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NUMERICAL CALCULATION OF STABILITY DERIVATIVES OF AN AIRCRAFT

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Work presents application of methods based on the Prandtl-Glauert potential model of fluid to stability derivatives calculation of an aircraft in subsonic (Vortex Lattice Method) and supersonic (Characteristic Box Method) flow. Modeling of velocity field, next used in calculating the set of stability derivatives was shown. Results of calculation were compared with experimental data.

Notations

a	–	speed of sound
a_∞	–	speed of sound in a free stream flow
b	–	wing span
c	–	reference chord
c_a	–	mean aerodynamic chord
C_l	–	dimensionless rolling moment coefficient
C_m	–	dimensionless pitching moment coefficient
C_n	–	dimensionless yawing moment coefficient
C_p	–	dimensionless pressure coefficient
C_X	–	drag coefficient
C_y	–	lateral force coefficient
C_z	–	lift force coefficient
h	–	grid parameter
Ma_∞	–	Mach number of a free stream flow, $\mathrm{Ma}_\infty = V_\infty/a_\infty$
p	–	pressure

p_∞	–	pressure of a free stream flow
Δp	–	pressure drop between lower and upper surfaces
P	–	rolling velocity
P_z	–	lift force
q	–	dynamic pressure, $q = \rho_\infty V_\infty^2/2$
Q	–	pitching velocity
R	–	yawing velocity
S	–	lifting surface (also reference surface)
r, s	–	characteristic coordinates
t	–	time
V_∞	–	free stream velocity
$\boldsymbol{V}$	–	velocity vector
u, v, w	–	coordinates of velocity (in stability axis system)
x, y, z	–	Cartesian coordinates
α	–	angle of attack
β	–	Prandtl-Glauert coefficient, $\beta = \sqrt{\mathrm{Ma}_\infty^2 - 1}$
$\varphi(x, y, z)$	–	disturbance potential
κ	–	isentropic exponent
ρ	–	air density
ρ_∞	–	air density in a free stream flow
ξ, η, ζ	–	Cartesian coordinates

1. Introduction

Stability derivatives are a very important group of data necessary in dynamic analysis of aircraft. Calculation of them is a very important question. The most often used numerical methods of aerodynamics base on a potential model of flow. Potential methods, in spite of many simplifications in comparison with the CFD method yield very good results (cf Bertin and Smith (1989)). The Prandtl-Glauert potential model has the form

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = \mathrm{Ma}^2 \frac{\partial^2 \varphi}{\partial x^2} \tag{1.1}$$

Methods based on this model could be divided into two groups: finite element methods and Kernel function methods. The first group (finite element) contains calculation of the integral

$$\varphi(x,y,z) = -\frac{1}{\pi} \iint\limits_{\Sigma(x,y,z)} \frac{w(\xi,\eta)\, d\xi d\eta}{\sqrt{(x-\xi)^2 + (1-\mathrm{Ma}^2)[(y-\eta)^2 + z^2]}} \tag{1.2}$$

where

$$w = \frac{\partial \varphi}{\partial z}\Big|_{z=0}$$

is the solution of equation (1.1) (cf Anderson (1985), Krasil'shchikova (1986)). These methods (FEM) yield the distribution of disturbance velocity potential and next the distribution of load pressure and aerodynamic coefficients. Kernel function methods give pressure distribution from the integral equation

$$\frac{w(x,y)}{V_\infty} = \frac{1}{4\pi\rho V^2} \int\limits_S \Delta p(\xi,\eta) K(x,\xi,y,\eta,\mathrm{Ma})\, ds \tag{1.3}$$

where

$K(x,\xi,y,\eta,\mathrm{Ma})$ – Kernel function
$w(x,y)$ – normal velocity.

The procedure of solving the equation (1.3) and consequently determining the Kernel function form is the fundamental problem of these group of methods. It is not easy, yet these methods are treated as better ones then the finite element methods, specially for a subsonic unsteady flow. For supersonic flow there are more problems with determining a kernel function due to flow disturbing caused by Mach lines affected by planform discontinuities crossing the surface. There is also, a serious problem, since a planform with discontinuous pressure distribution is very difficult to treat. All these and the other problems result in a statement (Cunningham (1974a,b)): *"... the economic advantage of existing kernel function methods over finite element methods is not nearly so great (if any at all) in supersonic flow as it is in subsonic flow."*

The present work contains the application of the Vortex Lattice Method (VLM) in subsonic flow and the Characteristic Box method (CHB) in supersonic flow for calculating the stability derivatives of an aircraft.

