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THE STOCHASTIC SIMULATION OF A VISCOUS FLUID FLOW PAST AN AIRFOIL – PART I

JANUSZ BŁAŻEWICZ

ANDRZEJ STYCZEK

Warsaw University of Technology

The viscous incompressible fluid flow past a given airfoil is considered. The vorticity field is noticed as random process. It is formed by the large number of randomicaly walking and advecting small vortices. The vortices are created on the boundary of the airfoil. The boundary integral equation determines their intensity. The asymptotic consistency of velocity field to the potential field allows to control the stability of process

1. Introduction

The problem of determination of a turbulent motion is one of the most difficult task in fluid dynamics. The understanding of the essence of this motion has not been complete so far – the way in which relatively well-ordered macro-structures of such flow results from the micro-structural stochasticity is still unknown. Also the mechanism of the appearance of almost periodical phenomena at moderate and, possibly, very high Reynolds numbers has not yet been explained with satisfaction.

Classical formulations of the problem of viscous fluid flows are difficult to solve and always need some severe simplifications and restrictions concerning the geometry and flow parameters.

All these circumstances imply that there are no universal and computationally cheap methods of solving turbulent flows. However, there is a large number of methods based on various simplifying assumptions. Since Reynolds's works, averaging procedures have been introduced. Unfortunately, they need applications of so called closure models. These models allow to solve some particular problems (even quite complicated), but they are usually based on some empirical data.

When the Reynolds number is high, it is possible to decouple a flow field into parts of strong and weak influence of viscosity. In this approach, known since the times of Ludwig Prandtl, the flow field is divided into the boundary layer and the outer flow. Providing that the former is thin, many quite efficient methods have been elaborated.

However, these methods usually fail when the separation of flow appears. The boundary layer cannot be assumed to be thin and some feed-back with the upstream flow must be taken into account. To deal with these problems, some methods have been invented, for example the method proposed by Jacob (1969). This method is, however, derived on the basis on some additional hypothesis which are not in any way implied by the fundamental laws of fluid dynamics.

Probably, many of the difficulties mentioned above will be overcome as soon as the computational power of the available computers is increased. According to certain estimations (Jameson, 1983; Kutler, 1985) it would require devices with computational efficiency several orders higher than such supercomputers as CRAY-3. At that time, the discretization of Navier-Stokes equation could be done on a fine grid that allows to calculate the fully turbulent structures of the flow. This conjecture is based on the general belief that the Navier-Stokes equations are the proper mathematical model for real-flow turbulence. Some examples of the calculated flows in simple geometries seem to confirm it.

All the above mentioned problems account for more research. It seems that certain simulation techniques have prove to be quite efficient in the numerical analysis of flow with high Reynolds numbers. Such algorithms are relatively simple and do not require very big computational power. One of these methods is based on the numerical simulation of diffusive processes. The theory of such processes was developed in the beginning of this century by Einstein and Smoluchowski (see Gardiner, 1985). Its essence is to apply the random motion of particles to the model on a microscopic level, while the macroscopic description of the model is the diffusion equation. The characteristic feature of methods utilizing this idea is the lack of wave effects. Therefore, methods based on "diffusive simulation" are evidently not proper for hyperbolic problems. However, in most cases of stationary, incompressible flows around bodies, wave effects can be neglected.

The two-dimensional motion of viscous liquids can be described by the Helmholtz equation

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu \Delta \omega \tag{1.1}$$

which is in fact the equation of transport of the vorticity ω in the velocity field given by its cartesian components u and v. Now, following the idea of Chorin (1973) and (1978), we introduce elementary "vorticity particles", which movement is composed of the advection determined by the velocity field and the random motion according to the diffusion coefficient ν (viscosity). On the microscopic level "vorticity particles" are analogues of particles of diffunding substance. In the macroscopic scale, vorticity ω is a density of this substance. In other words, the motion of hypothetical particles, determined by the equations

$$dx = udt + dR_x \qquad\qquad dy = vdt + dR_y \tag{1.2}$$

where dR_x and dR_y are infinitesimal random displacements (of proper kind), constitutes the microscopic description of the phenomenon modeled in macroscale by Eq (1.1). The existence of vortical particles influences the velocity field through induction

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad\qquad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega \tag{1.3}$$

When the number of vortical particles is high, one can expect the velocity field obtained to be an approximation of the field resulting from the macroscopic description.

Paragraph a similar approach has being used by Spalart (1981). The number of vortical particles used was not very great ($\sim$ 1000) and strict boundary conditions for the normal component of velocity was not incorporated. The improvement of the latter requires the explicit form of the connection between normal and tangential velocity components for potential flow. Thus, the Neumann boundary problem should be solved. Suitable integral operators can be found explicitly (Styczek, 1987) which allows to reduce the problem to boundary integral equation with the vorticity distribution on the contour as an unknown function. The solution in the sense of local averages can be calculated numerically.

However, the set of vortical particles determined in that way is not stable (probably due to random effects). It seems that introducing a stabilizing component is possible. If one assumes the asymptotic consistence of viscous and inviscid velocity fields far away from the body, then an additional condition concerning the total circulation of vortical particles in the flow and the circulation of the potential body-connected vortices can be imposed. As a result, the global circulation of the flow does not increase indefinitely with time. The comprehensive survey of vortex methods (Sarpakaya, 1989) does not mention any methods of such circulation control.

On the other hand, Bielocerkovskiĭ (1988), in his book on vortex methods for ideal fluid, introduced some control of these quantities. It can be viewed as a natural way of generalizing the Kutta-Joukovsky condition. Obviously, this requirement does not apply to viscous fluid flows. It turns out, however, that the introduction of its analogue is necessary.

The aim of this paper is to describe the stochastic simulation in application to the flow around an airfoil. The description contains the following elements:

- relations between stochastic modeling (Itô equation) and diffusion modeling (Fokker-Planck equation),
- adaptation of the stochastic simulation to flow problems,

- algorithm of to determine the velocity and vorticity fields.

2. Selected elements of the theory of stochastic processes

In this section, we present some useful results to be used later in this paper. Details on these results can be found by Gardiner (1985).

2.1. The Wiener process

The stochastic process with the transition probability-density function

$$p(x,t\big|x_0,t_0) = \frac{1}{\sqrt{2\pi(t-t_0)}}\exp\Big\{-\frac{(x-x_0)^2}{2(t-t_0)}\Big\} \tag{2.1}$$

is called the Wiener process denoted by W_t or $W(t)$.

It means the probability that the trajectory starting at (t_0, x_0) will drop inside the $A \subset R$ at $t > t_0$ is given by

$$P(A,t\big|x_0,t_0) = \int\limits_A \frac{1}{\sqrt{2\pi(t-t_0)}}\exp\Big\{-\frac{(x-x_0)^2}{2(t-t_0)}\Big\}dx$$

The above definition implies that the random variable $W_t - W_{t0}$ has a Gaussian (normal) distribution with average x_0 and variance $(t-t_0)$. Moreover, it can be proved that W_t is a Markovian process.. This means that the function $p(x,t|x_0,t_0)$ and the initial probability distribution $p(x_0,t_0)$ are sufficient to determine any characteristic of the process. Another important feature of W_t is the statistical independence of the increments, which means that random variables $W(t_1) - W(t_0)$, $W(t_2) - W(t_1)$ for $t_0 < t_1 < t_2 < \ldots$ are independent.

The trajectories of a Wiener process are continuous function of time. However, since the following formula can be calculated

$$P\Big\{|W(t+h) - W(t)|/h > k\Big\} = 2\int\limits_{kh}^{\infty}\frac{1}{\sqrt{2\pi h}}e^{-\frac{\xi^2}{2h}}d\xi \xrightarrow{h\to 0} 1$$
$$(k > 0)$$

they are no-where differentiable.

The Wiener process plays significant role in the theory of diffusive Markovian processes closely connected with the stochastic simulation of viscous flows. The diffusive Markov process satisfies the following conditions ($X(t)$ – n-dimensional process)

$$1) \qquad \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|x-x_0|>\varepsilon} p(x, t_0 + \Delta t \big| x_0, t_0) dx = 0$$

uniformly with respect to x, x_0, t; this condition assures the continuity of trajectories almost certainly,

$$2) \qquad \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|x-x_0|<\varepsilon} (x_i - x_{0i}) p(x, t_0 + \Delta t \big| x_0, t_0) dx = A_i(x_0, t_0) + o(\varepsilon)$$

$$3) \qquad \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|x-x_0|<\varepsilon} (x_i - x_{0i})(x_j - x_{0j}) p(x, t_0 + \Delta t \big| x_0, t_0) dx =$$

$$= B_{ij}(x_0, t_0) + o(\varepsilon)$$

In 2) and 3) the uniform convergence with respect to x_0, ε and t is required. The vector A is called the convective vector, while the matrix B is called the matrix of diffusion.

