MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
3, 32, 1994

ON 2D-MODELS OF STRATIFIED ELASTIC SUBGRADES

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2D-models of an elastic subsoil describe three dimensional problems of the elasticity theory for the subsoil layer $0 < x_3 < h$ in the averaged form which is independent of the coordinate x_3. 2D-representation of a subsoil behaviour are convenient mainly in the subsoil-structure interaction analysis which is reduced to the plane interface problem. In this paper two different 2D-models for periodically stratified subsoils are discussed and compared. The problem is studied in the framework of the linear elasticity theory and under assumption that the constituent homogeneous layers of a stratified subsoil are sufficiently thin and their number is large.

1. Introduction

Among different approaches to the structure-subgrade interaction analysis an important role play two dimensional models (2D-models) of the subgrade (cf Salvadurai (1979)). These models involve in the explicit form the subsoil response on the structure fundation footing and can be directly applied to the calculations of plates, beams and shells on elastic foundations. For homogeneous subsoils or those made of a few homogeneous layers the general approach to the formation of 2D-models was investigated by Vlasov and Leontev (1960) and in a series of related papers. Woźniak M. and Woźniak C. (1988) and Woźniak M. (1991) proposed a method of 2D-modelling for a thick elastic layer made of a large number of thin arbitrary distributed homogeneous sublayers. In this paper we are to study the formulation of 2D-models for stratified subgrades made of a large number of thin periodically distributed homogeneous layers underlain by a rigid stratum. Such situations are met in subgrades made of a glacial clays (warwed clays) which are composed of a great number of homogeneous thin silt and clay layers (cf Jumikis (1962)). A general description of the subgrade is given in Section 2. In Section 3 we consider the simplest 2D-model obtained by the direct averaging of three-dimensional equations of the linear elasticity theory. In Section 4 we apply a procedure proposed by Woźniak C. (1987) and formulate 2D-models based on the concept of effective modulae of a subsoil. In Section 5 we summarize the results and discuss some special applications. Throughout the paper subscripts i, j, k, l and α, β run over $1, 2, 3$ and $1, 2$, respectively. Superscripts A, B and a, b run over $1, ..., N$ and $1, ..., n$, respectively, unless otherwise stated. Summation convention holds for all aforementioned indices. The physical space is parameterized by the cartesian orthogonal coordinate system $0x_1x_2x_3$ and by τ we denote a time coordinate. Points situated on the coordinate plane $x_3 = 0$ are denoted by $\boldsymbol{x}$, $\boldsymbol{x} = (x_1, x_2)$.

2. Preliminaries

Let the part of a subsoil layer $0 < x_3 < h$, interacting with the structure, occupy region $\Pi \times (0, h)$ of the physical space, where Π is a regular region on the boundary plane $x_3 = 0$ and let $x_3 = h$ be a rigid basis plane. It means that the components $u_i = u_i(x_1, x_2, x_3, \tau)$ of the displacement vector for $x_3 \in (0, h)$, $\boldsymbol{x} = (x_1, x_2) \in \partial\Pi$ as well as for $x_3 = h$, $\boldsymbol{x} \in \Pi$ will be neglected. By $\sigma_{ij} = \sigma_{ij}(x_1, x_2, x_3, \tau)$, $b_i, t_i = t_i(\boldsymbol{x}, \tau)$ and $\rho = \rho(x_3)$ we denote stress components, body forces (which are assumed to be constant), tractions on a boundary plane $x_3 = 0$ and mass density (independent of x_1, x_2), respectively. The governing relations will be based on the principle of virtual work in the form

$$\int_{\Pi}\int_{0}^{h} \sigma_{ij}\delta u_{i,j}\; dx_3 da = \int_{\Pi} b_i \int_{0}^{h} \rho\delta u_i\; dx_3 da - \int_{\Pi}\int_{0}^{h} \rho\ddot{u}_i\delta u_i\; dx_3 da + \int_{\Pi} t_i\delta u_i^0\; da \tag{2.1}$$

