MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
3, 32, 1994

ASYMPTOTIC-BASED METHOD FOR A PLANE ELASTICITY MIXED BOUNDARY EIGENVALUE PROBLEM

I.V. Andrianov

V. Chernetsky

Department of Higher Mathematics

Dnepropetrovsk Institute of Civil Engineering

E.G. Kholod

Department of Higher Mathematics

Dnepropetrovsk Metallurgical Institute

Investigation of plane plates under mixed boundary conditions is of significant practical value: a lot of problems, arisen in machine design, civil engineering etc., are reduced to similar ones. The problems mentioned are usually solved using numerical methods such as the finite element procedure. Nevertheless, numerical approach fails to meet adequately the requirements of optimal structural design methodology. The approximate analytical relation, accurate enough, will be of great practical importance for these needs. The effective analytical approach, combining boundary conditions, perturbation technique and the Padé approximates (PA) of perturbation series, are presented in this paper.
Various problems of dynamics and statics of plates, under mixed boundary conditions, may be solved effectively on the basis of the approach presented.

1. Introduction

The basic idea of presented method may be described as follows. Parameter ε is introduced into boundary conditions in such a way that $\varepsilon = 0$ case corresponds to the common problem under consideration (cf Dorodnitsyn (1969), Andrianov (1991), Andrianov and Ivankov (1992a,b), (1993)). Then the ε-expansion of the solution is obtained. As a rule, just at the point $\varepsilon = 1$

the expansion of solution is obtained. The PA may be used to eliminate that drawback.

In connection with the procedur used below let us discuss the following important question.

Every scientist now must have repeatedly asked himself a question: are the asymptotic methods of any practical use at all having computers? Perhaps it would be simpler to write numerical program of an original problem and to find its solution with the help at universal numerical procedures.

The answer may be like this: first to apply the asymptotic methods at the preliminary stage of solving a problem even in cases where the principal aim is to obtain the numerical results. The asymptotic analysis makes it possible to choose the best numerical method. Secondly the asymptotic methods are especially effective in those regions of parameter values where machine computations are faced with serious difficulties. Laplace used to say, not without reason, that the asymptotic methods are "the more accurate, the more they are needed". Moreover, the possibility exist of developing such algorithms wherein smooth solutions are obtained numerically, and the asymptotic approaches are applied to that parameter value regions where these solutions change drastically, say, within boundary layers. Therefore, it would be more properly to consider the asymptotic and numerical methods not as competitive, but as mutually complementary ones.

Notations

$u(v)$ – displacement in the $x(y)$ direction
$\rho = \rho_1/G$, ρ_1 – plate density, G – shear modulus
ω – frequency
$c = (1+\nu)/(1-\nu)$, ν – Poisson ratio
$k(l)$ – number of parts of the mixed boundary conditions along sides $x = \pm a/2$ $(y = \pm b/2)$
$H(x)$ – Heaviside function

$$H_1 = \sum_{i=1}^{k} H(x - a_{1i}) - H(x - a_{2i})$$
$$H_2 = \sum_{i=1}^{k} H(x - a'_{1i}) - H(x - a'_{2i})$$

$$H_3 = \sum_{i=1}^{l} H(x - b_{1i}) - H(x - b_{2i})$$
$$H_4 = \sum_{i=1}^{l} H(x - b'_{1i}) - H(x - b'_{1i})$$

2. Boundary conditions perturbation technique

Let us consider the application of the approach presented to the free oscillations analysis of rectangular plate $(-0.5a \le x \le 0.5a\,; -0.5b \le y \le 0.5b)$. Mixed boundary conditions along the sides (see Fig.1) are imposed on the plate.