2.2. Stochastic integration with respect to the Wiener process

The aim of the strict formulation of stochastic differential equation theory lead to the determination of the stochastic integral with respect to the increments of Wiener process.

The starting point is the following sum

$$S_n = \sum_{i=1}^{n} G(\tau_i) \Big[W(t_i) - W(t_{i-1}) \Big] \tag{2.2}$$

for $t_0 \le t_1 \le t_2 \le ... \le t_{n-1} < t$, $\tau_i \in <t_i, t_{i-1}>$, where $G(\tau)$ means integrated processes.

If one assumes that the following limit S exists

$$\lim_{n \to \infty} \langle \Big\{ S - \sum_{i=1}^{n} G(\tau_i) \Big[W(t_i) - W(t_{i-1}) \Big] \Big\}^2 \rangle = 0$$

then S is called stochastic integral of G. This construction has been so far identical to classical Riemann-Stieltjes integrals. It turns out, however, that the limit S, if it exists, depends on the choice of the points τ_i. One of the possible ones, the choice made by Itô: $\tau_i = t_{i-1}$ leads to reasonable and fruitful mathematical theory. In that way one obtains Itô integral given by the limit

$$\lim_{n\to\infty}\langle\Big\{\int_{t_0}^{t} G(\tau)dW(\tau) - \sum_{i=1}^{n} G(t_{i-1})\big[W(t_i) - W(t_{i-1})\big]\Big\}^2\rangle = 0$$

determined for the particular class of processes which satisfy the so called nonantycypation condition: $G(s)$ is independent on the $W(t_2) - W(t_1)$ for $t_2 > t_1 \geq s$.

The alternative choice is the definition given by Stratonovich

$$G(\tau_i) = \frac{1}{2}\Big(G(t_{i-1}) + G(t_i)\Big)$$

The detailed analysis shows that the Stratonovich-integral is computationally similar to the ordinary integral, while Itô integral is not. The classical example yields

— Itô integral

$$\int_{t_0}^{t} W(t)dW(t) = \frac{1}{2}\Big[W^2(t) - W^2(t_0) - (t - t_0)\Big]$$

— Stratonovich integral

$$\int_{t_0}^{t} W(t)dW(t) = \frac{1}{2}\Big[W^2(t) - W^2(t_0)\Big]$$

2.3. Stochastic differential equations

The following equation is called the Itô equation

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t) \tag{2.3}$$

Since the trajectories of a Wiener process are nondifferentiable, Eq (2.3) cannot be integrated as an ordinary differential equations for trajectories parameterized by random elements. The solution of (2.3) is defined in the integral sense as presented in the following.

The solution of (2.3) satisfying the initial condition $X(t_0) = X_0$ is the stochastic process which fulfills the integral equation of the form

$$X(t) = X_0 + \int_{t_0}^{t} a(\tau, X(\tau))d\tau + \int_{t_0}^{t} b(\tau, X(\tau))dW(\tau)$$

It is proven that when the conditions are satisfied

$$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \leq K_1|x-y|$$

and

$$|a(t,x)|^2 + |b(t,x)|^2 \leq K_2(1+|x|^2)$$

for certain $K_1, K_2 > 0$ then this initial value problem has a unique solution which is the diffusive Markov process. One of the fundamental result of the theory of such processes is the equation governing the time-space evolution of the transition probability distribution

$$\begin{aligned} &\frac{\partial}{\partial t}p(x,t|x_0,t_0) + \frac{\partial}{\partial t}\Big[A(x,t)p(x,t|x_0,t_0)\Big] = \\ &= \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big[B(x,t)p(x,t|x_0,t_0)\Big] \end{aligned} \tag{2.4}$$

called Fokker-Planck equation. Functions A and B are the coefficients of convection and diffusion, respectively.

Thus there must be some connection between the functions a and b in Eq (2.3) and A and B in Eq (2.4). Actually, it turn out that $a(x,t) = A(x,t)$ and $B(x,t) = b^2(x,t)$. It is the crucial result which allows to construct the Itô equation corresponding to the diffusion of the vorticity in viscous flows.

To prove the above equalities some auxiliary formulas are necessary

1.

$$\int_{t_0}^{t} G(\tau)\Big[dX(\tau)\Big]^{N+2} = \begin{cases} 0 & \text{for} \quad N>0 \\ \int_{t_0}^{t} G(\tau)d\tau & \text{for} \quad N=0 \end{cases}$$

where these integrals are defined by the limit

$$\lim_{n\to\infty}\langle\Big\{\int_{t_0}^{t} G(\tau)\Big[dX(\tau)\Big]^{N+2} - \sum_{i=1}^{n} G(t_{i-1})\Big[X(t_i) - X(t_{i-1})\Big]^{N+2}\Big\}^2\rangle = 0$$

The formal meaning of this limit is that

$$\Big[dX\Big]^2 = dt \qquad \text{and} \qquad \Big[dX\Big]^{N+2} = 0 \qquad \text{for} \qquad N>0$$

2. The formula for the change of variables (so called Itô formula)

Let $Y = v(X_t, t)$, then the following

$$dY(t) = dv(X_t,t) = \Big[\partial_t v(X_t,t) + a(X_t,t)\partial_x v(X_t,t) + $$
$$+ \frac{1}{2}b^2(X_t,t)\partial_{xx}v(X_t,t)\Big]dt + b(X_t,t)\partial_x v(X_t,t)dW_t$$

holds for a function v sufficiently regular and a process X_t satisfying the equation

$$dX_t = a(X_t,t)dt + b(X_t,t)dW_t$$

3. Let $\{Y_t(\omega) = \int\limits_{t_1}^{t} G_s(\omega)dW_s(\omega)$. If $\{\int\limits_{t_1}^{t} E\{G_s^2\}ds < \infty$ for $t \in R$ then $E\{Y_t\} = 0$

Now, let's consider a function v of variables x and t with bounded support, i.e. $v(x,t) = 0$ if

$$(x,t) \notin [x_1,x_2] \times [t_1,t_2] \qquad \text{for some} \quad x_1,x_2 \quad \text{and} \quad t_2,t_1 > t_0$$

then one can construct another stochastic process due to the formula $Y_t = v(X_t,t)$. On the basis of the Itô formula, we have

$$dY_t = \Big[\partial_t v(X_t,t) + a(X_t,t)\partial_x v(X_t,t) + \frac{1}{2}b^2(X_t,t)\partial_{xx}v(X_t,t)\Big]dt + $$
$$+ b(X_t,t)\partial_x v(X_t,t)dW_t$$

The equivalent integral form is

$$v(X_t,t) - v(X_{t0},t_0) =$$
$$= \int\limits_{t_0}^{t} \Big[\partial_t v(X_\tau,\tau) + a(X_\tau,\tau)\partial_x v(X_\tau,\tau) + \frac{1}{2}b^2(X_\tau,\tau)\partial_{xx}v(X_\tau,\tau)\Big]dt + \quad (2.5)$$
$$+ \int\limits_{t_0}^{t} b(X_\tau,\tau)\partial_x v(X_\tau,\tau)dW_\tau$$

Since supp $v(x,t)$ is bounded then

1) $v(x,t_0) = 0$

2) $v(x,\infty) = 0$

3) $\int\limits_{t_0}^{\infty} E\Big\{[b(X_\tau,\tau)\partial_x v(X_\tau,\tau)]^2\Big\}d\tau < \infty$

The last condition together with 3. yields

$$E\Big\{\int_{t_0}^{t} b(X_\tau,\tau)\partial_x v(X_\tau,\tau)dW_\tau\Big\} = 0$$

Taking the limit $t \to \infty$ and calculating the expectation of both sides of (2.5) we obtain

$$\int_0^\infty E\Big\{\Big[\partial_t v(X_\tau,\tau) + a(X_\tau,\tau)\partial_x v(X_\tau,\tau) + \frac{1}{2}b^2(X_\tau,\tau)\partial_{xx} v(X_\tau,\tau)\Big]\Big\}dt = 0$$

or

$$\int_0^\infty dt \int_R dx\ p(x,t\big|x_0,t_0)\Big[\partial_t v(X_t,t) + a(X_t,t)\partial_x v(X_t,t) + \\ + \frac{1}{2}b^2(X_t,t)\partial_{xx} v(X_t,t)\Big] = 0$$

Integration by parts results in

$$\int_0^\infty dt \int_R dx v(x,t)\Big[-\partial_t p(x,t\big|x_0,t_0) - \partial_x a(x,t)p(x,t\big|x_0,t_0) + \\ + \frac{1}{2}\partial_{xx} b^2(x,t)p(x,t\big|x_0,t_0)\Big] = 0$$

Components including the boundary values vanished because sup pv is bounded. Since v is an arbitrary (but sufficiently regular) function with compact support, the last equality implies that $p(x,t\big|x_0,t_0)$ must satisfy the following equation

$$\partial_t p(x,t\big|x_0,t_0) + \partial_x a(x,t)p(x,t\big|x_0,t_0) = \\ = \frac{1}{2}\partial_{xx} b^2(x,t)p(x,t\big|x_0,t_0)$$

i.e. the Fokker-Planck equation with coefficients $a(x,t)$ and $b^2(x,t)$. This ends the proof.