$$da \equiv dx_1 dx_2 \qquad\qquad \delta u_i^0 = \delta u_i(x_1, x_2, 0)$$

which holds for every specified virtual displacement field $\delta u_i = \delta u_i(x_1, x_2, x_3)$, such that $\delta u_i = 0$ both for $x_3 = h$ and for $\boldsymbol{x} = (x_1, x_2) \in \partial\Pi$. The part of the subsoil bounded by planes $x_3 = 0$, $x_3 = h$, consists of a large number of layers having the constant thickness δ. We assume that these layers are thin; it means that δ/h is neglegibly small compared to 1. Moreover, every layer is assumed to have identical inhomogeneous material structure being made of $n + 1$ homogeneous sublayers of thickness $\delta_1, \delta_2, ..., \delta_{n+1}$, where

$\delta_1+...+\delta_{n+1} = \delta$. Hence, we deal with a periodically stratified subsoil, made of $n+1$ constituents, $n \geq 1$. The material of every ath sublayer, $a = 1,...,n+1$, is isotropic and linear-elastic with the elasticity tensor components C^a_{ijkl} given by

$$C^a_{ijkl} = \mu^a(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda^a\delta_{ij}\delta_{kl} \qquad a = 1,...,n+1 \tag{2.2}$$

where μ^a, λ^a are Lamè constants. Hence, the stress-strain relations in an arbitrary but fixed ath sublayer of a medium have the form

$$\begin{aligned} \sigma_{\alpha 3} &= 2C^a_{\alpha 3\beta 3}u_{(\beta,3)} \\ \sigma_{33} &= C^a_{3333}u_{3,3} + C^a_{33\alpha\beta}u_{(\alpha,\beta)} \\ \sigma_{\alpha\beta} &= C^a_{\alpha\beta\gamma\delta}u_{(\gamma,\delta)} + C^a_{\alpha\beta 33}u_{3,3} \qquad a = 1,...,n+1 \end{aligned} \tag{2.3}$$

Moreover, a constant mass density in ath sublayer will be denoted by ρ_a, $a = 1,...,n+1$.

The modelling procedures in this paper will be based on some auxiliary concepts. A continuous function F defined on $[0,h]$ which can also depend on $\boldsymbol{x} \in \Pi$ will be called the δ-*macro function* if for every $z', z'' \in [0,h]$ condition $|z'-z''| < \delta$ implies $|F(z')-F(z'')| < \lambda$, where λ is an admissible computation accuracy related to the calculations involving function F. Similarly, a function F which is continuous in $[0,h]$ and has in $(0,h)$ continuous derivatives up to kth order is said to be δ-macro function if F together with all its derivatives are δ-macro functions (with admissible computational accuracies $\lambda_1,...,\lambda_k$ related to calculations of the pertinent derivatives of F). For an arbitrary function $f = f(x_3)$ which is δ-periodic, i.e. $f(x_3) = f(x_3+\delta)$ holds for every x_3, we shall introduce denotation

$$<f> \equiv \frac{1}{\delta}\int\limits_0^\delta f(x_3)\,dx_3$$

If function $f(x_3)$ in every ath sublayer is constant and equal to f_a, $a = 1,...,n+1$, then

$$<f> = \sum_{a=1}^{n+1}\alpha_a f_a \qquad \alpha_a \equiv \frac{\delta_a}{\delta} \tag{2.4}$$

For every δ-periodic function $f = f(x_3)$ and an arbitrary continuous function $F = F(x_3)$, which is independent of the parameter δ, we obtain

$$\int\limits_0^h fF\,dx_3 = <f>\int\limits_0^h F\,dx_3 + O(\delta) \tag{2.5}$$

where $O(\delta) \to 0$ as $\delta \to 0$. Under assumption that F is a δ-macro function we replace in Eq (2.5) terms $O(\delta)$ by $O(\lambda)$. In this case terms $O(\delta)$ in formula (2.5) will be treated as sufficiently small and neglected in the course of modelling. This statement will be called the *δ-approximation assumption.*