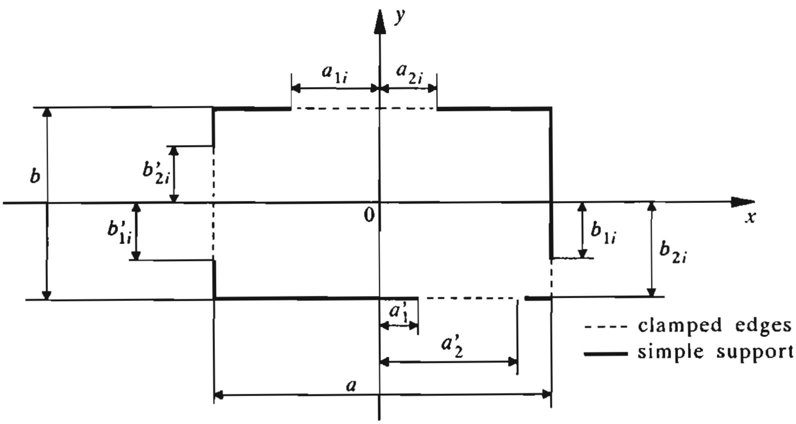

Fig. 1.

Governing differential equations may be written as follows

$$
\begin{aligned}
&(1+c)\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + c\frac{\partial^2 v}{\partial x \partial y} + \rho\omega^2 u = 0 \\
&(1+c)\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + c\frac{\partial^2 u}{\partial x \partial y} + \rho\omega^2 v = 0
\end{aligned}
\tag{2.1}
$$

Boundary conditions are written as follows

$$\begin{array}{llllll} u=0 & H_1 v=0 & (1-H_1)S=0 & \text{for} & x=a/2 \\ u=0 & H_2 v=0 & (1-H_2)S=0 & \text{for} & x=-a/2 \\ v=0 & H_3 u=0 & (1-H_3)S=0 & \text{for} & y=b/2 \\ v=0 & H_4 u=0 & (1-H_4)S=0 & \text{for} & y=-b/2 \end{array} \tag{2.2}$$

where

$$S = G\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big)$$

Introducing the parameter ε into the boundary conditions according to the before mentioned procedure one obtains

$$\begin{array}{llll} u=0 & \varepsilon[GH_1 v/a + (1-H_1)S] + (1-\varepsilon)S = 0 & \text{for} & x=a/2 \\ u=0 & \varepsilon[OH_2 v/a + (1-H_2)S] - (1-\varepsilon)S = 0 & \text{for} & x=-a/2 \\ v=0 & \varepsilon[OH_3 u/b + (1-H_3)S] + (1-\varepsilon)S = 0 & \text{for} & y=b/2 \\ v=0 & \varepsilon[GH_4 u/b + (1-H_4)S] - (1-\varepsilon)S = 0 & \text{for} & y=-b/2 \end{array} \tag{2.3}$$

The case $\varepsilon = 0$ corresponds to the plate, simply supported along the boundary; the case $\varepsilon = 1$ corresponds to the problem under consideration of Eqs (2.1) and (2.2).

Let us apply the perturbation technique to the Eq (2.1) and the boundary conditions (2.2), represented as the ε-expansions

$$\begin{aligned} u &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots \\ v &= v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \dots \\ \omega^2 &= \omega_0^2 + \varepsilon \omega_1^2 + \varepsilon^2 \omega_2^2 + \dots \end{aligned} \tag{2.4}$$

Substituting Eqs (2.3) and (2.4) into Eq (2.1) and boundary conditions (2.3) and splitting it into groups according to the powers of ε, one obtains the recurrence formula of eigenvalue problems

$$\begin{aligned} (1+c)\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + c\frac{\partial^2 v_0}{\partial x \partial y} + \rho\omega_0^2 u_0 = 0 \\ (1+c)\frac{\partial^2 v_0}{\partial y^2} + \frac{\partial^2 v_0}{\partial x^2} + c\frac{\partial^2 u_0}{\partial x \partial y} + \rho\omega_0^2 v_0 = 0 \end{aligned} \tag{2.5}$$

$$\begin{array}{llll} u_0 = 0 & \dfrac{\partial v_0}{\partial x} = 0 & \text{for} & x = \pm\dfrac{a}{2} \\ v_0 = 0 & \dfrac{\partial u_0}{\partial x} = 0 & \text{for} & y = \pm\dfrac{b}{2} \end{array} \tag{2.6}$$