The generalization to multidimensional case is easy. We define the n-dimensional Wiener process

$$\mathsf{W} = \{W_1(t), W_2(t), ..., W_n(t)\}$$

with transition probability given by

$$p(\mathsf{X},t|\mathsf{X}_0,t_0) = [2\pi(t-t_0)]^{-\frac{n}{2}} \exp\Big\{-\frac{(\mathsf{X}-\mathsf{X}_0)(\mathsf{X}-\mathsf{X}_0)^\mathsf{T}}{2(t-t_0)}\Big\}$$

The set of Itô equations has the form

$$dX_t = A(t, X_t)dt + B(t, X_t)dW_t$$

while the corresponding Fokker-Planck equation is the following

$$\frac{\partial}{\partial t}p(x,t\big|x_0,t_0) + \sum_{i=1}^{n}\frac{\partial}{\partial x_i}A_i(t,x)p(x,t\big|x_0,t_0) =$$
$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_i \partial x_j}(BB^{\mathsf{T}})_{ij}p(x,t\big|x_0,t_0)$$

3. The stochastic simulation of the motion of incompressible viscous fluids

3.1. The problem of a flow around a contour

The two-dimensional motion of incompressible viscous fluid around a given, fixed contour is described by the set consisting of the continuity equation and the Navier-Stokes equations

$$\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\ &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) + \nu\Delta u \\ &\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial}{\partial y}\left(\frac{p}{\rho}\right) + \nu\Delta v \end{aligned} \tag{3.1}$$

together with the following boundary conditions, imposed on the contour

$$V\Big|_w = 0 \tag{3.2}$$

an asymptotic condition describing the motion at infinity

$$\lim_{|r|\to\infty} V = V_\infty \tag{3.3}$$

and an initial condition. In addition, uniformity of the pressure field at infinity should be postulated.

The velocity field can be divided into a potential V_{pot} and a vorticity – induced component V_{cir}. Thus

$$V = V_{pot} + V_{cir} \tag{3.4}$$

while the vorticity is defined as follows

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial v_{cir}}{\partial x} - \frac{\partial u_{cir}}{\partial y} \tag{3.5}$$

The Navier-Stokes equations can be transformed into the Helmholtz equation for a vorticity field

$$\frac{\partial \omega}{\partial t} + \frac{\partial}{\partial x}(u\omega) + \frac{\partial}{\partial y}(v\omega) = \nu \Delta \omega \tag{3.6}$$

which is in the form of the Fokker-Planck equation.

Now the stream function ψ can be introduced

$$u_{cir} = \frac{\partial \psi}{\partial y} \qquad v_{cir} = -\frac{\partial \psi}{\partial x} \qquad \Delta \psi = -\omega \tag{3.7}$$

Providing that the function ω vanishes at infinity, the following asymptotic formula can be obtained

$$\psi\Big|_{\infty} \cong -\frac{1}{2\pi} \ln r \int\int \omega(B) dx_B dy_B + ... = -\frac{\Gamma_{\infty}}{2\pi} \ln r + ...$$

which comes from the formal inversion of the Laplace operator. As a result, the induced velocity field is asymptotically represented in the following way

$$u_{cir}\Big|_{\infty} = -\frac{\Gamma_w}{2\pi}\frac{y}{r^2} + ...$$
$$v_{cir}\Big|_{\infty} = \frac{\Gamma_w}{2\pi}\frac{x}{r^2} + ...$$

where

$$\Gamma_w = \int\int \omega dx dy = \int\int k \mathrm{rot} V dx dy = \oint V ds$$

It is very convenient to use complex notations in presenting the potential velocity. Let's denote by $V_z(z)$, $(z = x + iy)$ the complex velocity corresponding to V_{pot}, which is an analytical function in the flow domain. Then, using Laurent's expansion, we have

$$V_z = V_{z\infty} + \frac{Q - i\Gamma}{2\pi}\frac{1}{z} + \frac{\mathrm{const}}{z^2} + ...$$

where

$V_{z\infty}$ – complex velocity at infinity,

Q – total flux of sources located inside a contour,

Γ – circulation of the contour – connected vortex.

We represent V_{pot} as a sum of the following components

V_P – the velocity of the flow of an ideal fluid past the contour which is equal to V_∞ at infinity

V_Γ – the velocity field one to the potential vortex, V_Γ vanishes at infinity and $V \cdot n = 0$ on the contour

V_A – auxiliary, irrotational velocity field vanishing at infinity.

Thus

$$V_{pot} = V_P + V_\Gamma + V_A \tag{3.8}$$

and the total velocity field is in the form

$$V = V_P + V_\Gamma + V_A + V_{cir} \tag{3.9}$$

If one calculates the circulation Γ of (3.9) along the closed line containing the contour and infinitely distant from it, then

$$\Gamma = \Gamma_p + \Gamma_c + \Gamma_w \tag{3.10}$$

where

Γ_p – circulation of a contour-connected vortex which is selected due to the Kutta-Joukovsky condition at the trailing edge (in the case when there is no trailing edge this condition is imposed at the rear point with the maximal curvature)

Γ_c – circulation of the purely vortical potential flow

Γ_w – total vorticity flux in the flow domain

$$\Gamma_w = \int\int \omega dx dy \tag{3.11}$$

The velocity field V_P is a good approximation of the real flow outside the boundary layer and the wake behind the body. This observation is in fact the asis of the Prandtl scheme of a flow with a boundary layer. According to this scheme, when the Reynolds number is high, the fluid motion outside the boundary layer and outside the wake is close to the potential motion. Thus, it seems that the asymptotic expression for the velocity in viscous flows should have the form (r – very large)

$$\begin{aligned} u &= u_\infty - \frac{\Gamma_p}{2\pi}\frac{y}{r^2} + \dots \\ v &= v_\infty + \frac{\Gamma_p}{2\pi}\frac{x}{r^2} + \dots \end{aligned} \tag{3.12}$$

Assuming that at large distances from the body, the velocity field can be written as follows

$$V = iU_\infty + U'$$

and, assuming that U' is determined by Oseen's approximation, we can find the rate of decay of U'. The vanishing of U' instifies the hypothesis that higher order terms of the expansion (3.12) can be neglected.

For three-dimensional motion, this term diminishes very fast as $r \to \infty$. For two-dimensional motion, the convergence of $U' \to 0$ is weaker. On the symmetry line of the wake it is determined by

$$|V - U_\infty| = o\Big(\frac{1}{\sqrt{x}}\Big)$$

Beyond this line the convergence is very fast, which is reducted from Oseen's equation

$$U_\infty \frac{\partial U'}{\partial x} = \nu \frac{\partial^2 U'}{\partial y^2}$$

which fundamental solution is

$$U' \sim \frac{1}{\sqrt{x}} \exp\Big(-\frac{U_\infty y^2}{2\nu x}\Big)$$

Thus the wake is symmetric and vanishes very fast as soon as the distance from its symmetry line increases.

We introduce an additional hypothesis

- The asymptotic circulation of the velocity fields in a two-dimensional viscous flow past a given contour is identical to the one determined by the Kutta-Joukovsky condition for ideal flow.

In terms of the notation introduced previously we have

$$\Gamma_c + \int\int \omega dx dy = 0 \tag{3.13}$$

It can be added here, that in flows around three-dimensional bodies there is no ambiguity connected with the circulation mentioned above. On the other hand, this andiguity is typical for elliptic problems stated in multi-connected domains.