3. Direct 2D-modelling

Direct modelling will be based on the *kinematic 2D-modelling hypothesis*: we state that the distribution of displacements across the thickness of the layer (in x_3-axis direction) can be assumed in the form

$$u_i(x_1,x_2,x_3,\tau) = \gamma^A(x_3)W_i^A(\boldsymbol{x},\tau) \qquad \boldsymbol{x} \equiv (x_1,x_2) \tag{3.1}$$

where $\gamma^A(\cdot)$ are postulated *a priori* differentiable δ-macro functions defined in $[0,h]$ and $W_i^A(\cdot)$ are arbitrary sufficiently regular unknown fields, defined for every instant τ on the plane region Π. Because for $x_3 = h$ the displacement vector components can be treated as equal to zero we shall assume that $\gamma^A(h) = 0$ for $A = 1,...,N$. The postulated *a priori* δ-macro functions $\gamma^A(\cdot)$, $A = 1,...,N$, will be called the macro-shape functions; in the simplest case we can assume $N = 1$ and $\gamma^1(x_3) = (h - x_3)/h$ (cf Vlasov and Leontev (1960)). The new unknowns $W_i^A(\cdot,\tau)$ are said to be the generalized displacement fields and are independent of x_3-coordinate. Let us observe, that under the aforementioned kinematic hypothesis, functions $u_i(x_1,x_2,\cdot,\tau)$ are assumed to be δ-macro functions and hence, using Eqs (2.1) and (2.3) we can apply the δ-approximation assumption neglecting in formulae (2.5) terms $O(\delta)$. Denoting

$$T_{ij} \equiv \langle C_{ijkl} \rangle u_{(k,l)} \qquad \langle C_{ijkl} \rangle = \sum_{a=1}^{n+1} \alpha_a C_{ijkl}^a \tag{3.2}$$

where components of C_{ijkl}^a are given by Eq (2.2), we obtain from Eq (2.1) the variational condition

$$\begin{gathered}\int_\Pi \int_0^h (T_{\alpha\beta}\delta u_{(\alpha,\beta)} + T_{\alpha 3}\delta u_{\alpha,3} + T_{3\alpha}\delta u_{3,\alpha} + T_{33}\delta u_{3,3}) \, dx_3 da = \\ = \int_\Pi \int_0^h \langle \rho \rangle (b_i - \ddot{u}_i)\delta u_i \, dx_3 da + \int_\Pi t_i \delta u_i^0 \, da\end{gathered} \tag{3.3}$$

which holds for $\delta u_i = \gamma^A \delta W_i^A$. Under denotations

$$T_{i\alpha}^A \equiv \int_0^h T_{i\alpha}\gamma^A \, dx_3 \qquad T_i^A \equiv \int_0^h T_{i3}\gamma^A{}_{,3} \, dx_3$$
$$f_i^A \equiv <\rho> \int_0^h \gamma^A(b_i - \ddot{u}_i) \, dx_3 \qquad \gamma_0^A \equiv \gamma^A(0) \tag{3.4}$$

Eq (3.3) can be written down in the 2D-form (independent of x_3-coordinate)

$$\int_\Pi (T_{i\alpha}^A \delta W_i^A{}_{,\alpha} + T_i^A \delta W_i^A - f_i^A \delta W_i^A - t_i \gamma_0^A \delta W_i^A) \, da = 0 \tag{3.5}$$

which holds for every virtual field $\delta W_i^A = \delta W_i^A(x)$, $\boldsymbol{x} \equiv (x_1, x_2) \in \Pi$. Introducing constants

$$G^{AB} \equiv \int_0^h \gamma^A \gamma^B \, dx_3 \qquad G_3^{AB} \equiv \int_0^h \gamma^A{}_{,3} \gamma^B \, dx_3$$
$$G_{33}^{AB} \equiv \int_0^h \gamma^A{}_{,3} \gamma^B{}_{,3} \, dx_3 \qquad g^A \equiv \int_0^h \gamma^A \, dx_3 \tag{3.6}$$