$$
\begin{aligned}
&(1+c)\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + c\frac{\partial^2 v_1}{\partial x \partial y} + \rho\omega_0^2 u_1 = -\rho\omega_1^2 u_0\\
&(1+c)\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial x^2} + c\frac{\partial^2 u_1}{\partial x \partial y} + \rho\omega_0^2 v_1 = -\rho\omega_1^2 v_0
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
&u_1 = 0 \qquad \frac{\partial v_1}{\partial x} = -H_1\frac{v_0}{a} \qquad \text{for} \quad x = \frac{a}{2}\\
&u_1 = 0 \qquad \frac{\partial v_1}{\partial x} = H_2\frac{v_0}{a} \qquad \text{for} \quad x = -\frac{a}{2}\\
&v_1 = 0 \qquad \frac{\partial u_1}{\partial y} = -H_3\frac{u_0}{b} \qquad \text{for} \quad y = \frac{b}{2}\\
&v_1 = 0 \qquad \frac{\partial u_1}{\partial y} = H_4\frac{u_0}{b} \qquad \text{for} \quad y = -\frac{b}{2}
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
&(1+c)\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} + c\frac{\partial^2 v_2}{\partial x \partial y} + \rho\omega_0^2 u_2 = -\rho\omega_2^2 u_0 - \rho\omega_1^2 u_1\\
&(1+c)\frac{\partial^2 v_2}{\partial y^2} + \frac{\partial^2 v_2}{\partial x^2} + c\frac{\partial^2 u_2}{\partial x \partial y} + \rho\omega_0^2 v_2 = -\rho\omega_2^2 v_0 - \rho\omega_1^2 v_1
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
&u_2 = 0 \qquad \frac{\partial v_2}{\partial x} = -H_1\frac{v_0 + v_1}{a} \qquad \text{for} \quad x = \frac{a}{2}\\
&u_2 = 0 \qquad \frac{\partial v_2}{\partial x} = H_2\frac{v_0 + v_1}{a} \qquad \text{for} \quad x = -\frac{a}{2}\\
&v_2 = 0 \qquad \frac{\partial u_2}{\partial y} = -H_3\frac{u_0 + u_1}{b} \qquad \text{for} \quad y = \frac{b}{2}\\
&v_2 = 0 \qquad \frac{\partial u_2}{\partial y} = H_4\frac{u_0 + u_1}{b} \qquad \text{for} \quad y = -\frac{b}{2}
\end{aligned}
\tag{2.10}
$$

Solution of the nonperturbated eigenvalue problem (2.5) and (2.6) may be written in the form

$$
\begin{aligned}
u_0 &= A\sin\frac{2m\pi x}{a}\cos\frac{2n\pi y}{b}\\
v_0 &= B\cos\frac{2m\pi x}{a}\sin\frac{2n\pi y}{b}
\end{aligned}
\tag{2.11}
$$

$$\rho\omega_{0(1)}^2 = 4\pi^2\Big(\frac{m^2}{a^2}+\frac{n^2}{b^2}\Big) \qquad B_{(1)} = -A\frac{mb}{na}$$
$$\rho\omega_{0(2)}^2 = \rho\omega_{0(1)}^2(1+c) \qquad B_{(2)} = -A\frac{na}{mb} \tag{2.12}$$

The perturbation procedure (cf Nayfeh (1986)) leads to the following solutions of the systems (2.7) ÷ (2.10)

$$u_1 = A\frac{(-1)^n}{b^2}\Big[H_4\Big(\frac{by}{2}-\frac{y^2}{2}\Big) - H_3\Big(\frac{y^2}{2}+\frac{by}{2}\Big)\Big]\sin\frac{2n\pi x}{a}$$
$$v_1 = B\frac{(-1)^m}{a^2}\Big[H_2\Big(\frac{ax}{2}-\frac{x^2}{2}\Big) - H_1\Big(\frac{x^2}{2}+\frac{ax}{2}\Big)\sin\frac{2m\pi y}{b} \tag{2.13}$$

$$\rho\omega_1^2 = 2\Bigg\{\frac{n^2a}{b^2}\sum_{i=1}^{k}\Big[a_{2i}-a_{1i}-\frac{a}{4m\pi}\Big(\sin\frac{4m\pi a_{2i}}{a}-\sin\frac{4m\pi a_{1i}}{a}\Big)\Big]+$$
$$+\frac{m^2b}{a^2}\sum_{i=1}^{l}\Big[b_{2i}-b_{1i}-\frac{b}{4n\pi}\Big(\sin\frac{4n\pi b_{2i}}{b}-\sin\frac{4n\pi b_{1i}}{b}\Big)\Big]\Bigg\}\frac{1}{n^2a^2+m^2b^2} \tag{2.14}$$