2. Basis of the VLM

Integral equation (1.3) can be approximated with the system of linear equations

$$\overline{w}_r = \sum_S D_{rs} \overline{p}_s \tag{2.1}$$

where $\overline{w}_r$ and $\overline{p}_s$ denote a dimensionless normal velocity at the point R and a dimensionless pressure, respectively, and

$$D_{rs} = \frac{1}{8\pi} \int\limits_S K(x,\xi,y,\eta,\mathrm{Ma})\, ds \tag{2.2}$$

Transforming Eq (2.1) and using the Kutta-Zukowski condition one can show (cf Goraj and Molicki (1990)) that

$$D_{rs} = \frac{w_{rs}\Delta x_s}{2\Gamma_s} \tag{2.3}$$

where
Δx_s – mean geometric chord of panel
Γ_s – circulation of the horseshoe vortex.

The normal velocity w_{rs}, induced at the point R by an infinite horseshoe vortex filament Γ_s, was determined using the Biot-Savart law.

The VLM reduces the problem to calculation of coefficients D_{rs} of Eqs (2.1) and to calculating of the total normal velocity distribution, which is the sum of an induced velocity and components of a free stream velocity, and next to solve the system of equation (2.1). Details are given by Bertin and Smith (1989), Goraj and Molicki (1990).

3. Basis of the CHB method

The CHB method reduces the problem to calculation of the integral (1.2) with known $w(\xi,\eta)$. A new system of coordinates was defined by generators of the Mach cone (Fig.1)

$$r = \frac{x-\beta y}{\beta} \qquad\qquad s = \frac{x+\beta y}{\beta} \tag{3.1}$$

The integral given in Eq (1.2) in new coordinates has the form

$$\varphi(r,s) = -\frac{1}{\pi} \iint\limits_{S(r,s)} \frac{w(\xi^*,\eta^*)\, d\xi^* d\eta^*}{R^*} \tag{3.2}$$

where

$$R^* = \sqrt{(r-r^*)(s-s^*)}$$

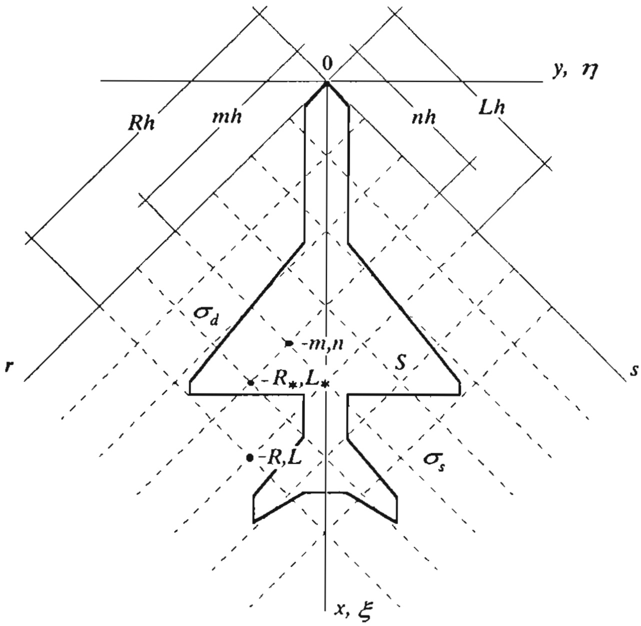

Fig. 1.

Discretization of integrating surface on elements – CHARACTERISTIC BOXes, is made by lines parallel to generators of the Mach cone. The area of integration is created by the generators of the Mach cone (coordinate system $0rs$ – Fig.1) and the generators of the "inverse" Mach cone in the way ensuring the aircraft planform to be inside. After some transformations (cf Belocerkovskii et al. (1983)) and making all the variables dimensionless, Eq (3.2) has the form

$$\varphi(Lh, Rh) = -\frac{2h}{\pi} \sum_{n=1}^{L} \sum_{m=1}^{R} w(nh, mh) B(n, m, L, R) \tag{3.3}$$

where

$$B(n, m, L, R) = \frac{1}{\left(\sqrt{L-n+1} + \sqrt{L-n}\right)\left(\sqrt{R-m+1} + \sqrt{R-m}\right)}$$