we obtain from Eqs (3.4) and (3.2)

$$\begin{aligned}
T_{\beta\alpha}^A(\boldsymbol{x},\tau) &= <C_{\beta\alpha\gamma\delta}> G^{AB} W_{(\gamma,\delta)}^B(\boldsymbol{x},\tau) + <C_{\beta\alpha 33}> G_3^{BA} W_3^B(\boldsymbol{x},\tau) \\
T_{3\alpha}^A(\boldsymbol{x},\tau) &= <C_{3\alpha3\beta}> \left[G_3^{BA} W_\beta^B(\boldsymbol{x},\tau) + G^{AB} W_3^B{}_{,\beta}(\boldsymbol{x},\tau)\right] \\
T_\alpha^A(\boldsymbol{x},\tau) &= <C_{3\alpha3\beta}> \left[G_{33}^{AB} W_\beta^B(\boldsymbol{x},\tau) + G_3^{AB} W_3^B{}_{,\beta}(\boldsymbol{x},\tau)\right] \\
T_3^A(\boldsymbol{x},\tau) &= <C_{3333}> G_{33}^{AB} W_3^B(\boldsymbol{x},\tau) + <C_{33\alpha\beta}> G_3^{AB} W_\alpha^B{}_{,\beta}(\boldsymbol{x},\tau)
\end{aligned} \tag{3.7}$$

and

$$f_i^A = <\rho> g^A b_i - <\rho> G^{AB} \ddot{W}^B \tag{3.8}$$

Eqs (3.7) will be referred to as the generalized constitutive equations. Let us also assume that the generalized displacements W_i^A are independent functions. Then the variational condition (3.5), after substituting Eq (3.8), yields

$$T_{i\alpha}^A{}_{,\alpha}(\boldsymbol{x},\tau) - T_i^A(\boldsymbol{x},\tau) + <\rho> g^A b_i + t_i(\boldsymbol{x},\tau)\gamma_0^A = <\rho> G^{AB} \ddot{W}_i^B(\boldsymbol{x},\tau) \tag{3.9}$$

Eqs (3.9) will be called the generalized equations of motion; together with the generalized constitutive equations (3.7) they represent the 2D-model of a stratified subsoil under consideration since all fields that enter Eqs (3.7), (3.9) are independent of x_3-coordinate. Substituting the right-hand sides from Eqs (3.7) into Eqs (3.9) we obtain the system of $3N$ linear partial differential equations in $3N$ unknown generalized displacements $W_i^A(x,\tau)$, $\boldsymbol{x} \equiv (x_1,x_2) \in \Pi$, $\tau \in [\tau_0,\tau_f]$.

The main drawback of the direct 2D-modelling method described in this section lies in the kinematic hypothesis (3.1) in which $\gamma^A(\cdot)$ are differentiable δ-macro functions. It follows that also displacement gradients $u_{i,3}$ are δ-macro functions. On the other hand these gradients suffer jump discontinuities across the interfaces between adjacent homogeneous sublayers of the stratified medium. This fact gives a motivation for introducing an alternative 2D-model which will be proposed in the subsequent section.

4. Effective 2D-modelling

Effective modelling will be based on the approach to elastic media with periodic material structure which was proposed by Woźniak C. (1987) and analysed in a series of papers. The idea of this approach is based on two assumptions. First, we postulate the *micro-macro kinematic hypothesis* by imposing on the displacement fields in a subsoil the constraints of the form

$$u_i(x_1,x_2,x_3,\tau) = U_i(x_1,x_2,x_3,\tau) + h_a(x_3)Q_i^a(x_1,x_2,x_3,\tau) \tag{4.1}$$

where U_i, Q_i^a are independent regular δ-macro functions and $h_a(\cdot)$ are continuous δ-periodic functions defined in $[0,\delta]$ by means of