3.2. The Vortex Blobs method

We are going to solve the Helmholtz equation (3.6) by constructing the families of stochastic processes satisfying the following Itô equations

$$\begin{aligned} dx &= udt + \sqrt{2\nu}dX & \qquad dy &= vdt + \sqrt{2\nu}dY \\ x\Big|_{t=0} &= x_0 & \qquad y\Big|_{t=0} &= y_0 \end{aligned} \tag{3.14}$$

with x_0, y_0 being arbitrary point inside the fixed domain of the flow, and satisfying the following second set of Itô equations

$$\begin{aligned} dx &= udt + \sqrt{2\nu}dX & \qquad dy &= vdt + \sqrt{2\nu}dY \\ x\Big|_{t=\tau} &= x_w & \qquad y\Big|_{t=\tau} &= y_w \end{aligned} \tag{3.15}$$

with x_w, y_w being arbitrary points on the contour of the airfoil for every $\tau \in [0,t]$. In other words, the first family of the stochastic processes concerns the evolution of an initial state, while the second one describes the effects arising on the boundary. If one assumes that the infinitesimal circulation $d\Gamma$ can be related to each processes of the families, the following formula can be obtained

$$\begin{aligned} \omega(t,x,y) = \int\int p(x,y,t\big|x_0,y_0,0)d\Gamma(x_0,y_0) + \\ + \int_0^t \oint p(x,y,t\big|x(s),y(s),\tau)\frac{\partial\Gamma(s,\tau)}{\partial\tau}dsd\tau \end{aligned} \tag{3.16}$$

The first integration (carried on the whole flow domain) assumes that the initial condition is determined by $d\Gamma = \omega_0(x,y)dxdy$. The second integral introduces the boundary distribution of vorticity $d\frac{\partial\Gamma}{\partial t} = \frac{\partial\omega}{\partial t}(t,x,y)ds$ connected with an airfoil. This distribution should have a linear density

$$\frac{\partial}{\partial t}\omega(t,s) = \frac{\partial}{\partial t}\omega(t,x(s),y(s)) \ . \tag{3.17}$$

so that the velocity on the boundary resulting from the total vorticity field fulfills (3.2).

The expression for the boundary component

$$\omega_B(x,y,t) = \int_0^t \oint p(x,y,t\big|x(s),y(s),\tau)\frac{\partial\Gamma(s,\tau)}{\partial\tau}dsd\tau$$

can be viewed as a generalization of Duhamel's formula. Indeed, due to the uniformity of the function p

$$p(x,y,t\big|x_0,y_0,\tau) = p_1(x,y,t-\tau,x_0,y_0)$$

we have

$$\frac{\partial p}{\partial t} = -\frac{\partial p}{\partial\tau}$$

when the derivatives exist.

Providing that functions p and Γ are sufficiently regular, one can write

$$\omega_B = \oint \int_0^t \frac{\partial}{\partial \tau}\left\{p(x,y,t \middle| x(s),y(s),\tau)\Gamma(s,\tau)\right\}d\tau ds - $$

$$- \oint \int_0^t \frac{\partial p}{\partial \tau}\Gamma(s,\tau)d\tau ds$$

$$\lim_{\tau \to t} p(x,y,t|x',y',\tau) = \delta(x,y,x',y')$$

and the following formula can be derived

$$\begin{aligned} \omega_B &= \oint \delta(x,y,x(s),y(s))\Gamma(s,t)ds - \oint p(x,y,t,x(s),y(s),0)\Gamma(s,0)ds + \\ &+ \int_0^t \frac{\partial}{\partial t} \oint p(x,y,t|x(s),y(s),\tau)\Gamma(s,\tau)dsd\tau \end{aligned}$$

Let's introduce discrete equivalents for the stochastic families constructed above. If at an initial moment there is no vorticity in the domain, only the second family is sufficient

$$\begin{aligned} dx_{i,j} &= u(t,x_{i,j},y_{i,j})dt + \sqrt{2\nu}dY \\ dy_{i,j} &= v(t,x_{i,j},y_{i,j})dt + \sqrt{2\nu}dY \\ x_{i,j}\Big|_{t=\tau_j} &= x_{wi} \qquad\qquad y_{i,j}\Big|_{t=\tau_j} = y_{wi} \end{aligned} \tag{3.18}$$

The processes $x_{i,j}$, $y_{i,j}$ start on the boundary at τ_j.

The starting points for the process indexed by (i,j) are x_{wi}, y_{wi} independent of the initial time. To simplify the calculations they are fixed. For each process, a fixed and constant circulation is selected and is obtained from the discretization of the distribution of vorticity on the boundary

$$\Gamma_{i,j} = \int_{\Delta S_i} \omega(\tau_j, x(s), y(s))ds$$

The question is: how do we determine $\Gamma_{i,j}$ or equivalently to choose the vorticity distribution? This problem can be solved as follows: the vortical velocity field is divided into the field induced by the set of vortices previously created and existing in the flow and the field induces by new vortices currently created on the boundary. We denote these fields by V_0 and V_W, respectively. Thus

$$V = V_P + V_A + V_\Gamma + V_W + V_0 \tag{3.19}$$

On the boundary of an airfoil, V vanishes. Taking the tangent and normal components for every point on the boundary we can write

$$
\begin{aligned}
V_P^t + V_A^t + V_\Gamma^t + V_W^t + V_0^t &= 0 \\
0 + V_A^n + 0 + V_W^n + V_0^n &= 0
\end{aligned}
\tag{3.20}
$$

Let's notice that V_P^t, V_0^t and V_0^n are known. The first one describes the boundary velocity of the inviscid fluid flow, while the others result from the known vorticity distribution in the flow domain. This distribution is determined from the location and circulation of vortices created previously. The function V_Γ^t is determined from the total circulation of all vortices. Furthermore V_A^t and V_A^n are boundary values of the potential velocity field. This field, vanishing at infinity, is completely determined by the normal component V_A^n. Thus

$$
V_A^t = \mathcal{L} V_A^n \tag{3.21}
$$

where $\mathcal{L}$ is a linear integral operator with Cauchy-type kernel which will be calculated later.

The use of (3.21) and (3.20) yields the following integral equation

$$
V_P^t + V_\Gamma^t + V_0^t - \mathcal{L}V_0^n + V_W^t - \mathcal{L}V_W^n = 0 \tag{3.22}
$$

with unknown vorticity on the boundary. Chorin (1967) replaced this distribution by a function which is constant on small segments of the contour. He then connected the corresponding circulation to very small circular vortices in comparison with the flow domain dimensions, which he called "vortex blobs". These blobs move with accordance to the Itô equations (3.18).

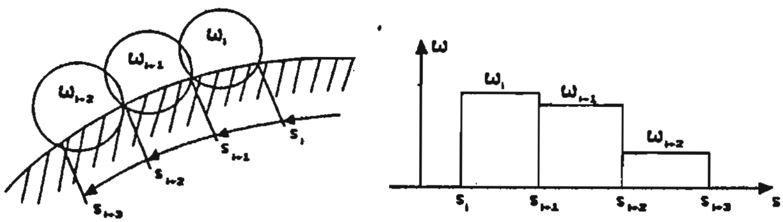

Fig. 1.

The blobs are created in the vicinity of boundaries, so that $\omega \neq 0$ on the contour line (see Fig.1). The way the blobs are created, determinated by (3.22), assures that the total tangential velocity at any point on the contour vanishes. At the same time, the potential velocity V_A satisfies the second equation of (3.20); thus the total normal velocity also vanishes. (In the original work of Chorin, and also from other authors, condition (3.22) assuring the strict fulfilling of (3.20) does not exist.)

It should be added that besides circular blobs, in some work (Chorin, 1978), linear vortical structure (vortex sheet) were used. In other words, the vorticity field in the vicinity of the boundary was approximated by a constant function on straight segments parallel to the local tangential direction. However, such modification is not essential. Accordingly to diffusive interpretation, the vortex sheets should be small compared to the scale of dimensions of the velocity field, while their shape and internal structure is not important. Moreover the self-induction of a radially symmetric vortex does not change its shape which is not true in case of linear structure. Finally, the induction field of a circular vortex is particularly simple.