Unknown function w(nh,mh) can be determined by boundary conditions. This function has the form:

- For the lifting surface – S (in Fig.1)

$$w_S(nh, mh) = -\sin\alpha(x, y) \tag{3.4}$$

where $\alpha(x,y)$ is the local angle of attack

- For the diaphragm surface – δ_d

on the left-hand side of axis $0X$

$$w_{dl}(Lh,Rh) = -\sum_{n=1}^{L-1} w(nh,mh)B(n,L) \tag{3.5}$$

on the right-hand side of the axis $0X$

$$w_{dr}(Lh,Rh) = -\sum_{m=1}^{R-1} w(nh,mh)B(m,R) \tag{3.6}$$

where

$$B(n,L) = \frac{1}{\sqrt{L-n+1}+\sqrt{L-n}}$$

$$B(m,R) = \frac{1}{\sqrt{R-m+1}+\sqrt{R-m}}$$

- For the wake – δ_s

$$w_s(Lh,Rh) = -\frac{2h}{\pi}\varphi(L_*h,R_*h) - \sum_{n=1}^{L}\sum_{m=1}^{R}{}' w(nh,mh)B(n,m,L,R) \tag{3.7}$$

where (L_*h, R_*h) denote box on the trailing edge which satisfies the condition: $L-R = L_* - R_*$, and $\sum\sum'$ denotes the sum without an element (L,R).

4. Calculations of stability derivatives

Numerical methods presented here show the way of calculating the pressure distribution or the disturbance potential distribution, from which the pressure distribution is easy to obtain. So the load calculation is now straigh forward, but stability derivatives (aerodynamic load derivatives with respect to linear and angular velocities) are much more important in dynamic analysis than aerodynamic loads. Definitions of these derivatives and the way of calculating them in various coordinate systems are given by Goraj (1984). In the present work stability derivatives are defined in "body axis system" (right-handed system, with origin at 1/4 of MAC (Mean Aerodynamical Chord), $0X$ axis is parallel to MAC directed to the aircraft nose, $0Y$ axis directed towards the right wing). Stability derivatives noted as coefficients of linear components of the Taylor series, could be obtained, from

$$\frac{\partial C_i}{\partial P_j} = \frac{C_i(P_{j2}) - C_i(P_{j1})}{P_{j2} - P_{j1}} \tag{4.1}$$

where

C_i – coefficient of the ith load (e.g. lift force)

P_j – jth dimensionless velocity (e.g. pitching velocity).

To obtain loads in various flight conditions, normal velocity distribution should be determined.

It could be consider:

- Flight with a constant angle of attack

  In this case the global angle of attack is constant

  $$\alpha(x,y) = \text{const} \qquad \text{so} \qquad w_s(x,y) = -\sin(\alpha + \alpha_{z(i)})$$

  where: $\alpha_{z(i)}$ – setting angle of the ith panel (box) in relation to the reference surface

- Flight with a constant pitching velocity

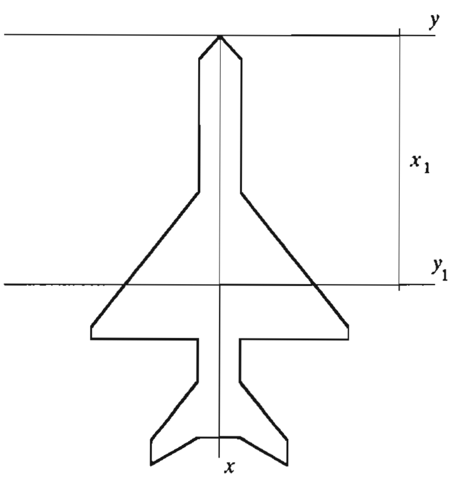

Fig. 2.