$$h_a(x_3) = \begin{cases} 0 & \text{if} \quad x_3 \in [0,z_{a-1}] \cup [z_{a+1},\delta] \\ (x_3 - z_{a-1})/\alpha_a & \text{if} \quad x_3 \in [z_{a-1},z_a] \\ (z_{a+1} - x_3)/\alpha_{a+1} & \text{if} \quad x_3 \in [z_a,z_{a+1}], \qquad a = 1,...,n \end{cases} \tag{4.2}$$

where $x_3 = z_a$ is the interface between ath and $(a+1)$th sublayers and where we have denoted $z_0 \equiv 0$, $z_{n+1} \equiv \delta$. Since of $h_a(x_3) = h_a(x_3 \mp n\delta)$, $n = 1,2,...$, functions $h_a(\cdot)$ defined by Eq (4.2) in $[0,\delta]$ are uniquely defined for every $x_3 \in \mathcal{R}$ and are called the *micro-shape functions*. At the same time fields U_i and Q_i^a are said to be macro-displacements and correctors (or microlocal parameters), respectively. The physical sense of the kinematic assumption (4.1) is evident; terms $h_aQ_i^a$ describe the disturbances of displacements due to the inhomogeneous periodic structure of the subsoil. From Eqs (4.1), (4.2) it follows that

$$
\begin{aligned}
u_i &= U_i + O(\delta) \\
u_{i,j} &= U_{i,j} + h_{a,3} Q_i^a \delta_{3j} + O(\delta)
\end{aligned}
\tag{4.3}
$$

The second assumption of the proposed method is that in the course of modelling terms $O(\delta)$ in Eq (4.3) can be neglected as sufficiently small ones; this assumption has a physical sense for subsoils made of a large number of thin homogeneous sublayers and will be called the *micro-inhomogeneity approximation.*

Substituting Eq (2.3) into the principle of virtual work (2.1), using the kinematic hypothesis (4.1) together with the micro-inhomogeneity approximation, and applying formula (2.5), after simple reculations we arrive at the principle of virtual work in the form

$$
\begin{aligned}
&\int\limits_{\Pi}\int\limits_{0}^{h} (< C_{ijkl} > U_{k,l} + < C_{ijk3} h_{a,3} > Q_k^a) \delta U_{i,j} \; dx_3 da = \\
&= \int\limits_{\Pi} b_i < \rho > \int\limits_{0}^{h} \delta U_i \; dx_3 da - \int\limits_{\Pi} < \rho > \int\limits_{0}^{h} \ddot{U}_i \delta U_i \; dx_3 da + \int\limits_{\Pi} t_i \delta U_i \Big|_{x_3=0} da
\end{aligned}
\tag{4.4}
$$

and we obtain the following variational condition for correctors

$$
\int\limits_{\Pi}\int\limits_{0}^{h} (< C_{3jkl} h_{a,3} > U_{k,l} + < C_{j3k3} h_{a,3} h_{b,3} > Q_k^b) \delta Q_j^a \; dx_3 da = 0 \tag{4.5}
$$

Under assumption that δQ_j^a are arbitrary independent functions, Eqs (4.5) imply the system of linear algebraic equations for correctors

$$
< C_{j3k3} h_{a,3} h_{b,3} > Q_k^b = - < C_{3jkl} h_{a,3} > U_{k,l} \tag{4.6}
$$

The general solution of Eqs (4.6) can be written down in the form

$$
Q_k^a = - K_{kj}^{ab} < C_{3jkl} h_{b,3} > U_{k,l} \tag{4.7}
$$

where K_{ij}^{ab} are determined by

$$
< C_{j3k3} h_{a,3} h_{b,3} > K_{kl}^{bc} = \delta_a^c \delta_{jl} \tag{4.8}
$$

