MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
3, 32, 1994

ON THE EQUILIBRIUM EQUATIONS FOR A CERTAIN TYPE OF SHELLS

JERZY KRAWCZYK

Department of Mathematics

Pedagogical University of Opole

In this paper it is shown that the symmetric parametrization of a shell midsurface can be succesively applied to the analysis of a certain class of thin walls investigations in the framework of the Kirchhoff shell theory.

1. Preliminaries

In the paper a thin, linear elastic, isotropic shell of constant thickness is considered. In this section, following Green (1954) we introduce the basic relations of the problem under investigation. In the undeformed shell region the normal coordinates u^1, u^2, z are introduced in the known way

$$\boldsymbol{R} = \boldsymbol{r} + z\boldsymbol{m}$$

where $\boldsymbol{R}$ is a position vector, $(u^1, u^2) \in D$, D being the regular region on $\mathcal{R}^2$, $2h$ is the shell thickness $|z| \leq h$ and $\boldsymbol{m}$ is a unit vector normal to the shell midsurface S given by

$$\boldsymbol{r} = [x_1(u^1, u^2), x_2(u^1, u^2), x_3(u^1, u^2)]$$

We define $\boldsymbol{r}_i \equiv \partial\boldsymbol{r}/\partial u^i$ (here and in the sequel $i, j, ...$ run over $1, 2$)

External loadings acting on the shell were denoted by (cf Krawczyk (1993))

$$\boldsymbol{P} = P^i\boldsymbol{r}_i + P^3\boldsymbol{m} \qquad i = 1,2 \tag{1.1}$$

It is assumed that the shell undergoes deformation of the Kirchoff-Love type. Let S' be the deformed shell midsurface and

$$\boldsymbol{w} = w^i\boldsymbol{r}_i + w^3\boldsymbol{m} \tag{1.2}$$

be the displacement vector field for $z = 0$. Therefore S' is given by

$$\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{w} \tag{1.3}$$

and endowed with metric tensors (all "primes" are related to a deformed shell)

$$\begin{aligned} g'_{ij} &= \boldsymbol{r}'_i \cdot \boldsymbol{r}'_j = g_{ij} + 2\gamma_{ij} \\ b'_{ij} &= \boldsymbol{m}' \cdot \boldsymbol{r}'_{ij} = b_{ij} + 2\rho_{ij} \\ c'_{ij} &= \boldsymbol{m}'_i \cdot \boldsymbol{m}'_j = c_{ij} + 2\eta_{ij} \end{aligned}$$

where g_{ij}, b_{ij}, c_{ij} are related to an undeformed shell midsurface.

For the linear theory

$$\begin{aligned} 2\gamma_{ij} &= w^k\Big|_i g_{kj} + w^k\Big|_j g_{ki} - 2w^3 b_{ij} \\ 2\rho_{ij} &= w^k\Big|_i b_{kj} + w^k\Big|_j b_{ki} + w^k b_{ij}\Big|_k + w^3\Big|_{ij} - w^3 c_{ij} \\ 2\eta_{ij} &= w^k\Big|_i c_{kj} + w^k\Big|_j c_{ki} + w^k c_{ij}\Big|_k + b_i^k w^3\Big|_{kj} + b_j^k w^3\Big|_{ki} \end{aligned} \tag{1.4}$$

where $|$ stands for a covariant differentiation on the shell midsurface.

For any surface $\overline{S}$, parallel to S, we have

$$\overline{g}_{ij} = (1 - Kz^2)g_{ij} - 2z(1 - Hz)b_{ij} \qquad \overline{g} = Z^2 g \tag{1.5}$$

where K, H are the Gauss curvature and mean curvature of the surface S, respectively and

$$\begin{aligned} Z &= 1 - 2Hz + Kz^2 \\ \overline{g} &= \overline{g}_{11}\overline{g}_{22} - (\overline{g}_{12})^2 \\ g &= g_{11}g_{22} - (g_{12})^2 \end{aligned}$$

The deformation, being considered, tranforms surface $\overline{S}$ onto surface $\overline{S}'$ for which

$$\overline{g}'_{ij} = \overline{g}_{ij} + 2\overline{\gamma}_{ij}$$

where

$$\overline{\gamma}_{ij} = \gamma_{ij} - 2z\rho_{ij} + z^2\eta_{ij} \tag{1.6}$$

Let τ^{jk} be a stress tensor and define internal forces by means of

$$\begin{aligned} N^{ij} &= \int\limits_{-h}^{h} \sqrt{\frac{\overline{g}}{g}}(\delta_k^j - zb_k^j)\tau^{ik}\,dz \\ M^{ij} &= \int\limits_{-h}^{h} \sqrt{\frac{\overline{g}}{g}}(\delta_k^j - zb_k^j)\tau^{ik}z\,dz \end{aligned} \tag{1.7}$$

where for isotropic materials

$$\tau^{ij} = \frac{E}{1-\nu^2}[(1-\nu)\overline{g}^{ir}\overline{g}^{js} + \nu\overline{g}^{ij}\overline{g}^{rs}]\overline{\gamma}_{rs} \tag{1.8}$$