In this case, the origin of pitch y_1 (Fig.2) should be defined, and next the distribution of normal velocity can be calculated from the formula

$$w_s(x,y) = \frac{(x-x_1)Q}{V_\infty} \tag{4.2}$$

- Flight with a constant yawing velocity

  In this case, the origin of roll z_1 (Fig.3) must be defined, and next the distribution of normal velocity can be calculated from the formula

$$w_s(x,z) = \frac{(x-x_1)R}{V_\infty} \tag{4.3}$$

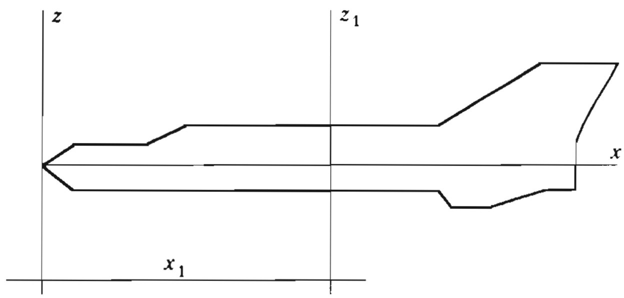

Fig. 3.

- Flight with a constant rolling velocity

  In this case the normal velocity is as follows:

  – in the symmetric case

$$w_s(x,y) = \frac{Py}{V_\infty} \tag{4.4}$$

  – in the antisymmetric case – lateral planform

$$w_s(x,z) = \frac{Pz}{V_\infty} \tag{4.5}$$

5. Modification of the CHB method for stability derivatives calculation

If we solve a problem for the stability derivatives only, the CHB method could be modified in order to calculate the derivatives directly. Eq (1.1) can be transformed to the form

$$\beta^2 \frac{\partial^2 \varphi^w}{\partial x^2} - \frac{\partial^2 \varphi^w}{\partial y^2} - \frac{\partial^2 \varphi^w}{\partial z^2} = 0 \tag{5.1}$$

where φ^w denotes the derivative of velocity potential with respect to the defined velocity. Then solution of equation (5.1) has the form

$$\varphi^w(x,y,z) = -\frac{1}{\pi} \iint\limits_{S(x,y,z)} \frac{w^w(\xi,\eta)\, d\xi d\eta}{R} \tag{5.2}$$

where w^w denotes the derivative of normal velocity with respect to the defined velocity.

Next procedure is analogical to the method presented in Section 3. Derivatives of normal velocity for various cases should be determined now:

- Flight with a constant angle of attack

$$w_s^w(nh,mh) = -\cos(\alpha + \alpha_{z(i)}) \tag{5.3}$$

- Flight with a constant pitching velocity (or yawing velocity in the anti-symmetric case)

$$w_s^w(nh,mh) = \beta \frac{(n+m-1)h}{2} - \xi_1 \tag{5.4}$$

  where ξ_1 is the dimensionless translation of the rotation pole

- Flight with a constant rolling velocity

$$w_s^w(nh,mh) = \frac{(n-m)h}{2} \tag{5.5}$$

The presented procedure does not contain calculation method for stability derivatives with respect to acceleration. Partial derivative with respect to $\dot{\alpha} = \dot{w}/V$ is very important in dynamic analysis. To obtain this derivative Eq (1.1) is transformed to the form

$$\beta^2 \frac{\partial^2 \varphi^{\dot{\alpha}}}{\partial x^2} - \frac{\partial^2 \varphi^{\dot{\alpha}}}{\partial y^2} - \frac{\partial^2 \varphi^{\dot{\alpha}}}{\partial z^2} = 0 \tag{5.6}$$

where $\varphi^{\dot{\alpha}}$ denotes the derivative of velocity potential with respect to $\dot{\alpha}$. Solution of Eq (5.6) has the form (cf Belocerkovskiĭ et al. (1983))

$$\varphi^{\dot{\alpha}}(x,y,z) = -\frac{1}{\pi} \iint\limits_{S(x,y,z)} \Big[\frac{w^{\dot{\alpha}}(\xi,\eta)}{R} - \frac{\mathrm{Ma}^2}{\beta} \frac{w^{\alpha}(\xi,\eta)}{R}\Big]\, d\xi d\eta \tag{5.7}$$

Eq (5.7) could be solved with the procedure presented in Section 3. An unknown function $w^{\dot{\alpha}}(nh,mh)$ depending on the wing area, has the form:

- For the lifting surface (index S on Fig.1)

$$w_S^{\dot{\alpha}}(nh,mh) = -\frac{\mathrm{Ma}^2}{\beta}\frac{(n+m-1)h}{2} \tag{5.8}$$