Now, substituting Eq (4.7) into Eq (4.4), denoting

$$E_{ijkl} \equiv <C_{ijkl}> - <C_{ijm3}h_{a,3}> K_{mn}^{ab} <C_{3nkl}h_{b,3}> \tag{4.9}$$

and

$$S_{ij} \equiv E_{ijkl}U_{(k,l)} \tag{4.10}$$

we obtain the following form of the principle of virtual work

$$\begin{aligned}\int_{\Pi}\int_0^h S_{ij}\delta U_{i,j}\ dx_3 da &= \int_{\Pi} b_i <\rho> \int_0^h \delta U_i\ dx_3 da + \\ &- \int_{\Pi} <\rho> \int_0^h \ddot{U}_i\delta U_i\ dx_3 da + \int_{\Pi} t_i\delta U_i^0\ da \\ \delta U_i^0 &\equiv \delta U_i(x_1,x_2,0)\end{aligned} \tag{4.11}$$

The proposed 2D-modelling approach will be based on the *kinematic 2D-modelling hypothesis* assumed in the form

$$U_i(x_1,x_2,x_3,\tau) = \gamma^A(x_3)W_i^A(\boldsymbol{x},\tau) \qquad \boldsymbol{x} \equiv (x_1,x_2) \tag{4.12}$$

which from a formal point of view is similar to that appearing in Eq (3.1) but is related now to the macro-displacements U_i which are δ-macro functions. Hence, the drawback of direct 2D-modelling, discussed at the end of Section 3 has been removed from the proposed approach. Using Eqs (4.11), (4.12) and following the line of modelling applied in Section 3, we obtain the generalized equations of motion in the form

$$S_{i\alpha,\alpha}^A(\boldsymbol{x},\tau) - S_i^A(\boldsymbol{x},\tau) + <\rho> g^A b_i + t_i(\boldsymbol{x},\tau)\gamma_0^A = <\rho> G^{AB}\ddot{W}_i^B(\boldsymbol{x},\tau) \tag{4.13}$$

where we have denoted

$$S_{i\alpha}^A \equiv \int_0^h S_{i\alpha}\gamma^A\ dx_3 \qquad\qquad S_i^A \equiv \int_0^h S_{i3}\gamma^A{}_{,3}\ dx_3$$

The aforementioned definitions combined with Eqs (4.10) and with the kinematic hypothesis (4.12), under denotations (3.6) lead to the following generalized constitutive equations

$$
\begin{aligned}
S^A_{\beta\alpha}(\mathbf{x},\tau) &= E_{\beta\alpha\gamma\delta}G^{AB}W^B_{(\gamma},_{\delta)}(\mathbf{x},\tau) + E_{\beta\alpha33}G^{BA}_3W^B_3(\mathbf{x},\tau)\\
S^A_{3\alpha}(\mathbf{x},\tau) &= E_{3\alpha3\beta}\Big[G^{BA}_3W^B_\beta(\mathbf{x},\tau) + G^{AB}W^B_3,_\beta(\mathbf{x},\tau)\Big]\\
S^A_\alpha(\mathbf{x},\tau) &= E_{3\alpha3\beta}\Big[G^{AB}_{33}W^B_\beta(\mathbf{x},\tau) + G^{AB}_3W^B_3,_\beta(\mathbf{x},\tau)\Big]\\
S^A_3(\mathbf{x},\tau) &= E_{3333}G^{AB}_{33}W^B_3(\mathbf{x},\tau) + E_{33\alpha\beta}G^{AB}_3W^B_{(\alpha},_{\beta)}(\mathbf{x},\tau)
\end{aligned}
\tag{4.14}
$$

The constant coefficients E_{ijkl} defined by Eqs (4.9) are called the effective elastic modulae. That is why the proposed approach will be referred to as the effective 2D-modelling approach. Eqs (4.13), (4.14) involve only fields independent of x_3-coordinate and hence constitute 2D-model of a stratified subsoil under consideration. This model leads to the system of $3N$ linear partial differential equations in $3N$ unknown generalized displacements $W^A_i(\mathbf{x},\tau)$, $\mathbf{x} = (x_1,x_2) \in \Pi$, $\tau \in [\tau_0,\tau_f]$. It has to be emphasized that the 2D-model described by Eqs (4.13), (4.14) is free of physical inconsistency discussed at the end of Section 3 taking into account the jumps of displacement gradients $u_i,_3$ across the interfaces of the stratified medium. This result was obtained by using the micro-macro kinematic hypothesis (4.1) leading to formulae (4.3) with discontinuities of $u_i,_3$ across interfaces.

