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ELASTIC-PLASTIC STRAIN ANALYSIS BY PHOTOELASTIC COATING METHOD

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In the paper the method of determining of strain and stress distribution in plastified zones of the construction by using isochromatics pattern obtained in testing by the method of photoelastic coating is presented. Calculation of the strain components (strain separation) requires introduction of both the schematization of the material characteristics and the physical relations valid in regions, where the yield point was exceeded. The introduced relations were applied to the strain separation along the axis of symmetry. The process of calculations was illustrated by an example.

1. Introduction

The method of photoelastic coating may be applied to the analysis of the strain fields, in which the material of the tested area of the construction remaines partly in the elastic state and partly it is plastified. Such a possibility may occur due to the assumption of the linear relation between the photoelastic effect and strain in a wide range.

In the range of strain, where the material of the tested construction is in the state of plastic flow, the material of photoelastic coating still keeps linear characteristics. Therefore it may be concluded, that the constant of photoelastic coating f_ε, determined for the elastic strain, may be applied to the analysis of results for the plastified zones.

In testing of the construction, in which some areas are plastified and the rest of it remains in the elastic range, strain separation proceeds differently in both zones. It is because of different physical connections between stress and strain. The methods of strain separation using isochromatics distribution in the elastic range were described in many publications (cf Kapkowski, 1977 and 1988; Kapkowski and Stupnicki, 1971). Below, the method of determining of the strain components in plastified regions using also isochromatics pattern is presented. It is the development of the procedure introduced in the monograph by Kapkowski et al. (1987).

Notation

α	–	angle between the tangent to the border at the boundary point and the x axis
α_n	–	angle between the direction of the greater principal strain component and the x axis
e	–	half of the sum of the principal strain components
E	–	modulus of elasticity
E_p	–	modulus of strain hardening (bilinear model)
ε_1, ε_2	–	principal strain components
ε_0	–	first plastic strain (bilinear model – simple tension)
f_ε	–	strain constant of the photoelastic coating
k_1, k_2, k_{pl}	–	plastification coefficients
N	–	value of the isochromatic order
N_{pl}	–	value of the isochromatic order at the moment of entering into the plastic state
ν	–	Poisson ratio
σ_0	–	yield point of the material under simple tension (bilinear model)
σ_1, σ_2	–	principal stress components
$(\sigma_1)_K$, $(\sigma_2)_K$	–	principal stress components in the point of achievement of the plastic strain for plane stress (bilinear model)
σ_t, σ_b, τ_b	–	boundary loading
i	–	index denoting every intermediate overelastic range (multisectional model)
m	–	index denoting the overelastic range at the calculation point (multisectional model)
b	–	index denoting the boundary point
$(K_1)_i$, $(K_2)_i$	–	index denoting the point of conversion from the i overelastic state to the state $i+1$ for plane stress (multisectional model)

$(K)_i$ – index denoting the point of conversion from the i overelastic state to the state $i+1$ for simple tension (multisectional model)

2. Schematization of the material characteristics

The strain separation within the elastic range of material does not cause difficulties, because the relations between stress and strain have simple structure. However in regions, where not only elastic deformations appear, these relations, corresponding with the accepted theory of plasticity, are much less universal. Therefore the results of analysis of the strain and stress field in plastified zones may be different depending on the accepted hypothesis and theories of nonelastic strain.

One of the methods of description of the material behaviour in the nonelastic range is the schematization of the material characteristics (a relationship between stress and strain). This term defines the curve simplification and the description of such constructed scheme by analytical functions.

The first, practicaly useful trial of the schematization of the material characteristics, was the introduction of the bilinear model (Kapkowski et al., 1987), which approximately corresponds with the behaviour of many real materials, e.g. aluminum alloys.

The charakteristics of bilinear material consists of two line segments intersected at the point of the achievement of plastic strain K, which is defined by coordinates $(\sigma_0, \varepsilon_0)$ (Fig.1).

Parameter σ_0 determines the yield point of the material under simple tension, while ε_0 is the strain, which then occurs at the point of achievement of the first plastic strain.

In the first range $(0 \le \varepsilon \le \varepsilon_0)$ the material satisfies the Hooke's law, the modulus of elasticity is E and the Poisson ratio is ν. In the second range $(\varepsilon \ge \varepsilon_0)$ the modulus of elasticity changes for E_p (the modulus of strain hardening) and also the Poisson ratio assumes the value of $\nu_p = 0.5$. The acceptance of such a kind of the material characteristics permits making the assumption that the principal directions of strain components are invariable beyond the yield point. Experiments made on n duralumin samples (Aleksandrov and Akhmetzyanov, 1973) show that the changes of the principal directions of strain components are negligible.