E is the Young modulus, ν is the Poisson ratio, N^{ij} and M^{ij} satisfy the well-known shell equilibrium equations

$$\begin{aligned} N^{ij}\Big|_i - b^j_k M^{ki}\Big|_i + P^j &= 0 \\ b_{ij}N^{ij} + M^{ij}\Big|_{ij} + P^3 &= 0 \end{aligned} \tag{1.9}$$

where P^j, P^3 are given by Eq (1.1) and $b^j_k = g^{jr}b_{rk}$.

2. Symmetrical parametrization

We assume that the parametrization u^1, u^2 of the surface S satisfies the following conditions

$$g_{12} = b_{12} = 0 \tag{2.1}$$

An arbitrary and smooth and invertible transformation of parameters

$$u^1 = u^1(v^1, v^2) \qquad\qquad u^2 = u^2(v^1, v^2)$$

yields (cf Willmore (1969))

$$\begin{aligned} \widetilde{g}_{ij} &= g_{ks}\frac{\partial u^k}{\partial v^i}\frac{\partial u^s}{\partial v^j} \\ \widetilde{b}_{ij} &= b_{ks}\frac{\partial u^k}{\partial v^i}\frac{\partial u^s}{\partial v^j} \\ \det&\Big[\frac{\partial u^m}{\partial v^n}\Big] \neq 0 \end{aligned} \tag{2.2}$$

Parametrization of the surface S is called a symmetrical one if

$$\widetilde{g}_{11} = \widetilde{g}_{22} \qquad\qquad \widetilde{b}_{11} = \widetilde{b}_{22} \tag{2.3}$$

We shall show the conditions leading to the symmetrical parametrization. From Eqs (2.1), (2.2) and (2.3) it is obvious that

$$\begin{aligned} g_{11}\Big[\Big(\frac{\partial u^1}{\partial v^1}\Big)^2 - \Big(\frac{\partial u^1}{\partial v^2}\Big)^2\Big] + g_{22}\Big[\Big(\frac{\partial u^2}{\partial v^1}\Big)^2 - \Big(\frac{\partial u^2}{\partial v^2}\Big)^2\Big] &= 0 \\ b_{11}\Big[\Big(\frac{\partial u^1}{\partial v^1}\Big)^2 - \Big(\frac{\partial u^1}{\partial v^2}\Big)^2\Big] + b_{22}\Big[\Big(\frac{\partial u^2}{\partial v^1}\Big)^2 - \Big(\frac{\partial u^2}{\partial v^2}\Big)^2\Big] &= 0 \end{aligned} \tag{2.4}$$

• If $b_{ij} \neq \lambda g_{ij}$ then $g_{11}b_{22} - g_{22}b_{11} \neq 0$ and Eq (2.4) has only a trivial solution

$$\Big(\frac{\partial u^1}{\partial v^1}\Big)^2 - \Big(\frac{\partial u^1}{\partial v^2}\Big)^2 = 0 \qquad \Big(\frac{\partial u^2}{\partial v^1}\Big)^2 - \Big(\frac{\partial u^2}{\partial v^2}\Big)^2 = 0 \tag{2.5}$$

• If $b_{ij} = \lambda g_{ij}$ then Eqs (2.4) reduce to one equation

$$g_{11}\Big[\Big(\frac{\partial u^1}{\partial v^1}\Big)^2 - \Big(\frac{\partial u^1}{\partial v^2}\Big)^2\Big] + g_{22}\Big[\Big(\frac{\partial u^2}{\partial v^1}\Big)^2 - \Big(\frac{\partial u^2}{\partial v^2}\Big)^2\Big] = 0$$

In both cases the following functions

$$u^1 = \frac{1}{\sqrt{2}}(v^1 - v^2) \qquad u^2 = \frac{1}{\sqrt{2}}(v^1 + v^2) \tag{2.6}$$

are solutions to the above equations.

It is easy to show that for a symmetrical parametrization

$$\begin{aligned} b_{11} &= b_{22} = Hg_{11} + \delta g_{12} \qquad & b_1^1 &= b_2^2 = H \\ b_{12} &= b_{21} = Hg_{12} + \delta g_{11} \qquad & b_2^1 &= b_1^2 = \delta \end{aligned} \tag{2.7}$$

where $\delta = e\sqrt{H^2 - K}$, $|e| = 1$.