5. Conclusions

It can be seen that the 2D-model obtained in Section 4 has a mathematical form similar to that derived in Section 3. The difference lies in the generalized constitutive equations; in the direct approach we deal with the averaged modulae $< C_{ijkl} >$ while in the effective approach we have to introduce the effective modulae E_{ijkl}. The interrelation between these modulae is given by Eqs (4.9) and (4.8) which can be also written down in the explicit form

$$
E_{ijkl} = < C_{ijkl} > - \sum_{a=1}^{n}\sum_{b=1}^{n}(C^a_{ijm3} - C^{a+1}_{ijm3})(C^b_{3nkl} - C^{b+1}_{3nkl})K^{ab}_{mn} \tag{5.1}
$$

where (no summation over a!)

$$
-\frac{1}{\alpha_a}C^a_{j3k3}K^{a-1,c}_{kl} + \Big(\frac{1}{\alpha_a}C^a_{j3k3} + \frac{1}{\alpha_{a+1}}C^{a+1}_{j3k3}\Big)K^{ac}_{kl} - \frac{1}{\alpha_{a+1}}C^{a+1}_{j3k3}K^{a+1,c}_{kl} = \delta^{ac}\delta_{jl} \tag{5.2}
$$

and where we have assumed $K^{ac}_{kl} \equiv 0$ if $a = 0$ or $a = n + 1$. Hence, we conclude that 2D-models proposed in Section 4 can be used if $C^a_{ijm3} \cong C^{a+1}_{ijm3}$ for $a = 1, ..., n$. The components C^a_{ijkl} of the elasticity tensor in ath sublayer are given by Eqs (2.2) since we have assumed that every sublayer is made of an isotropic material. Formulae (5.1), (5.2) are the basis for calculations of the effective modulae and constitute a special case of more general approach proposed by Woźniak C. and Woźniak M. (1993). If $n = 1$, i.e. every periodic layer is made of two isotropic homogeneous sublayers, then from Eqs (5.1), (5.2) we obtain the known result (no summation over i!)

$$E_{i3i3} = \Big(\frac{\alpha_1}{C^1_{i3i3}} + \frac{\alpha_2}{C^2_{i3i3}}\Big)^{-1} \qquad\qquad E_{ij\alpha\beta} = \alpha_1 C^1_{ij\alpha\beta} + \alpha_2 C^2_{ij\alpha\beta}$$

in which the elasticity tensor components C^a_{ijkl}, $a = 1, 2$, are determined by Eqs (2.2).

In most cases we introduce in Eqs (3.1) and (4.12) one macro-shape function and hence, one generalized displacement vector field $W^1_i(x, \tau)$. In this case superscripts A, B take the value 1 (since $N = 1$). Let us also assume that $\gamma^1_0 \equiv \gamma^1(0) = 1$, $b_\alpha = 0$ and denote $G \equiv G^{11}$, $G_3 \equiv G^{11}_3$, $G_{33} \equiv G^{11}_{33}$. Introducing new unknown $W_3(x, \tau) \equiv W^1_3(x, \tau) - <\rho> g b_3 (G_{33} E_{3333})^{-1}$ and setting $W_\alpha(x, \tau) \equiv W^1_\alpha(x, \tau)$, we obtain from Eqs (4.13), (4.14) the following system of governing equations for W_i

