3\theta_1\sin\theta_2\cos^2\varphi \\
&\quad -p_{12}\cos\theta_1\cos\theta_2\sin^2\varphi\Big(p_{11}\sin 2\theta_1 - \frac{p_{22}}{2}\sin 2\theta_2 + p_{12}\sin\theta_1\cos\theta_2\tan\varphi\Big) \\
a_5 &= \frac{1}{4}\cos^2\theta_2\sin 2\varphi[p_{22}^2\sin 2\theta_2 - (p_{11}p_{22} + p_{12}p_{21})\sin 2\theta_1] \\
&\quad -p_{22}p_{12}\sin\theta_1\cos^3\theta_2\sin^2\varphi \\
&\quad -p_{21}\cos\theta_1\cos\theta_2\cos^2\varphi\Big(\frac{p_{11}}{2}\sin 2\theta_1 + p_{22}\sin 2\theta_2 + p_{21}\cos\theta_1\sin\theta_2\cot\varphi\Big) \\
a_6 &= (p_{11}p_{22} + p_{12}p_{21})\cos^2\theta_1\cos^2\theta_2 + \cos\theta_1\cos\theta_2(p_{11}p_{21}\cos^2\theta_1\cot\varphi \\
&\quad +p_{22}p_{12}\cos^2\theta_2\tan\varphi) \\
b_1 &= (\beta_1\sin^2\theta_1 - \beta_2\sin^2\theta_2)\sin 2\varphi + \frac{p-1}{16}(p_{11}p_{22} + p_{12}p_{21})\sin 2\theta_1\sin 2\theta_2\sin 4\varphi \\
&\quad -\Big(\frac{1}{4}p_{11}^2\cos\theta_1\sin 2\varphi + p_{11}p_{12}\cos\theta_2\sin^2\varphi\Big)[\cos^2\theta_1 + 2(p-1)\cos^2\varphi\sin^2\theta_1]\cos\theta_1
\end{aligned}
$$

$$+\Big(\frac{1}{4}p_{22}^2\cos\theta_2\sin 2\varphi+p_{22}p_{21}\cos\theta_1\cos^2\varphi\Big)[\cos^2\theta_2+2(p-1)\sin^2\varphi\sin^2\theta_2]\cos\theta_2$$
$$-\frac{1}{2}p_{12}^2\cos^2\theta_2\sin^2\varphi[(p-1)\sin^2\theta_1\sin 2\varphi+\cos^2\theta_1\tan\varphi]$$
$$+\frac{1}{2}p_{21}^2\cos^2\theta_1\cos^2\varphi[(p-1)\sin^2\theta_2\sin 2\varphi+\cos^2\theta_2\cot\varphi]$$
$$-(p-1)\sin\theta_1\sin\theta_2\cos 2\varphi(p_{11}p_{21}\cos^2\theta_1\cos^2\varphi+p_{22}p_{12}\cos^2\theta_2\sin^2\varphi)$$

$$b_2=-\beta_1\sin 2\theta_1+\frac{1}{2}(p_{11}^2\cos\theta_1+p_{11}p_{12}\cos\theta_2\tan\varphi)[(p-1)\cos^2\varphi$$
$$-\sin^2\varphi]\sin 2\theta_1\cos\theta_1+\frac{p}{2}(p_{11}p_{22}+p_{12}p_{21})\cos^2\theta_1\sin 2\theta_2\sin^2\varphi$$
$$+\frac{1}{2}p_{12}^2\sin 2\theta_1\cos^2\theta_2[(p-1)\cos^2\varphi-\sin^2\varphi]\tan^2\varphi$$
$$+\frac{p}{2}p_{11}p_{21}\cos^3\theta_1\sin\theta_2\sin 2\varphi+\frac{p}{2}p_{22}p_{12}\cos\theta_1\cos\theta_2\sin 2\theta_2\sin^2\varphi\tan\varphi$$