We also have

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2g}\Big[g_{11}\frac{\partial g_{11}}{\partial v^1} + g_{12}\Big(\frac{\partial g_{11}}{\partial v^2} - 2\frac{\partial g_{12}}{\partial v^1}\Big)\Big] \\
\Gamma^2_{22} &= \frac{1}{2g}\Big[g_{11}\frac{\partial g_{11}}{\partial v^2} + g_{12}\Big(\frac{\partial g_{11}}{\partial v^1} - 2\frac{\partial g_{12}}{\partial v^2}\Big)\Big] \\
\Gamma^1_{22} &= \frac{1}{2g}\Big[g_{11}\Big(2\frac{\partial g_{12}}{\partial v^2} - \frac{\partial g_{11}}{\partial v^1}\Big) - g_{12}\frac{\partial g_{11}}{\partial v^2}\Big] \\
\Gamma^2_{11} &= \frac{1}{2g}\Big[g_{11}\Big(2\frac{\partial g_{12}}{\partial v^1} - \frac{\partial g_{11}}{\partial v^2}\Big) - g_{12}\frac{\partial g_{11}}{\partial v^1}\Big] \\
\Gamma^1_{12} &= \frac{1}{2g}\Big[g_{11}\frac{\partial g_{11}}{\partial v^2} - g_{12}\frac{\partial g_{11}}{\partial v^1}\Big] \\
\Gamma^2_{12} &= \frac{1}{2g}\Big[g_{11}\frac{\partial g_{11}}{\partial v^1} - g_{12}\frac{\partial g_{11}}{\partial v^2}\Big]
\end{aligned}$$

The Gauss and Codazzi formulae for symmetrical parametrization are

$$\begin{aligned}
&l^2 - m^2 = K \\
&\frac{\partial H}{\partial v^2}g_{11} - \frac{\partial H}{\partial v^1}g_{12} = \frac{\partial \delta g_{11}}{\partial v^1} - \frac{\partial \delta g_{12}}{\partial v^2} \\
&\frac{\partial H}{\partial v^1}g_{11} - \frac{\partial H}{\partial v^2}g_{12} = \frac{\partial \delta g_{11}}{\partial v^2} - \frac{\partial \delta g_{12}}{\partial v^1}
\end{aligned}$$

where

$$l = \frac{1}{\sqrt{g}}(Hg_{11} + \delta g_{12}) \qquad m = \frac{1}{\sqrt{g}}(Hg_{12} + \delta g_{11})$$

Example

Let

$$\boldsymbol{r} = \left[a \cosh \frac{u^1}{a} \cos u^2, \ a \cosh \frac{u^1}{a} \sin u^2, \ u^1\right]$$

and assume

$$u^1 = a(v^1 + v^2) \qquad u^2 = v^1 - v^2$$

Hence

$$\begin{aligned} &g_{11} = g_{22} = 2a^2 \cosh^2(v^1 + v^2) && g_{12} = 0 \\ &b_{11} = b_{22} = 0 && b_{12} = 2a \\ &K = -\left(\frac{b_{12}}{g_{12}}\right)^2 && H = 0 \\ &\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^2_{12} = \Gamma^2_{22} = -\Gamma^1_{22} = -\Gamma^2_{11} = \tanh(v^1 + v^2) \end{aligned}$$

3. Equilibrium equations

Let us assume that a symmetrical parametrization on the surface S is given. From Eq (1.5)

$$\begin{aligned} \overline{g}^{11} = \overline{g}^{22} &= \frac{1}{Z}g^{11} + \frac{2\delta z}{Z^2}\left[\delta z g^{11} + (1 - Hz)g^{12}\right] \\ \overline{g}^{12} = \overline{g}^{21} &= \frac{1}{Z}g^{12} + \frac{2\delta z}{Z^2}\left[\delta z g^{12} + (1 - Hz)g^{11}\right] \end{aligned}$$

Since

$$\delta z = ze\sqrt{H^2 - K} = ze|k_1 - k_2|$$

where k_1, k_2 are main curvatures of the surface S, then at the points in which $b_{ij} = \lambda g_{ij}$, it is

$$\overline{g}^{ij} = \frac{1}{Z}g^{ij}$$

For thin shells and $k_1 \cong k_2$ we obtain $\delta z \cong 0$. Moreover, assume

$$\overline{g}^{ij} \approx \frac{1}{Z}g^{ij} \tag{3.1}$$

From Eqs (1.8) and (3.1) we obtain

$$\tau^{ij} = \frac{1}{Z^2}\Big[\overline{\tau}^{ij} - 2z\widehat{\tau}^{ij} + z^2\check{\tau}^{ij}\Big] \tag{3.2}$$

where

$$\begin{aligned}
\overline{\tau}^{ij} &= \frac{E}{1-\nu^2}\Big[(1-\nu)\gamma^{ij} + \nu\gamma_k^k g^{ij}\Big]\\
\widehat{\tau}^{ij} &= \frac{E}{1-\nu^2}\Big[(1-\nu)\rho^{ij} + \nu\rho_k^k g^{ij}\Big]\\
\check{\tau}^{ij} &= \frac{E}{1-\nu^2}\Big[(1-\nu)\eta^{ij} + \nu\eta_k^k g^{ij}\Big]
\end{aligned} \tag{3.3}$$