- For the diaphragm surface

  on the left-hand side of $0X$ axis

$$w_{dl}^{\dot{\alpha}}(Lh,Rh) = -\sum_{n=1}^{L-1} w^{\dot{\alpha}}(nh,mh)B(n,L) \tag{5.9}$$

  on the right-hand side of $0X$ axis

$$w_{dr}^{\dot{\alpha}}(Lh,Rh) = -\sum_{m=1}^{R-1} w^{\dot{\alpha}}(nh,mh)B(m,R) \tag{5.10}$$

- For the wake (index δ_s)

$$w_s^{\dot{\alpha}}(Lh,Rh) = -\frac{2h}{\pi}\Big[\varphi^{\dot{\alpha}}(L_*h,R_*h) + \frac{h(L+R-L_*-R_*)}{2\beta}\varphi^{\alpha}(L_*h,R_*h)\Big] + \tag{5.11}$$
$$-\sum_{n=1}^{L}\sum_{m=1}^{R}{}' w^{\dot{\alpha}}(nh,mh)B(n,m,L,R)$$

where (L_*h,R_*h) denotes the element on the trailing edge which satisfies the condition: $L-R = L_*-R_*$, and $\sum\sum'$ denotes the sum without an element (L,R).

6. Results of test calculations

Test calculations are made for isolated trapezoidal wings and DELTA wings and results are compared with results of other numerical methods (cf Neni and Chee Tung (1971)) and with experimental data from the wind tunnel (cf Hill (1957), Rather et al. (1948)). The lift force coefficient derivative with respect to the angle of attack for DELTA wing of aspect ratio $AR = 1$ is shown in

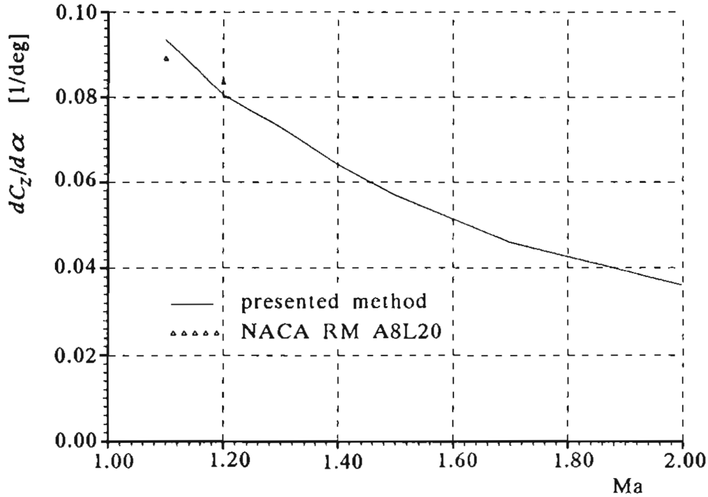

Fig. 4. Derivative $dC_z/d\alpha$ of trapezoidal wing with the aspect ratio $AR = 4$ versus the Mach number

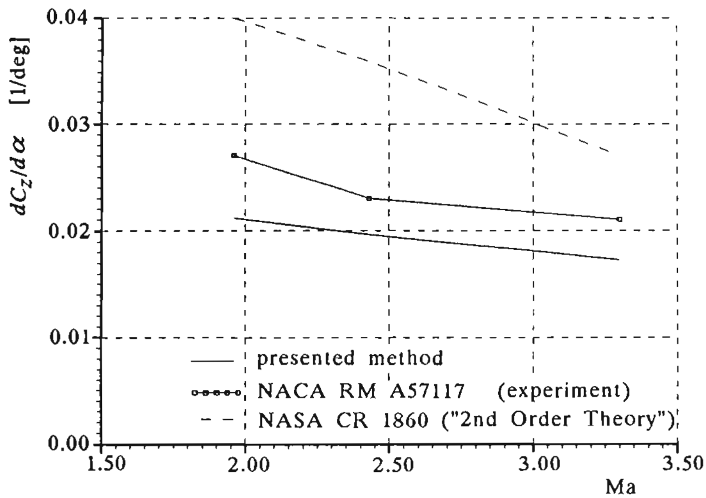

Fig. 5. Derivative $dC_z/d\alpha$ of DELTA wing with the aspect ratio $AR = 1$ versus the Mach number