$$b_3=-\beta_2\sin 2\theta_2+\frac{1}{2}(p_{22}^2\cos\theta_2+p_{22}p_{21}\cos\theta_1\cot\varphi)[(p-1)\sin^2\varphi$$
$$-\cos^2\varphi]\sin 2\theta_2\cos\theta_2+\frac{p}{2}(p_{11}p_{22}+p_{12}p_{21})\sin 2\theta_1\cos^2\theta_2\cos^2\varphi$$
$$+\frac{1}{2}p_{21}^2\sin 2\theta_2\cos^2\theta_1[(p-1)\sin^2\varphi-\cos^2\varphi]\cot^2\varphi$$
$$+\frac{p}{2}p_{11}p_{21}\sin 2\theta_1\cos\theta_1\cos\theta_2\cos^2\varphi\cot\varphi+\frac{p}{2}p_{22}p_{12}\sin\theta_1\cos^3\theta_2\sin 2\varphi$$

$$c=-2p(\beta_1\sin^2\theta_1\cos^2\varphi+\beta_2\sin^2\theta_2\sin^2\varphi)$$
$$+\frac{p(p-2)}{16}(p_{11}p_{22}+p_{12}p_{21})\sin 2\theta_1\sin 2\theta_2\sin^2 2\varphi$$
$$+\frac{p(p-2)}{2}(p_{11}p_{21}\cos^2\theta_1\cos^2\varphi+p_{22}p_{12}\cos^2\theta_2\sin^2\varphi)\sin\theta_1\sin\theta_2\sin 2\varphi$$
$$+\frac{p}{2}(p_{11}\cos\theta_1\cos\varphi+p_{12}\cos\theta_2\sin\varphi)^2\{\cos^2\theta_1+[(p-1)\cos^2\varphi+\sin^2\varphi]\sin^2\theta_1\}$$
$$+\frac{p}{2}(p_{22}\cos\theta_2\sin\varphi+p_{21}\cos\theta_1\cos\varphi)^2\{\cos^2\theta_2+[(p-1)\sin^2\varphi+\cos^2\varphi]\sin^2\theta_2\}$$

Appendix 3

$$A_{00}=-\Lambda_1(p)-\frac{p}{2}(\beta_1+\beta_2)+\frac{p(10+3p)}{128}(p_{11}^2+p_{22}^2)+\frac{p(6+p)}{64}(p_{12}^2+p_{21}^2)$$
$$A_{10}=-\frac{p+2}{4}(\beta_1-\beta_2)+\frac{1}{64}(p+2)^2(p_{11}^2-p_{22}^2)+\frac{1}{4}(p_{12}^2-p_{21}^2)$$
$$A_{20}=\frac{(p+2)(p+4)}{256}[p_{11}^2+p_{22}^2-2(p_{12}^2+p_{21}^2)]-\frac{17}{32}(p_{12}^2+p_{21}^2)$$
$$A_{30}=\frac{3}{4}(p_{12}^2-p_{21}^2)$$
$$A_{01}=-\frac{p}{4}(\beta_1-\beta_2)+\frac{p(p+2)}{64}(p_{11}^2-p_{22}^2)-\frac{p}{16}(p_{12}^2-p_{21}^2)$$
$$A_{11}=-\frac{1}{2}\Lambda_1(p)-\frac{p}{4}(\beta_1+\beta_2)+\frac{7p^2+22p-8}{512}(p_{11}^2+p_{22}^2)+\frac{p^2+10p-56}{256}(p_{12}^2+p_{21}^2)$$