Performing integration of Eq (1.7), taking into account Eqs (2.7), (3.1) and (3.2) we obtain

$$\begin{aligned}
N^{ij} &= 2h\overline{\tau}^{ij} + \frac{2}{3}h^3\Big[H^2\overline{\tau}^{ij} - 2H\widehat{\tau} + \check{\tau}^{ij}\Big] + 0(h^5)\\
M^{ij} &= \frac{2}{3}h^3\Big[H\overline{\tau}^{ij} - 2\widehat{\tau}^{ij}\Big] + 0(h^5)
\end{aligned} \tag{3.4}$$

where $0(h^5)$ are terms of an order h^5. Substituting Eq (3.3) into Eq (3.4) and neglecting $0(h^5)$ we obtain

$$\begin{aligned}
N^{ij} &= \frac{2Eh}{1-\nu^2}\Big[\frac{1-\nu}{2}\Big(w^i\Big|^j + w^j\Big|^i\Big) + \Big(\nu w^k\Big|_k + H(1-\nu)w^3\Big)g^{ij}\Big]\\
M^{ij} &= -\frac{2Eh^3}{1-\nu^2}\Big[(1-\nu)\Big(\frac{1}{2}H\Big(w^i\Big|^j + w^j\Big|^i\Big) + \frac{\partial H}{\partial v^k}w^k g^{ij} + w^3\Big|^{ij}\Big) +\\
&+ \nu\Big(Hw^k\Big|_k + 2\frac{\partial H}{\partial v^k}w^k + w\Big)g^{ij}\Big]
\end{aligned} \tag{3.5}$$

where

$$w = g^{ij}w^3\Big|_{ij} \qquad w^3\Big|^{ij} = g^{ir}g^{js}w^3\Big|_{rs} \qquad w^m\Big|^n = g^{nr}w^m\Big|_r$$

Taking into account the above results, the equalibrium eqiations (1.9) can be written down in the form

$$\begin{aligned}
N^{ij}\Big|_i - HM^{ij}\Big|_i + P^j = 0\\
Hg_{ij}N^{ij} + M^{ij}\Big|_{ij} + P^3 = 0
\end{aligned} \tag{3.6}$$

For $H = 0$ Eqs (3.6) reduce then to

$$N^{ij}\Big|_i + P^j = 0 \qquad\qquad M^{ij}\Big|_{ij} + P^3 = 0$$

where

$$N^{ij} = \frac{2Eh}{1-\nu^2}\Big[\frac{1-\nu}{2}\Big(w^i\Big|^j + w^j\Big|^i\Big) + \nu w^k\Big|_k g^{ij}\Big]$$

$$M^{ij} = -\frac{2Eh^3}{3(1-\nu^2)}\Big[(1-\nu)w^3\Big|^{ij} + \nu w g^{ij}\Big]$$

In this case, for w^3 we obtain the following partial differential equation of the fourth order

$$\Big[(1-\nu)g^{ik}g^{jr} + \nu g^{ij}g^{kr}\Big]w^3\Big|_{krij} = \frac{3(1-\nu)}{2Eh^3}P^3 \tag{3.7}$$

The above equation has a form similar to that of the well known Kirchhoff plate theory. For further analysis and applications of the results derived in this contribution the reader is referred to Krawczyk (1993) and the related papers.

References

1. GREEN A.E., ZERNA W., 1954, *Theoretical Elasticity*, Oxford
2. WILLMORE T.J., 1969, *An Introduction to Differential Geometry*, Oxford
3. KRAWCZYK J., 1993, *Teoria powłok. Ujęcie symetryczne nieliniowości geometrycznej*, Wydawn. WSP Opole, Mon. 212

O równaniach równowagi pewnych typów powłok

Streszczenie

W pracy rozpatruje się parametryzację symetryczną powierzchni środkowej. Przy założeniu $H^2 \approx K$ rozpatruje się równania równowagi liniowej teorii cienkich powłok przyjmując więzy Kirchhoffa-Love'a. Ponadto, podano wyrażenia dla sil i momentów. Otrzymane wyniki mogą być zastosowane w praktyce inżynierskiej.

Manuscript received December 22, 1993; accepted for print December 22, 1993