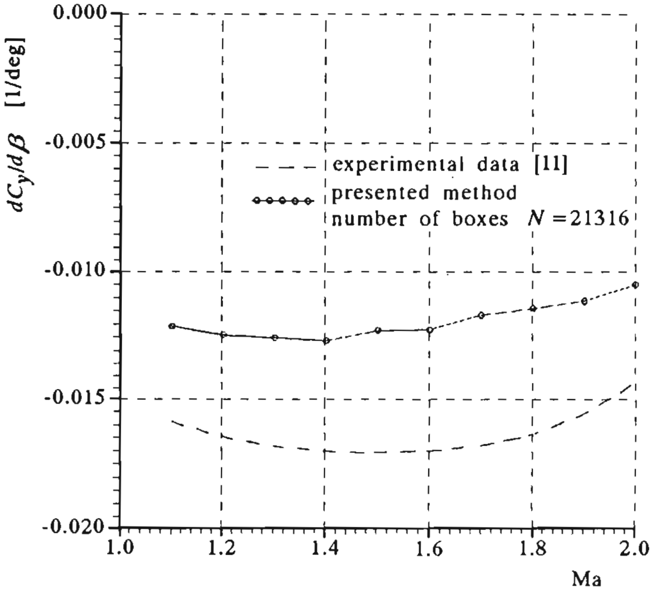

Fig. 6. Aircraft MiG-21 – derivative $dC_y/d\beta$

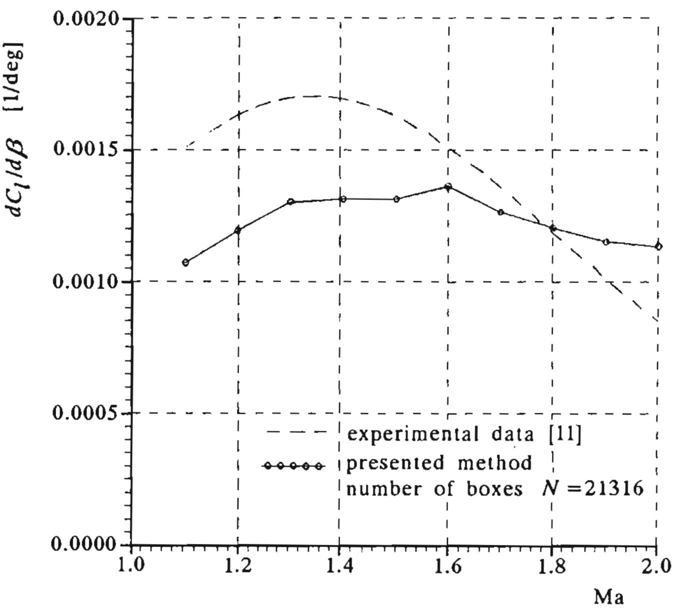

Fig. 7. Aircraft MiG-21 – derivative $dC_l/d\beta$

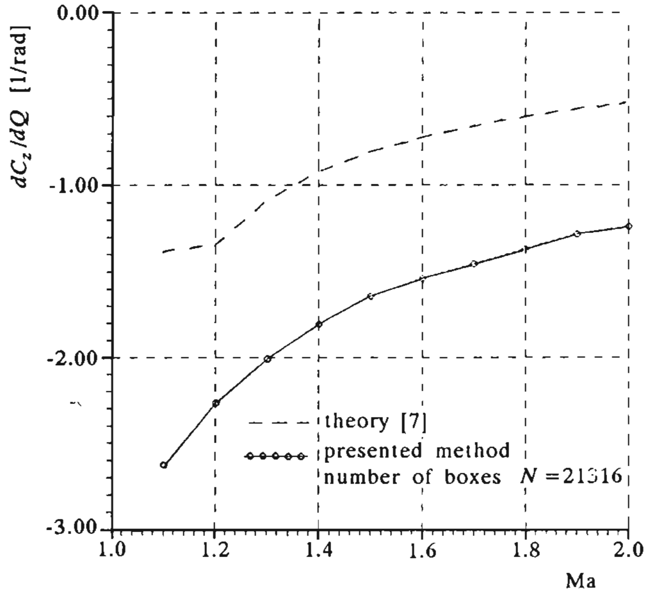

Fig. 8. Aircraft MiG-21 – derivative dC_z/dQ

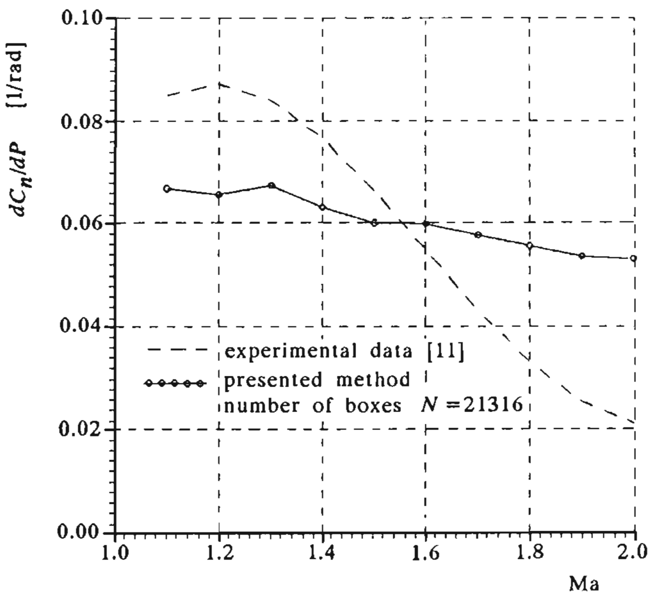

Fig. 9. Aircraft MiG-21 – derivative dC_n/dP, angle of attack $\alpha = 7.5°$

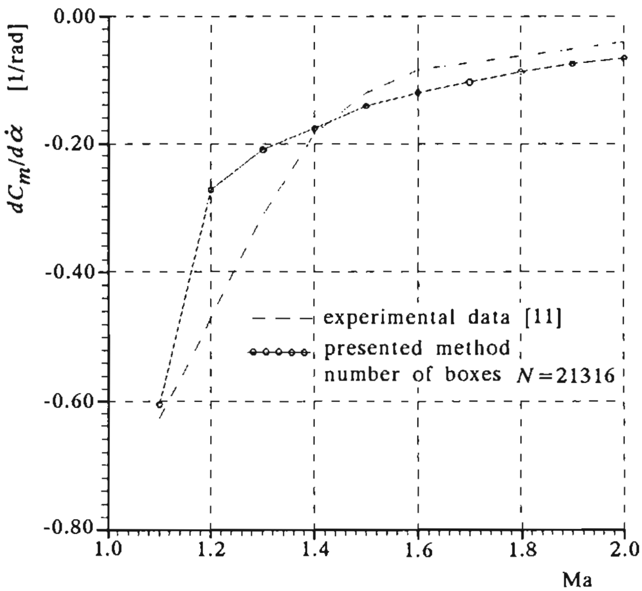

Fig. 10. Aircraft MiG-21 – derivative $dC_m/d\dot{\alpha}$

Fig.4 as a function of the Mach number. Gradient $dC_z/d\alpha$ for a trapezoidal wing of aspect ratio $AR = 4$ is shown in Fig.5.

Stability derivatives for MiG-21 aircraft are presented in Fig.6 ÷ Fig.10. There are respectively: derivative of lateral force with respect to the angle of lateral flow, derivative of rolling moment with respect to the angle of lateral flow, derivative of lift force with respect to the pitching velocity, derivative of yawing moment with respect to the rolling velocity and derivative of pitching moment with respect to the acceleration $\dot{\alpha}$. All results are compared with data from testing flight (cf Manerowski et al. (1989)).

7. Concluding remarks

Results in terms of the presented methods show a good agreement with the experimental results in the cases, where a body modeled by thin surface could be treated as the lifting surface. In the cases of wide fuselage, engine nacelle etc. serious errors appear. Especially bad influence of the wide elements above mentioned can easily be perceived when lateral stability derivatives are computed, or in case of extremely complex aerodynamic configurations. Advantages of these methods however, especially their relatively low cost, fully constitute their application inspite of existence of more sophisticated methods.

Acknowledgements

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Numeryczne wyznaczanie pochodnych aerodynamicznych samolotu

Streszczenie

W pracy pokazano zastosowanie metod bazujących na modelu potencjalnym Prandtla-Glauerta do wyznaczania pochodnych aerodynamicznych samolotu w zakresie poddźwiękowym (Vortex Lattice Method) i naddźwiękowym (Characteristic Box Method). Przedstawiono modelowanie pola prędkości umożliwiające wyznaczenie kompletu pochodnych potrzebnych do badania dynamiki samolotu. Wyniki obliczeń porównano z danymi doświadczalnymi.

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