Let's assume that the vorticity field connected to a simple blob is radially symmetric and equal to $\omega(r)$. Then

$$\frac{1}{r}\frac{d}{dr}r\frac{d\psi_{cir}}{dr} = -\omega(r) \tag{3.23}$$

The components of the velocity field are easy to calculate

$$\begin{aligned} u(x,y) &= -\frac{\partial\psi_{cir}}{\partial y} = -\frac{\partial r}{\partial y}\frac{d\psi}{dr} = -\frac{y}{r^2}\int_0^r \omega(\xi)\xi d\xi \\ v(x,y) &= -\frac{\partial\psi_{cir}}{\partial x} = -\frac{\partial r}{\partial x}\frac{d\psi}{dr} = \frac{x}{r^2}\int_0^r \omega(\xi)\xi d\xi \end{aligned} \tag{3.24}$$

For $r \geq \sigma$, where σ is the radius of the given blob we obtain

$$u = -\frac{\Gamma}{2\pi}\frac{y}{r^2} \qquad v = \frac{\Gamma}{2\pi}\frac{x}{r^2} \tag{3.25}$$

The circulation Γ is determined by the integral

$$\Gamma = \int_0^{2\pi}\int_0^{\sigma} \omega(\xi)\xi d\xi d\varphi = 2\pi\int_0^{\sigma}\omega(\xi)\xi d\xi \tag{3.26}$$

Next, for $r < \sigma$ the velocity depends on the vorticity distribution $\omega(r)$. If $\omega(r) = \text{const} = \omega_0 = \frac{\Gamma}{\pi\sigma^2}$ then

$$u = -\frac{\Gamma}{2\pi\sigma^2}y \qquad v = \frac{\Gamma}{2\pi\sigma^2}x \tag{3.27}$$

The expressions (3.25) and (3.27) yield the induction formula. Let's assume that the center of a blob is located at (x_0, by_0). Then

$$u(x,y) = \begin{cases} -\frac{\Gamma}{2\pi}\frac{y-y_0}{\sigma^2} & \text{for} \quad r^2 \leq \sigma^2 \\ -\frac{\Gamma}{2\pi}\frac{y-y_0}{r^2} & \text{for} \quad r^2 \geq \sigma^2 \end{cases}$$

$$v(x,y) = \begin{cases} \dfrac{\Gamma}{2\pi}\dfrac{x-x_0}{\sigma^2} & \text{for} \quad r^2 \le \sigma^2 \\ \dfrac{\Gamma}{2\pi}\dfrac{x-x_0}{r^2} & \text{for} \quad r^2 \ge \sigma^2 \end{cases} \tag{3.28}$$

$$r^2 = (x-x_0)^2 + (y-y_0)^2$$

In should be pointed out that (3.28) is only one of many possible choices (see Chorin, 1973; Spalart and Leonard, 1981). Other distributions of vorticity inside a blob can be admitted, including the case when supp $\omega(r)$ is unbounded, for example $\omega \sim \exp(-\beta r^2)$ with β being a constant.

Actually the vorticity field is approximated by the combination of local distributions resulting from replacing the continuous family of stochastic processes with a finite family. It seems that the approximation of vorticity with constant functions of bounded support covering the whole domain is appropriate. The approximation error (in the sense of L_P norm) diminishes in proportion with the diameter of the largest blob. Also, the induction formula is particularly simple.

We now apply the above concepts to the flowfield around an airfoil. First, assume that the contour of an airfoil is divided into N arcs located between nodes such that

$$0 = s_0, s_1, s_2, ..., s_N = \text{periphery}$$

where s denotes the arc – length coordinates. Next, one vortex blob is located above the center of the cord of each arc. The tangential and normal components of the induced velocity by the blob located near (s_k, s_{k+1}) at the contour point s can be expressed as

$$V_W^t(s,k) = \Gamma_k T(s,k) \qquad , \qquad V_W^n(s,k) = \Gamma_k N(s,k)$$

The functions $T(s,k)$ and $N(s,k)$ are completely determined by the geometry of the contour.

In application to all the vortex blobs, the induced velocity at the contour points is obtained by taking the sum or all the influences and thus yields

$$V_W^t(s) = \sum_{k=1}^{N} \Gamma_k T(s,k) \qquad\qquad V_W^n(s) = \sum_{k=1}^{N} \Gamma_k N(s,k) \tag{3.29}$$

Then, by substitution in (3.22)

$$V_P^t(s) + V_\Gamma^t(s) + V_0^t(s) - \mathcal{L}V_0^n(s) = \sum_{k=1}^{N} \Gamma_k \Big\{ \mathcal{L}N(s,k) - T(s,k) \Big\}$$

The component V_Γ^t depends on the sum of all the vortex blobs existing in the flow field, including these being created on the boundary. Thus

$$V_\Gamma^t(s) = V_1^t(s)\Big\{-\sum_{k=1}^{N} \Gamma_k - \Gamma_0\Big\} \tag{3.30}$$

where

Γ_0 – total circulation of the blobs previously created,

$V_1^t(s)$ – tangential velocity induced by the unitary, airfoil – connected vortex.

As a resultwe have the equation

$$\begin{aligned} &V_P^t(s) + V_0^t(s) - \mathcal{L}V_0^n(s) - \Gamma_0 V_1^t(s) = \\ &= \sum_{k=1}^{N} \Gamma_k \Big\{\mathcal{L}N(s,k) - T(s,k) + V_1^t(s)\Big\} \end{aligned} \tag{3.31}$$

We require that (3.31) is satisfied in the mean sense on each arc – segment s_{j-1}, s_j. This averaging is a consequence of the discretization of a continuous vorticity distribution. After introducing (3.29) one is not able to fulfill (3.31) "every-where". Hence, as a solution of (3.31) in the mean sense, we regard such set of circulations Γ_1, Γ_2, ..., Γ_N, which satisfies the equation

$$\begin{aligned} &\int_{m_{j-1}}^{m_j} \Big\{V_P^t(s) + V_0^t(s) - \mathcal{L}V_0^n(s) - \Gamma_0 V_1^t(s)\Big\} ds = \\ &= \sum_{k=1}^{N} \Gamma_k \Big\{ \int_{m_{j-1}}^{m_j} \mathcal{L}N(s,k) ds - \int_{m_{j-1}}^{m_j} T(s,k) ds + \int_{m_{j-1}}^{m_j} V_1^t(s) ds \Big\} \\ &\qquad j = 1, 2, ..., N \end{aligned} \tag{3.32}$$

It can be written shortly

$$\mathbf{R}\mathbf{\Gamma} = \mathbf{B} \tag{3.33}$$

where $\mathbf{R}$ is a fixed matrix. The right – hand side vector $\mathbf{B}$ depends on the current values of V_0^t, V_0^n and the total value of circulation of all previously created blobs. The determination of V_0^t and V_0^n is possible – the location and circulation of each "old" blob are known. Also, calculating the integrals from V_P^t and V_1^t is easy. However, to use equation (3.32) one must determine explicitly the operator L.

3.3. Operator $\mathcal{L}$

Let $F_*(\zeta)$ be an analytic function of the variable $\zeta = \xi + i\eta$ defined in the exterior of the unit circle. Let $\lim_{|\zeta|\to\infty} F_*(\zeta) = 0$ and $F_*(\zeta)$ has a limit where $\zeta \to \exp(i\theta)$ and denoted by $F_*(\theta) = a(\theta) + ib(\theta)$.

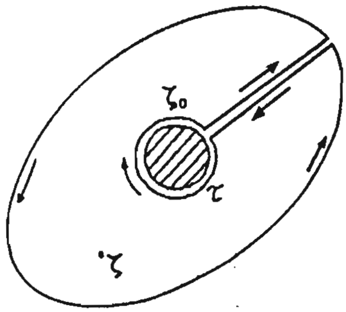

Fig. 2.