$$G E_{i\alpha j\beta} W_{j,\alpha\beta} + G_3 (E_{i\beta j3} - E_{i3j\beta}) W_{j,\beta} - G_{33} E_{3i3j} W_j + t_i = <\rho> G \ddot{W}_i \tag{5.3}$$

where components E_{ijkl} labelled odd number of subscripts "3" are equal to zero. Let us observe that under aforementioned assumptions the generalized displacements W_i have a simple interpretation coinciding with the displacements of the boundary plane $x_3 = 0$, provided that in formulae (4.3) terms $O(\delta)$ are neglected. Hence, Eqs (5.3) represent the interrelation between displacements $W_i(x, \tau)$ of the boundary plane and loadings $t_i(x, \tau)$ acting on this plane. Introducing into Eqs (5.3) the response $r_i \equiv -t_i$ of the subsoil we shall describe the structure-subgrade interactions which take into account the periodically stratified character of the subsoil. For vertical r_3 and horizontal r_α components of the subgrade response, respectively, we obtain formulae

$$\begin{aligned} r_3 &= G E_{3\alpha 3\beta} W_{3,\alpha\beta} + G_3 (E_{3\alpha 3\beta} - E_{33\alpha\beta}) W_{\alpha,\beta} - G_{33} E_{3333} W_3 - <\rho> G \ddot{W}_3 \\ r_\alpha &= G E_{\alpha\beta\gamma\delta} W_{\gamma,\beta\delta} + G_3 (E_{\alpha\beta 33} - E_{3\alpha 3\beta}) W_{3,\beta} - G_{33} E_{3\alpha 3\beta} W_\beta - <\rho> G \ddot{W}_\alpha \end{aligned} \tag{5.4}$$

In many special problems horizontal components W_α of the boundary displacements can be neglected and hence the subsoil response given by Eqs (5.4) reduces to the form

$$r_3 = GE_{3\alpha 3\beta}W_{3,\alpha\beta} - G_{33}E_{3333}W_3 - <\rho> \ddot{W}_3 \tag{5.5}$$

For subgrades made of two periodically distributed isotropic sublayers formula (5.5) yields

$$r_3 = G\Big(\frac{\alpha_1}{\mu_1}+\frac{\alpha_2}{\mu_2}\Big)^{-1}W_{3,\alpha\alpha} - G_{33}\Big(\frac{\alpha_1}{\lambda_1+2\mu_1}+\frac{\alpha_2}{\lambda_2+2\mu_2}\Big)^{-1}W_3 - (\alpha_1\rho_1+\alpha_2\rho_2)\ddot{W}_3$$

and describes in the explicit form the effect of periodic inhomogeneity of the stratified subgrade on its response. It has to be remembered that the resulting relations have a physical sense only if the thickness δ of a single representative layer is sufficiently small compared to the thickness h of the part of a subgrade interacting with the structure.

The applications of effective 2D-modelling of stratified subsoils to the subgrade-structure interaction analysis, which were restricted here to the case described by Eqs (5.4), can be also extended on situations in which we have to deal with a few macro-shape functions γ^A. More detailed treatment of this problem including numerical calculations will be subject to separate investigations.

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Dwuwymiarowe modele uwarstwionego podłoża sprężystego

Streszczenie

Dwuwymiarowe modele podłoża sprężystego opisują trójwymiarowe zagadnienia teorii sprężystości dla warstwy podłoża $0 < x_3 < h$ w sposób przybliżony, niezależny od współrzędnej x_3. Dwuwymiarowe reprezentacje zachowania podłoża są wygodne, szczególnie w analizie oddziaływań podłoże-struktura, zredukowanej do płaskiego zagadnienia kontaktu. W pracy omówiono i porównano dwa różne modele dwuwymiarowe periodycznie uwarstwionego podłoża. Problem rozważany jest w zakresie liniowej teorii sprężystości, przy założeniu, że jednorodne warstwy składowe uwarstwionego podłoża są dostatecznie cienkie i liczba ich jest duża.

Manuscript received November 17, 1993; accepted for print January 6, 1994