$$A_{21} = -\frac{p+4}{8}(\beta_1 - \beta_2) + \frac{p^2+6p+8}{128}(p_{11}^2 - p_{22}^2) + \frac{p+20}{32}(p_{12}^2 - p_{21}^2)$$
$$A_{31} = \frac{p^2+10p+24}{512}(p_{11}^2 + p_{22}^2) - \frac{p^2+10p+216}{256}(p_{12}^2 + p_{21}^2)$$
$$A_{02} = \frac{p(p-2)}{256}[(p_{11}^2 + p_{22}^2) - 2(p_{12}^2 + p_{21}^2)]$$
$$A_{12} = -\frac{p-2}{8}(\beta_1 - \beta_2) + \frac{(p+2)(p-2)}{128}(p_{11}^2 - p_{22}^2) + \frac{p-2}{16}(p_{12}^2 - p_{21}^2)$$
$$A_{22} = -\frac{1}{2}\Lambda_1(p) - \frac{p}{4}(\beta_1 + \beta_2) + \frac{3p^2+10p-16}{256}(p_{11}^2 + p_{22}^2) + \frac{p^2+6p-80}{128}(p_{12}^2 + p_{21}^2)$$
$$A_{32} = -\frac{p+6}{8}(\beta_1 - \beta_2) + \frac{7p^2+8p+12}{512}(p_{11}^2 - p_{22}^2) + \frac{p+18}{16}(p_{12}^2 - p_{21}^2)$$
$$A_{03} = 0$$
$$A_{13} = \frac{p^2-6p+8}{512}[(p_{11}^2 + p_{22}^2) - 2(p_{12}^2 + p_{21}^2)]$$
$$A_{23} = -\frac{p-4}{8}\Big[(\beta_1 - \beta_2) - \frac{1}{16}(p+2)(p_{11}^2 - p_{22}^2) + \frac{3}{4}(p_{12}^2 - p_{21}^2)\Big]$$
$$A_{33} = -\frac{1}{2}\Lambda_1(p) - \frac{p}{4}(\beta_1 + \beta_2) + \frac{3p^2+10p-36}{256}(p_{11}^2 + p_{22}^2) + \frac{2p^2+12p-312}{256}(p_{12}^2 + p_{21}^2)$$

Appendix 4

$$d_1^{(1)} = p(\beta_1 + \beta_2) + \Big(\frac{1}{32} - \frac{21p}{28} - \frac{13p^2}{256}\Big)(p_{11}^2 + p_{22}^2) + \Big(\frac{7}{16} - \frac{11p}{64} - \frac{3p^2}{128}\Big)(p_{12}^2 + p_{21}^2)$$
$$\begin{aligned} d_0^{(1)} = {} & \frac{1}{8}p(p-2)(\beta_1^2 + \beta_2^2) + \frac{1}{4}p(2+3p)\beta_1\beta_2 + \Big(-\frac{13p}{2048} + \frac{p^2}{8192} + \frac{5p^3}{4096} \\ & + \frac{5p^4}{32768}\Big)(p_{11}^4 + p_{22}^4) + \Big(-\frac{5p}{512} + \frac{p^2}{2048} + \frac{p^3}{512} + \frac{p^4}{8192}\Big)(p_{12}^4 + p_{21}^4) + \Big(\frac{3p}{1024} \\ & + \frac{97p^2}{4096} + \frac{29p^3}{2048} + \frac{37p^4}{16384}\Big)p_{11}^2 p_{22}^2 + \Big(-\frac{37p}{256} + \frac{p^2}{1024} + \frac{p^3}{256} + \frac{p^4}{4096}\Big)p_{12}^2 p_{21}^2 \\ & + \Big(-\frac{23p}{512} + \frac{7p^2}{2048} + \frac{17p^3}{2048} + \frac{5p^4}{8192}\Big)(p_{11}^2 p_{12}^2 + p_{22}^2 p_{21}^2) + \Big(-\frac{15p}{512} + \frac{7p^2}{2048} \\ & + \frac{9p^3}{2048} + \frac{5p^4}{8192}\Big)(p_{11}^2 p_{21}^2 + p_{22}^2 p_{12}^2) + \Big(-\frac{3p}{64} - \frac{37p^2}{256} - \frac{21p^3}{512}\Big)(\beta_1 p_{22}^2 + \beta_2 p_{11}^2) \\ & + \Big(\frac{5p}{64} - \frac{5p^2}{256} - \frac{5p^3}{512}\Big)(\beta_1 p_{11}^2 + \beta_2 p_{22}^2) + \Big(\frac{5p}{32} - \frac{7p^2}{128} - \frac{3p^3}{256}\Big)(\beta_1 p_{21}^2 + \beta_2 p_{12}^2) \\ & + \Big(\frac{9p}{32} - \frac{15p^2}{128} - \frac{3p^3}{256}\Big)(\beta_1 p_{12}^2 + \beta_2 p_{21}^2) \end{aligned}$$