$$\rho\omega_2^2 = \Bigg\{\frac{m^2b^2}{a}\Big\{5\sum_{i=1}^{l}\Big[\frac{b_{2i}-b_{1i}}{2}-\frac{b}{8n\pi}\Big(\sin\frac{4n\pi b_{2i}}{b}-\sin\frac{4n\pi b_{1i}}{b}\Big)\Big]+$$
$$+\sum_{i=1}^{l}\Big[b'_{2i}-b'_{1i}-\frac{b}{4n\pi}\Big(\sin\frac{4n\pi b'_{2i}}{b}-\sin\frac{4n\pi b'_{1i}}{b}\Big)\Big]\Big\}+$$
$$+\frac{n^2a^2}{b}\Big\{5\sum_{i=1}^{k}\Big[\frac{a_{2i}-a_{1i}}{2}-\frac{a}{8m\pi}\Big(\sin\frac{4m\pi a_{2i}}{a}-\sin\frac{4m\pi a_{1i}}{a}\Big)\Big]+\sum_{i=1}^{k}\Big[a'_{2i}+$$
$$-a'_{1i}-\frac{a}{4m\pi}\Big(\sin\frac{4m\pi a'_{2i}}{a}-\sin\frac{4m\pi a'_{1i}}{a}\Big)\Big]\Big\}\Bigg\}\frac{1}{2ab(n^2a^2+m^2b^2)}+ \tag{2.15}$$
$$-\rho\omega_1^2\Bigg\{1-\Big\{n^2a^2b\sum_{i=1}^{k}\Big[a_{2i}-a_{1i}-\frac{a}{4m\pi}\Big(\sin\frac{4m\pi a_{2i}}{a}-\sin\frac{4m\pi a_{1i}}{a}\Big)\Big]+$$
$$-m^2b^2a\sum_{i=1}^{l}\Big[b_{2i}-b_{1i}-\frac{b}{4n\pi}\Big(\sin\frac{4n\pi b_{2i}}{b}-\sin\frac{4n\pi b_{1i}}{b}\Big)\Big]\Big\}\cdot$$
$$\cdot\frac{1}{ab(n^2a^2+m^2b^2)\pi^2}\Bigg\}$$

where $a'_{1(2)i}$, $b'_{1(2)i}$ – coordinates of end points for clamped areas, whose coordinates coincide on opposite sides.

The truncated series (2.4) in our case may be represented in the following form

$$\omega^2 = \omega_0^2 + \varepsilon\omega_1^2 + \varepsilon^2\omega_2^2 \tag{2.16}$$

For the error estimation we investigated the problem, which has the exact solution: vibration of a square plate with two opposite sides simply supported and another two clamped.

The numerical results for the frequency $\rho\omega^2$ are plotted in Table 1.

Table 1.

a/b	Formula (2.16)	Error [%]	Formula (3.4)	Error [%]	Exact solution
0.5	220.951	15.8	277.650	5.8	262.459
1	82.625	4.7	91.036	4.7	86.726
1.5	58.029	6.6	60.379	2.8	62.152
2	49.716	4.0	50.602	2.2	51.791

The formulae (2.16) represent poor approximation of the true value of ω, and we can use the Padé approximates technique to eliminate this drawback.

3. Application of the Padé-approximates

Let us formulate the PA definition (cf Baker abd Graves-Morris (1990)). For expansion given by

$$F(\varepsilon) = \sum_{i=0}^{\infty} c_i\varepsilon^i \tag{3.1}$$

the fractional-rational function $F(\varepsilon)[m/n]$

$$F(\varepsilon)[m/n] = \frac{\sum\limits_{i=0}^{m} a_i\varepsilon^i}{\sum\limits_{i=0}^{n} b_i\varepsilon^i} \tag{3.2}$$

represents PA of Eq (3.1) as well as the power series (2.16), if the McLoran series of the formula for $F(\varepsilon)$ shows the coincidence of its coefficients with the corresponding ones of Eq (2.16) up to the terms of the $(m+n+1)$th order. The advantages of PA are: for m and n chosen it is uniquely defined; it creates meromorphic continuation of a function; besides, the expansion (2.16) can be treated as a linear algebraic problem (cf Baker abd Graves-Morris (1990)).