On the basis of Cauchy theorem $F_*(\zeta)$ can be expressed as the following integral

$$F_*(\zeta) = \frac{1}{2\pi i} \oint \frac{F_*(\tau)d\tau}{\tau - \zeta}$$

where the contour of integration is presented on Fig.2. Since F_* vanishes for $|\zeta| \to \infty$ the following estimation is valid

$$\Big| \int\limits_{\substack{\text{outher}\\ \text{line}}} \frac{F_*(\tau)d\tau}{\tau - \zeta}\Big| \le |F_*(\tau)| \Big| \oint \frac{d\tau}{\tau - \zeta}\Big| = 2\pi |F_*(\tau)| \to 0$$

Hence

$$F_*(\zeta) = \frac{-1}{2\pi i} \int_0^{2\pi} \frac{F_*(\tau)d\tau}{\tau - \zeta} = -\frac{1}{2\pi} \int_0^{2\pi} \frac{F_*(\tau)e^{i\varphi}d\varphi}{e^{i\varphi} - \zeta} \tag{3.34}$$

Next, for $\zeta \to \zeta_0 \in \{\text{contour of integration}\}$ Plemeliy's formula yields

$$F_*(\zeta_0) = \frac{1}{2}F_*(\zeta_0) + \frac{1}{2\pi i} VP \oint \frac{F_*(\tau)d\tau}{\tau - \zeta_0}$$

or

$$F_*(\zeta_0) = \frac{1}{\pi i} VP \oint \frac{F_*(\tau)d\tau}{\tau - \zeta_0} \tag{3.35}$$

where the integral is taken in the sense of principal values. For $\zeta_0 = \exp(i\theta)$ and after elimination of superfluous elements of the integration path we obtain

$$a(\theta) + ib(\theta) = \frac{-1}{2\pi} VP \int_0^{2\pi} [a(\varphi) + ib(\varphi)]\left[1 + i\cot\left(\frac{\theta}{2} - \frac{\varphi}{2}\right)\right] d\varphi$$

or after detaching the real and imaginary parts

$$a(\theta) = -\frac{1}{2\pi}\int_0^{2\pi} a(\varphi)d\varphi + \frac{1}{2\pi} VP \int_0^{2\pi} b(\varphi)\cot\left(\frac{\theta}{2} - \frac{\varphi}{2}\right)d\varphi$$

$$b(\theta) = -\frac{1}{2\pi}\int_0^{2\pi} b(\varphi)d\varphi - \frac{1}{2\pi} VP \int_0^{2\pi} a(\varphi)\cot\left(\frac{\theta}{2} - \frac{\varphi}{2}\right)d\varphi$$

Integration eliminates constants (singular integrals with respect to φ are integrable functions of the variable θ). As a result we have Hilbert transformations

$$\begin{aligned} a(\theta) &= \frac{1}{2\pi} VP \int_0^{2\pi} b(\varphi)\cot\left(\frac{\theta}{2} - \frac{\varphi}{2}\right)d\varphi \\ b(\theta) &= -\frac{1}{2\pi} VP \int_0^{2\pi} a(\varphi)\cot\left(\frac{\theta}{2} - \frac{\varphi}{2}\right)d\varphi \end{aligned} \tag{3.36}$$

Let's assume that the normal component of the velocity field, which is potential outside the unit circle and vanishes at infinity is given on the boundary of this circle. The aim is to determine the velocity outside and on the boundary.

The potential velocity field is determined by the equations

$$u_\xi + v_\eta = 0 \qquad\qquad v_\xi - u_\eta = 0$$

Thus $V_z(\zeta) = u - iv$ is an analytic function of the variable ζ. For $|\zeta| > 1$, and considering that $V_z(\infty) = 0$, we have the following Laurent expansion

$$V_z(\zeta) = \frac{Q - i\Gamma_{pot}}{2\pi}\frac{1}{\zeta} + \frac{B_2}{\zeta^2} + \dots$$

Q denotes the flux flowing through the contour of the circle while Γ is the circulation of the circle – connected vortex. Now we introduce another analytic function of ζ

$$F(\zeta) = \zeta V_z(\zeta) \tag{3.37}$$

On the boundary we have

$$F(\theta) = e^{i\theta}(u - iv) = V_r(\theta) - iV_\theta(\theta) \tag{3.38}$$

The function $F(\zeta)$ vanishes at infinity only when $Q - \mathrm{i}\Gamma_{pot} = 0$, i.e. when for the parameters

$$Q = \int_0^{2\pi} V_r(\theta) d\theta \qquad \Gamma_{pot} = \int_0^{2\pi} V_\theta(\theta) d\theta$$

the following equalities arise: $Q = 0$ and $\Gamma_{pot} = 0$. Otherwise we can write

$$\Big(V_r(\theta) - \frac{Q}{2\pi}\Big) - \mathrm{i}\Big(V_\theta(\theta) - \frac{\Gamma_{pot}}{2\pi}\Big) = F_*(\theta) \tag{3.39}$$

where $F_*(\theta)$ is a boundary value of $F_*(\zeta)$, which is analytic outside the circle and vanishes at infinity. Using Hilbert transformation (with consideration that the transformation of constants is equal to zero) we have

$$\begin{aligned} V_\theta(\theta) &= \frac{1}{2\pi} VP \int_0^{2\pi} V_r(\varphi) \cot\Big(\frac{\theta}{2} - \frac{\varphi}{2}\Big) d\varphi + \frac{\Gamma_{pot}}{2\pi} \\ V_r(\theta) &= -\frac{1}{2\pi} VP \int_0^{2\pi} V_\theta(\varphi) \cot\Big(\frac{\theta}{2} - \frac{\varphi}{2}\Big) d\varphi + \frac{Q}{2\pi} \end{aligned} \tag{3.40}$$

For $V_r(\theta)$ given, the function $V_\theta(\theta)$ is determined up to a constant. This constant is defined by the arbitrary circulation Γ_{pot}. In particular Γ_{pot} can be zero.

Thus for a given $V_r(\theta)$ we determine $V_\theta(\theta)$, or, which is the same, the boundary value of F_*. Next, using (3.34), we calculate $F_*(\zeta)$ and finally, on the basis of (3.37), we calculate the complex velocity $V_z(\zeta)$.

Let's now assume, that the contour of an airfoil is given by the boundary values of $z_c(\theta)$ where $z = z(\zeta)$ is an analytic (conformal) transformation of the exterior of the unit circle into the exterior of the airfoil. The function $z(\zeta)$ satisfies the condition

$$\frac{dz}{d\zeta}\Big|_{|\zeta|\to\infty} = 1$$

The derivative $dz/d\zeta$ is different from zero for $|\zeta| > 1$, while on the boundary of the circle it vanishes for $\zeta = 1$

$$\frac{dz}{d\zeta}\Big|_{\zeta=1} = 0$$

The image of this point is the trailing edge of the airfoil. The potential velocity field around the airfoil can be obtained from the velocity field around the circle by using the following transformation (Prosnak, 1970)

$$V_z(z) = V_z(\zeta)\Big(\frac{dz}{d\zeta}\Big)^{-1}$$

Let's assume that the normal component of the potential velocity field, which is noncirculative and vanishes at infinity, is given on the airfoil. We want to find the velocity on the contour and on the exterior domain of the.

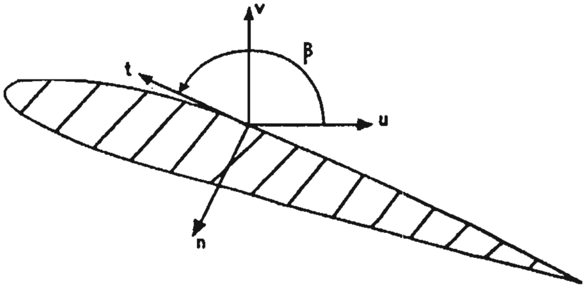

Fig. 3.

The following formulas hold (see Fig.3)

$$v^t = u\cos\beta + v\sin\beta$$

$$v^n = -u\sin\beta + v\cos\beta$$

$$v^t - \mathrm{i}v^n = (u - \mathrm{i}v)\mathrm{e}^{\mathrm{i}\beta}$$

Moreover

$$\mathrm{e}^{\mathrm{i}\beta} = \frac{dz}{ds} \qquad\qquad \frac{d\zeta}{d\theta} = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}$$

$$V_r - \mathrm{i}V_\theta = V_z(\zeta)\mathrm{e}^{\mathrm{i}\theta}$$

Simple calculations allow to determine $V_r - \mathrm{i}V_\theta$

$$V_r - \mathrm{i}V_\theta = (V_\xi - \mathrm{i}V_\eta)\mathrm{e}^{\mathrm{i}\theta} = (u - \mathrm{i}v)\frac{dz}{d\zeta}\mathrm{e}^{\mathrm{i}\theta} =$$

$$= (v^t - \mathrm{i}v^n)\mathrm{e}^{-\mathrm{i}\beta}\frac{dz}{d\zeta}\mathrm{e}^{\mathrm{i}\theta} = (v^t - \mathrm{i}v^n)\Big(-\mathrm{i}\frac{ds}{d\theta}\Big) = -v^n\frac{ds}{d\theta} - \mathrm{i}v^t\frac{ds}{d\theta}$$

Hence

$$V_r = -V^n\frac{ds}{d\theta} \qquad\qquad V_\theta = V^t\frac{ds}{d\theta} \tag{3.41}$$

Having $V^n(s)$ and considering that $s = s(\theta) = \int\limits_0^\theta |\dot{z}(\varphi)|d\varphi$ we find $V_r(\theta)$. Next, with use of (3.40), we determine $V_\theta(\theta)$. Finally, after the calculation of $V_z(\zeta)$, we find $V_z(z)$.