Appendix 5

$$d_2^{(2)} = \frac{3p}{2}(\beta_1 + \beta_2) + \Big(\frac{5}{32} - \frac{31p}{128} - \frac{19p^2}{256}\Big)(p_{11}^2 + p_{22}^2) + \Big(\frac{27}{16} - \frac{17p}{64} - \frac{5p^2}{128}\Big)(p_{12}^2 + p_{21}^2)$$

$$d_1^{(2)} = \Big(\frac{1}{2}-\frac{3p}{8}-\frac{9p^2}{16}\Big)(\beta_1^2+\beta_2^2)-\Big(1-\frac{3p}{4}-\frac{15p^2}{8}\Big)\beta_1\beta_2+\Big[\Big(\frac{3}{256}-\frac{47p}{2048}+\frac{41p^2}{32768}$$
$$+\frac{61p^3}{8192}+\frac{35p^4}{32768}\Big)(p_{11}^4+p_{22}^4)+\Big(-\frac{1}{128}-\frac{55p}{1024}+\frac{153p^2}{4096}+\frac{133p^3}{4096}+\frac{83p^4}{16384}\Big)p_{11}^2p_{22}^2\Big]$$
$$+\Big[\Big(\frac{15}{64}-\frac{55p}{512}-\frac{39p^2}{2048}+\frac{13p^3}{2048}+\frac{3p^4}{8192}\Big)(p_{12}^4+p_{21}^4)+\Big(\frac{55}{32}-\frac{231p}{256}-\frac{71p^2}{1024}$$
$$+\frac{13p^3}{1024}+\frac{3p^4}{4096}\Big)p_{12}^2p_{21}^2]+\Big[\Big(\frac{1}{8}+\frac{p}{8}-\frac{43p^2}{128}-\frac{25p^3}{256}\Big)(\beta_1p_{22}^2+\beta_2p_{11}^2)$$
$$+\Big(-\frac{1}{8}+\frac{3p}{16}-\frac{19p^2}{128}-\frac{13p^3}{256}\Big)(\beta_1p_{11}^2+\beta_2p_{22}^2)\Big]+\Big[\Big(\frac{3}{8}+\frac{45p}{32}-\frac{7p^2}{32}-\frac{5p^3}{128}\Big)$$
$$\cdot(\beta_1p_{21}^2+\beta_2p_{12}^2)+\Big(-\frac{3}{8}+\frac{63p}{32}-\frac{5p^2}{16}-\frac{5p^3}{128}\Big)(\beta_1p_{12}^2+\beta_2p_{21}^2)\Big]$$