We have in our case for the truncated series (3.2)

$$N_{[1/1]}(\varepsilon) = \frac{\omega_0^2(\omega_1^2 - \varepsilon\omega_2^2) + \varepsilon\omega_1^4}{\omega_1^2 - \varepsilon\omega_2^2} \tag{3.3}$$

For $\varepsilon = 1$ one obtains

$$\omega^2 = \frac{\omega_0^2(\omega_1^2 - \omega_2^2) + \omega_1^4}{\omega_1^2 - \omega_2^2} \tag{3.4}$$

Numerical results for the formula (3.4) for the above mentioned problem are plotted in Table 1.

The discrepancy of frequencies does not exceed 5%, which confirms acceptable accuracy of the method presented.

4. Conclusions

In conclusion let us compare numerical and asymptotic procedures on the example of R-function method (RFM), which was proposed by Professor V.L.Rvachov, and the method of boundary conditions form perturbation (MBCFP) – asymptotic method, which was proposed by the authors of present report.

The aforementioned methods have been used for solving the boundary value problems for plates with mixed boundary conditions.

Merits (M) and demerits (D) of this approaches may be formulated as follows:

MBCFP M – Analytical solutions for any numbers of mixed boundary conditions may be obtained.

Approximate analytical formula, accurate enough, will be of great practical advantage for optimal structural design needs.

MBCFP D – It is difficult to use this method for the areas of complex form and to formulate further approximation in terms of the asymptotic series.

RFM M – Generality with respect to the plate form.

RFM D – Numerical approach fails to meet adequately the requirements of optimal structural design methodology.

The most adequate way consist in using both the RFM and the MBCFP simultaneously.

References

1. ANDRIANOV I.V., 1991, Application of Padé-Approximates in Perturbation Methods, *Advances in Mechanics*, **14**, 2, 3-25
2. ANDRIANOV I.V., IVANKOV A.O., 1992a, New Asymptotic Method for Solving of Mixed Boundary Value Problem, *International Series of Numerical Mathematics*, **106**, 39-45
3. ANDRIANOV I.V., IVANKOV A.O., 1992b, Solution of Stability Problem for Rectangular Plates with Mixed Boundary Conditions on the Basis on New Asymptotic Method, The 5th Conference SHELL STRUCTURES: Theory and Applications, Warsaw, 43-46
4. ANDRIANOV I.V., IVANKOV A.O., On the Solution of Plate Bending Mixed Problems Using Modified Technique of Boundary Conditions Perturbation, *ZAMM*, **73**, 2, 120-122
5. BAKER G.A., GRAVES-MORRIS P.R., 1990, *Padé Approximates*, Addison-Wesley Publ. Company, New York
6. DORODNITSYN A.A., 1969, Numerical Method of Solution of Elastic Media Problems of Mechanics, (in Russian), 85
7. NAYFEH A.H., 1986, *Introduction to perturbation Techniques*, John Wiley & Sons, New York

Metoda asymptotyczna rozwiązania problemu własnego z mieszanymi warunkami brzegowymi dla płaskich zagadnień sprężystych

Streszczenie

Badanie plyt z mieszanymi warunkami brzegowymi ma duże znaczenie praktyczne: duża liczba problemów pojawiająca się przy projektowaniu maszyn, w inżynierii budowlanej itp. redukuje się do omawianego zagadnienia. Na ogól problemy te rozwiązywane są metodami numerycznymi, takimi jak metoda elementów skończonych. Tym niemniej numeryczne przybliżenia są niewystarczające do metodologii optymalnego projektowania konstrukcji. Duże znaczenie praktyczne mają więc dostatecznie dokladne przybliżone związki analityczne. W tej pracy prezentowane są efektywne przybliżenia analityczne z uwzględnieniem warunków brzegowych, perturbacji i aproksymacji Padé szeregów perturbacyjnych, umożliwiające efektywne rozwiązanie różnych zagadnień dynamiki i statyki plyt.

Manuscript received April 13, 1993; accepted for print April 14, 1994