All formulas derived here allow to define the operator $\mathcal{L}$. According to (3.21) we can write

$$V^t = \mathcal{L}V^n$$

so

$$V^t(\theta) = -\Big(\frac{ds}{d\theta}\Big)^{-1}\frac{1}{2\pi}VP\int_0^{2\pi} V^n(\varphi)\frac{ds(\varphi)}{d\varphi}\cot\Big(\frac{\theta}{2}-\frac{\varphi}{2}\Big)d\varphi \tag{3.42}$$

The angle θ is function of the arc – length coordinates $\theta = \theta(s)$.

The kernel of $\mathcal{L}$ has a Cauchy-type singularity. Thus the boundary equation (3.22) is substantially singular.

Now we can employ the transformation connected with the averaging on the segments of the airfoil (see [3.22]). Using (3.41) and (3.40) for $\Gamma = 0$ we obtain

$$\int_{s^{j-1}}^{s^j} V^t(s)ds = \int_{\theta^{j-1}}^{\theta^j} V^t(s(\theta))\frac{ds}{d\theta}d\theta = \int_{\theta^{j-1}}^{\theta^j} V_\theta(\theta)d\theta =$$

$$= \int_{\theta^{j-1}}^{\theta^j}\Big\{\frac{1}{2\pi}VP\int_0^{2\pi} V_r(\varphi)\cot\Big(\frac{\theta}{2}-\frac{\varphi}{2}\Big)d\varphi\Big\}d\theta$$

and next with accordance to the integral in the V.P. sense

$$\int_{s^{j-1}}^{s^j} V^t(s)ds = \frac{1}{2\pi}\int_{\theta^{j-1}}^{\theta^j}\Big\{\lim_{\varepsilon\to 0}\Big[\int_0^{\theta-\varepsilon} V_r(\varphi)\cot\Big(\frac{\theta}{2}-\frac{\varphi}{2}\Big)d\varphi +$$

$$+\int_{\theta+\varepsilon}^{2\pi} V_r(\varphi)\cot\Big(\frac{\theta}{2}-\frac{\varphi}{2}\Big)d\varphi\Big]\Big\}d\theta = \frac{1}{2\pi}\int_{\theta^{j-1}}^{\theta^j}\lim_{\varepsilon\to 0}\Big\{2\frac{d}{d\theta}\int_0^{\theta-\varepsilon} V_r'(\varphi)\ln\Big|\sin\frac{\theta-\varphi}{2}\Big|d\varphi - \tag{3.43}$$

$$-2V_r(\theta)\ln\frac{\varepsilon}{2} + 2\frac{d}{d\theta}\int_{\theta+\varepsilon}^{2\pi} V_r(\varphi)\ln\Big|\sin\frac{\theta-\varphi}{2}\Big|d\varphi + 2V_r(\theta)\ln\frac{\varepsilon}{2}\Big\}d\theta =$$

$$= \frac{1}{\pi}\Big[\int_0^{2\pi} V_r(\varphi)\ln\Big|\sin\frac{\theta-\varphi}{2}\Big|d\varphi\Big]_{\theta^{j-1}}^{\theta^j} = \frac{1}{\pi}\int_0^{2\pi} V_r(\varphi)\ln\Big|\frac{\sin\frac{\theta^j-\varphi}{2}}{\sin\frac{\theta^{j-1}-\varphi}{2}}\Big|d\varphi$$

The sign of V.P. disappeared since the kernel of the logarithm is weakly singular and the considered is singular in ordinary sense. The above formula can also be written as

$$\int_{s^{j-1}}^{s^j} = -\frac{1}{\pi}\int_0^{\text{periphery}} V^n(s)\ln\Big|\frac{\sin\frac{\theta(s^j)-\varphi}{2}}{\sin\frac{\theta(s^{j-1})-\varphi}{2}}\Big|ds \tag{3.44}$$

where (3.41) and the fact that $\varphi = \varphi(s)$ were taken into consideration.

4. The algorithm of the numerical simulation

The simulated motion is a free-stream flow around a fixed airfoil. The locations and radii of the blobs created on the boundary are also fixed. Thus the matrix R in (3.32) doesn't change during the simulation process. This allows to invert it only once and to calculate the new circulations from the formula

$$\Gamma = \mathsf{R}^{-1}\mathsf{B} \tag{4.1}$$

Starting from the moment of their creation, blobs with such circulations remain in the flow field and move according to Itô equations (3.18).

The right-hand side B depends on V_0 induced on the boundary by all blobs previously created and still remaining within the flow domain. Thus B is not fixed and should be evaluated in each two-step of the simulation. The Itô equation governing the blobs' movement have the following form

$$\begin{aligned} dx_i &= u(t, x_i, y_i)dt + \sqrt{2\nu}dX \\ dy_i &= v(t, x_i, y_i)dt + \sqrt{2\nu}dY \end{aligned} \tag{4.2}$$

The components of the advective velocity u and v are determined by the velocity field V at (x_i, y_i), where

$$V = V_P + V_\Gamma + V_A + V_{cir} \tag{4.3}$$

V_P and V_Γ are potential velocity fields and can be found easily using classical methods (Γ_0 is taken according to the hypothesis stated in (3.13)). V_{cir} is the velocity induced by all blobs within the flow domain and can be calculated with the use of (3.28). V_A is the additional potential field. Its normal component on the boundary is known from (3.20). It is sufficient to find its tangential component and then to reconstruct the field in any point of the flow domain. dX and dY are stochastic differentials of two independent Wiener processes.

The calculated velocity V allows to determine the time evolution of the vorticity field due to the Itô equations. It also yields the instant velocity field (which is unsteady).

Due to the conditions imposed on the boundary, the normal component of V vanishes on it. Thus, none of the blobs can penetrate the airfoil during its purely advectional motion. However, it is possible due to the existence of a random motion that simulates diffusion. Thee blobs that difuse inside the airfoil are eliminated. It is admissible since the boundary condition is stated for the velocity, not for the vorticity. If the disappearanig of any blob violates the boundary condition, then in the beginning of the next time-step, corresponding circulation will be created in the next generation of blobs on the boundary.

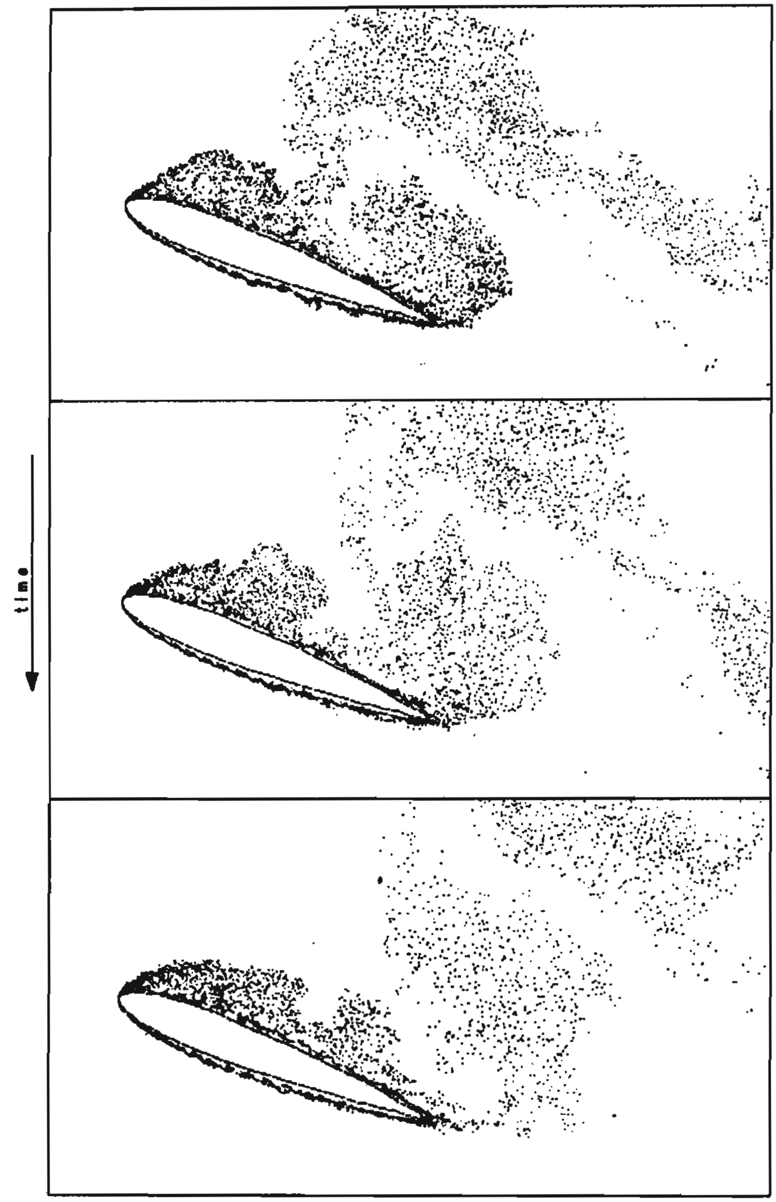

Fig. 4. The large vortex structure shedding on the NACA 0012 airfoil. The instantaneous locations of the vortex blobs centers