$$d_0^{(2)} = \Big(\frac{p}{4}-\frac{3p^2}{16}+\frac{p^3}{32}\Big)(\beta_1^3+\beta_2^3)+\Big(-\frac{p}{4}+\frac{3p^2}{16}+\frac{15p^3}{32}\Big)(\beta_1^2\beta_2+\beta_1\beta_2^2)$$
$$+\Big(-\frac{75p}{2^{16}}+\frac{9p^2}{2^{16}}+\frac{277p^3}{2^{20}}+\frac{7p^4}{2^{21}}-\frac{49p^5}{2^{22}}-\frac{7p^6}{2^{23}}\Big)(p_{11}^6+p_{22}^6)$$
$$+\Big(-\frac{p}{2^{16}}+\frac{227p^2}{2^{16}}+\frac{63p^3}{2^{20}}-\frac{3179p^4}{2^{21}}-\frac{2323p^5}{2^{22}}-\frac{469p^6}{2^{23}}\Big)(p_{11}^4p_{22}^2+p_{11}^2p_{22}^4)$$
$$+\Big(-\frac{639p}{2^{15}}+\frac{93p^2}{2^{15}}+\frac{2321p^3}{2^{19}}-\frac{61p^4}{2^{20}}-\frac{413p^5}{2^{21}}-\frac{35p^6}{2^{22}}\Big)(p_{11}^4p_{12}^2+p_{22}^4p_{21}^2)$$
$$+\Big(-\frac{339p}{2^{15}}+\frac{89p^2}{2^{15}}+\frac{1137p^3}{2^{19}}-\frac{29p^4}{2^{20}}-\frac{189p^5}{2^{21}}-\frac{35p^6}{2^{22}}\Big)(p_{11}^4p_{21}^2+p_{22}^4p_{12}^2)$$
$$+\Big(-\frac{1407p}{2^{13}}-\frac{7p^2}{2^{13}}+\frac{1633p^3}{2^{17}}+\frac{3p^4}{2^{18}}-\frac{45p^5}{2^{19}}-\frac{3p^6}{2^{20}}\Big)(p_{12}^4p_{21}^2+p_{21}^4p_{12}^2)$$
$$+\Big(-\frac{417p}{2^{14}}+\frac{55p^2}{2^{14}}+\frac{1551p^3}{2^{18}}-\frac{11p^4}{2^{19}}-\frac{291p^5}{2^{20}}-\frac{21p^6}{2^{21}}\Big)(p_{11}^2p_{12}^4+p_{22}^2p_{21}^4)$$
$$+\Big(-\frac{273p}{2^{14}}+\frac{59p^2}{2^{14}}+\frac{815p^3}{2^{18}}-\frac{43p^4}{2^{19}}-\frac{131p^5}{2^{20}}-\frac{21p^6}{2^{21}}\Big)(p_{11}^2p_{21}^4+p_{22}^2p_{12}^4)$$
$$+\Big(-\frac{1465p}{2^{13}}+\frac{321p^2}{2^{13}}+\frac{3807p^3}{2^{17}}+\frac{229p^4}{2^{18}}-\frac{211p^5}{2^{19}}-\frac{21p^6}{2^{20}}\Big)(p_{11}^2+p_{22}^2)p_{12}^2p_{21}^2$$
$$+\Big(\frac{9p}{2^{14}}+\frac{6951p^2}{2^{14}}+\frac{4545p^3}{2^{18}}-\frac{621p^4}{2^{19}}-\frac{1133p^5}{2^{20}}-\frac{195p^6}{2^{21}}\Big)(p_{12}^2+p_{21}^2)p_{11}^2p_{22}^2$$
$$+\Big(\frac{5p}{2^8}-\frac{25p^2}{2^{12}}-\frac{51p^3}{2^{14}}+\frac{7p^4}{2^{14}}+\frac{7p^5}{2^{16}}\Big)(\beta_1p_{11}^4+\beta_2p_{22}^4)$$
$$+\Big(-\frac{p}{2^7}-\frac{69p^2}{2^{12}}+\frac{133p^3}{2^{14}}+\frac{115p^4}{2^{14}}+\frac{63p^5}{2^{16}}\Big)(\beta_2p_{11}^4+\beta_1p_{22}^4)$$
$$+\Big(\frac{7p}{2^6}+\frac{17p^2}{2^8}-\frac{41p^3}{2^8}-\frac{45p^4}{2^{10}}\Big)\beta_1\beta_2(p_{11}^2+p_{22}^2)+\Big(-\frac{15p}{2^7}+\frac{31p^2}{2^9}+\frac{3p^3}{2^9}-\frac{7p^4}{2^{11}}\Big)$$
$$\cdot(p_{11}^2\beta_1^2+p_{22}^2\beta_2^2)+\Big(\frac{p}{2^7}+\frac{15p^2}{2^9}-\frac{45p^3}{2^9}-\frac{55p^4}{2^{11}}\Big)(p_{11}^2\beta_2^2+p_{22}^2\beta_1^2)$$
$$+\Big(-\frac{p}{2^8}-\frac{55p^2}{2^{11}}+\frac{153p^3}{2^{13}}+\frac{133p^4}{2^{13}}+\frac{83p^5}{2^{15}}\Big)p_{11}^2p_{22}^2(\beta_1+\beta_2)$$