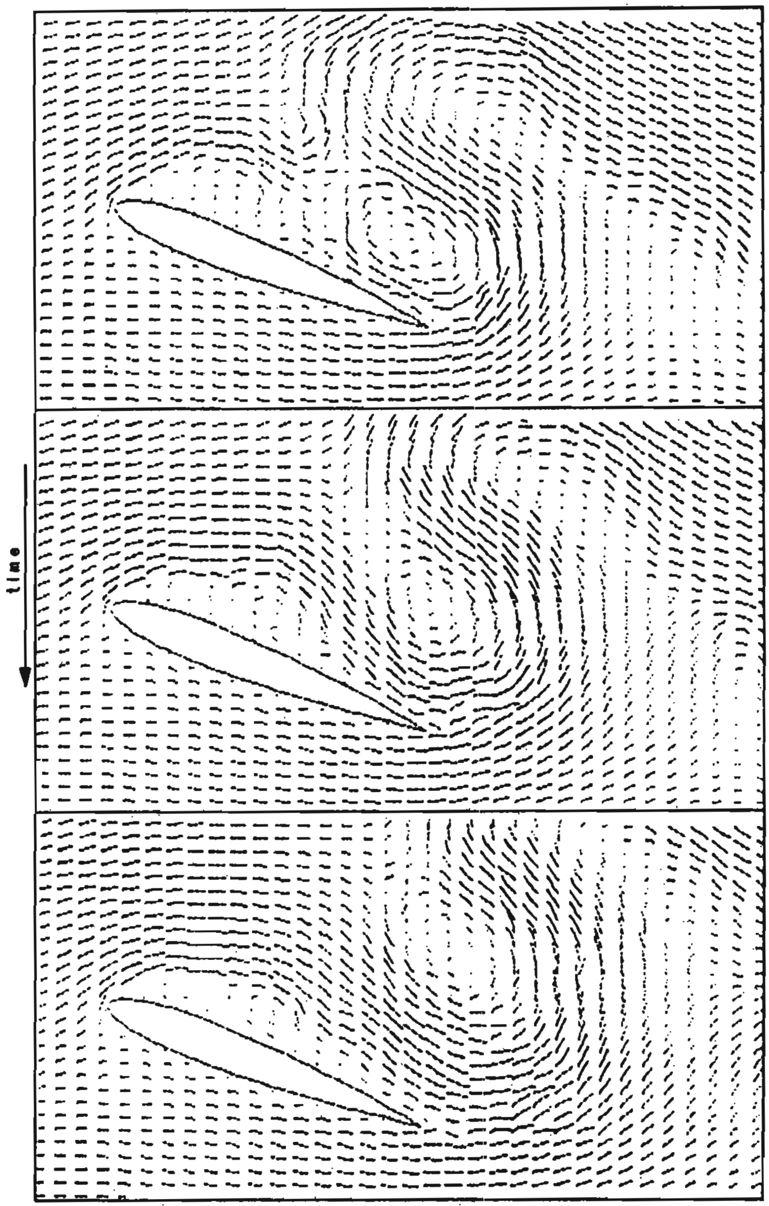

Fig. 5. The large vortex structure shedding on the NACA 0012 airfoil. Velocity field

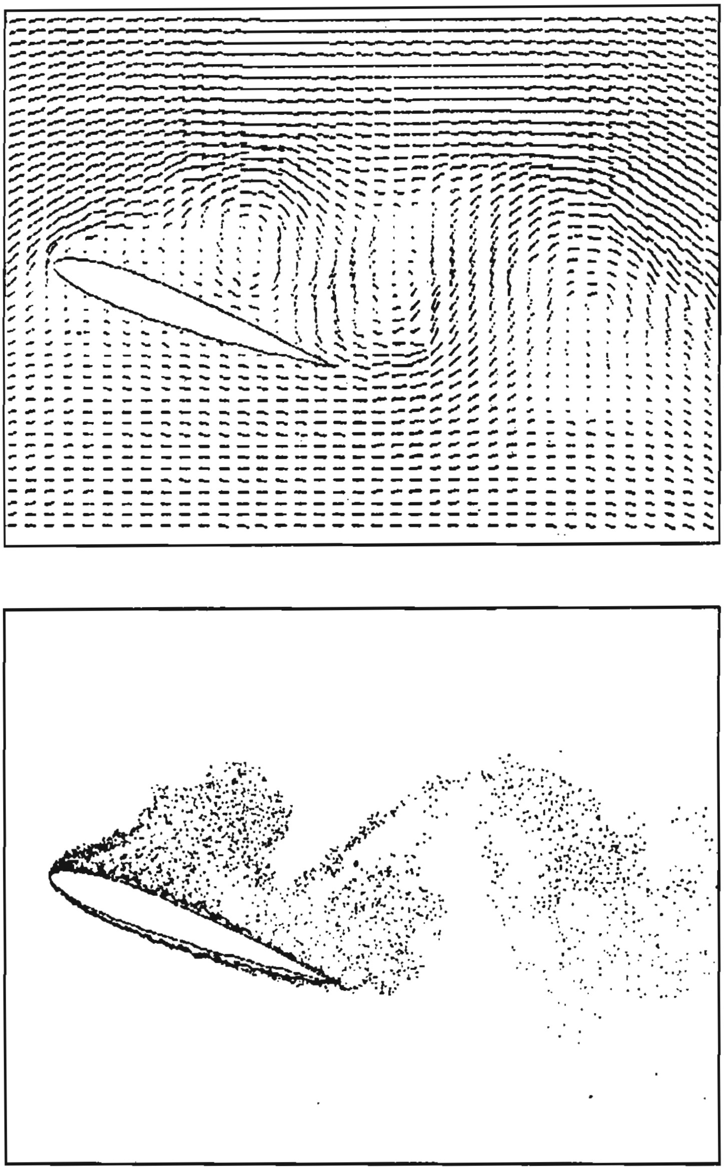

Fig. 6. The locations of vortex blobs and the velocity field at later time

In summary the simulation of the fluid motion starting from some initial state (it can be even purely potential flow) consists of two cyclical processes. At the first stage the circulations of the new generation of blobs (created near the boundary) are found. This mean that the velocity field satisfies the condition (3.2): $V\big|_w = 0$. Having $u(t, x_i, y_i)$ and $v(t, x_i, y_i)$ and the set of independent displacements dX and dY obtained from the random number generator with a $\mathcal{N}(0, \sqrt{\Delta t})$ distribution, we determine increments Δx_i and Δy_i. After elimination of the blobs penetrating the contour and, possibly, the blobs that are far from it we obtain the new vorticity and velocity fields. The later violates the boundary condition, so a new set of boundary blobs (modeling the boundary distribution of the vorticity) should be found. This ends the full cycle of numerical simulation in one time-step.

The method described in this paper has been applied to computing the unsteady flow past an airfoil. The computations showed to be quite successful, producing patterns of the velocity field with a great resemblance, at least qualitatively, to those observed in a real flow. The sample results obtained for a given Reynolds number and an arbitrary angle of incidence are presented in Fig.4 $\div$ 6. The details of the numerical implementation of the random-vortex algorithm will be presented in the next part of the paper.

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Symulacja stochastyczna opływu profilu płynem lepkim

Streszczenie

Rozważa się opływ profilu płynem lepkim nieściśliwym. Pole wirowości określa się w sposób losowy. Tworzy go zbiór poruszających się ruchem losowym i unoszonych niewielkich wirów. Wiry te tworzą się na brzegu opływanego profilu. Rozkład wirowości określający ich intensywność jest rozwiązaniem brzegowego równania całkowego. Warunek asymptotycznej zgodności pola prędkości z polem potencjalnym pozwala stabilizować proces.

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