$$
\begin{aligned}
&+\Big(\frac{117p}{2^9}-\frac{191p^2}{2^{11}}-\frac{109p^3}{2^{12}}+\frac{59p^4}{2^{13}}+\frac{7p^5}{2^{14}}\Big)(p_{11}^2p_{12}^2\beta_1+p_{22}^2p_{21}^2\beta_2)\\
&+\Big(-\frac{45p}{2^9}-\frac{469p^2}{2^{11}}-\frac{109p^3}{2^{12}}+\frac{129p^4}{2^{13}}+\frac{27p^5}{2^{14}}\Big)(p_{11}^2p_{12}^2\beta_2+p_{22}^2p_{21}^2\beta_1)\\
&+\Big(\frac{63p}{2^9}-\frac{105p^2}{2^{11}}-\frac{49p^3}{2^{12}}+\frac{21p^4}{2^{13}}+\frac{27p^5}{2^{14}}\Big)(p_{11}^2p_{21}^2\beta_1+p_{22}^2p_{12}^2\beta_2)\\
&+\Big(-\frac{39p}{2^9}-\frac{507p^2}{2^{11}}-\frac{73p^3}{2^{12}}+\frac{119p^4}{2^{13}}+\frac{27p^5}{2^{14}}\Big)(p_{11}^2p_{21}^2\beta_2+p_{22}^2p_{12}^2\beta_1)\\
&+\Big(-\frac{21p}{2^{13}}+\frac{3p^2}{2^{13}}+\frac{75p^3}{2^{17}}+\frac{p^4}{2^{18}}-\frac{15p^5}{2^{19}}-\frac{p^6}{2^{20}}\Big)(p_{12}^6+p_{21}^6)\\
&+\Big(\frac{39p}{2^8}-\frac{9p^2}{2^7}-\frac{51p^3}{2^{12}}+\frac{9p^4}{2^{11}}+\frac{3p^5}{2^{14}}\Big)(p_{12}^4\beta_1+p_{21}^4\beta_2)\\
&+\Big(\frac{21p}{2^7}-\frac{19p^2}{2^9}-\frac{27p^3}{2^{12}}+\frac{p^4}{2^9}+\frac{3p^5}{2^{14}}\Big)(p_{12}^4\beta_2+p_{21}^4\beta_1)\\
&+\Big(-\frac{45p}{2^6}+\frac{125p^2}{2^8}-\frac{15p^3}{2^8}-\frac{5p^4}{2^{10}}\Big)(p_{12}^2\beta_1^2+p_{21}^2\beta_2^2)\\
&+\Big(-\frac{21p}{2^6}+\frac{53p^2}{2^8}-\frac{3p^3}{2^8}-\frac{5p^4}{2^{10}}\Big)(p_{12}^2\beta_2^2+p_{21}^2\beta_1^2)\\
&+\Big(\frac{33p}{2^5}+\frac{127p^2}{2^7}-\frac{25p^3}{2^7}-\frac{15p^4}{2^9}\Big)(p_{12}^2+p_{21}^2)\beta_1\beta_2\\
&+\Big(\frac{55p}{2^6}-\frac{231p^2}{2^9}-\frac{71p^3}{2^{11}}+\frac{13p^4}{2^{11}}+\frac{3p^5}{2^{13}}\Big)p_{12}^2p_{21}^2(\beta_1+\beta_2)
\end{aligned}
$$

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Momentowe wykładniki Lapunowa i stateczność stochastyczna cienkościennej belki poddanej mimośrodowemu obciążeniu w kierunku osiowym

Streszczenie

W artykule zbadano wykładniki Lapunowa i momentowe wykładniki Lapunowa układów o dwóch stopniach swobody poddanych parametrycznemu wymuszeniu białym szumem. Zastosowano regularną metodę perturbacyjną do wyznaczenia jawnych wyrażeń na te wykładniki w obecności szumów o małej intensywności. Wykładniki Lapunowa i momentowe wykładniki Lapunowa są ważnymi wielkościami w określaniu prawie pewnej i momentowej stateczności stochastycznej układu dynamicznego. Jako przykład rozważono cienkościenną belkę poddaną mimośrodowemu obciążeniu osiowemu o charakterze losowym. Poprawność otrzymanych wyników przybliżenia momentowych wykładników Lapunowa sprawdzono w drodze symulacji numerycznej przy wykorzystaniu metody Monte Carlo.

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