In the case of two-dimensional state of stress, the relationship $\sigma(\varepsilon)$ for both principal components of thestress tensor (σ_1, σ_2) may proceed in different manner. That is because the enter into the plastic state may occur under different value of the σ_1/σ_2 ratio. For later considerations it was assumed that the entering of

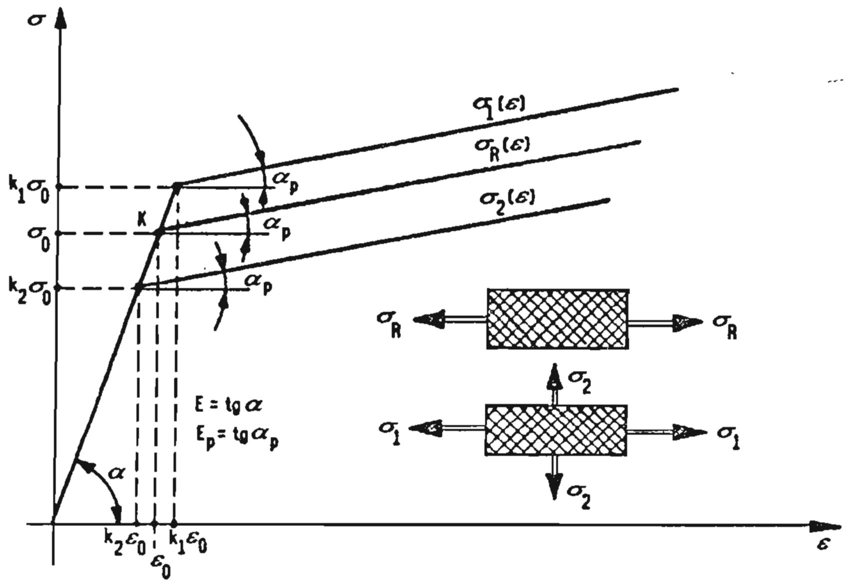

Fig. 1.

the material into the plastic state occurs at the moment, when the principal stress components have the following values

$$(\sigma_1)_K = k_1\sigma_0$$
$$(\sigma_2)_K = k_2\sigma_0 \tag{2.1}$$

After crossing the yield point in the considered (infinitesimal) element, the characteristics for particular stress components will proceed linearly, while the slope of these straight lines will correspond with the modulus of strain hardening (Fig.1) and also the Poisson ratio will change its value to $\nu_p = 0.5$.

The coefficients k_1, k_2 satisfy the Huber-Mises yield criterion

$$k_1^2 - k_1k_2 + k_2^2 = 1 \tag{2.2}$$

Introducing the coefficient

$$k_{pl} = \frac{k_1 - k_2}{2} \tag{2.3}$$

and taking under consideration Eq (2.2), the coefficients k_1, k_2 may be described as

$$k_1 = \sqrt{1 - 3k_{pl}^2} + k_{pl}$$
$$k_2 = \sqrt{1 - 3k_{pl}^2} - k_{pl} \tag{2.4}$$

The coefficient k_{pl} is connected by a simple relationship with the value of isochromatic order N_{pl}, which occurs at the considered point at the moment of the entering into the plastic state.

When the Huber-Mises yield criterion is satisfied at the considered point, the difference of the principal stress components may be expressed by the basic relationship of photoelasticity

$$k_1\sigma_0 - k_2\sigma_0 = N_{pl} f_\varepsilon \frac{E}{1+\nu} \tag{2.5}$$

In the above formula, by E and ν the modulus of elasticity and the Poisson ratio of the material of the tested construction are destignated, respectively, while f_ε is the strain constant of the photoelastic coating. Transforming the relationship (2.5) and including Eq (2.3), the following formula for the coefficient k_{pl} can be obtained

$$k_{pl} = \frac{E}{2(1+\nu)} \frac{f_\varepsilon}{\sigma_0} N_{pl} \tag{2.6}$$

This coefficient is the function of coordinates x, y in the area under consideration and its value is proportional to the isochromatics order N_{pl}.

Thus recapitulating the foregoing considerations it may be proved, that for the accepted material model, the determination of the stress components in the plastified zone requires the knowledge of the coefficient k_{pl} distribution in the investigated area. This coefficient is connected by the simple relationship (2.6) with the value of isochromatics order, which occurs at the considered point at the moment of meeting the yield criterion. The coefficient k_{pl} should be determined by the experimental testing of the plastifying process on the certain intermediate loading levels.

The relationship between stress and strain components for the bilinear material was presented in the monograph by Kapkowski et al. (1987).

The bilinear model of the material, though it approximately corresponds with the behaviour of many real construction materials, however is it not the universal model. The divergences occur especially at the point K (Fig.1), corresponding with the moment of the entering into the plastic state – the moment important from the point of view of the construction work-condition. The assumption of the multisectional model leads to the reduction of the above divergences and gives the possibility of better fitting the mathematical model with the real material characteristics.

In this model the $\sigma(\varepsilon)$ curve is replaced by n line segments (Fig.2).

Each of these segments describes the different state of the material and is characterized by the different modulus of elasticity E_i and the Poisson ratio ν_i. The points K_i $(i = 1 \div (n-1))$ are every time fitted for each material on the grounds of the real stress-strain curve.

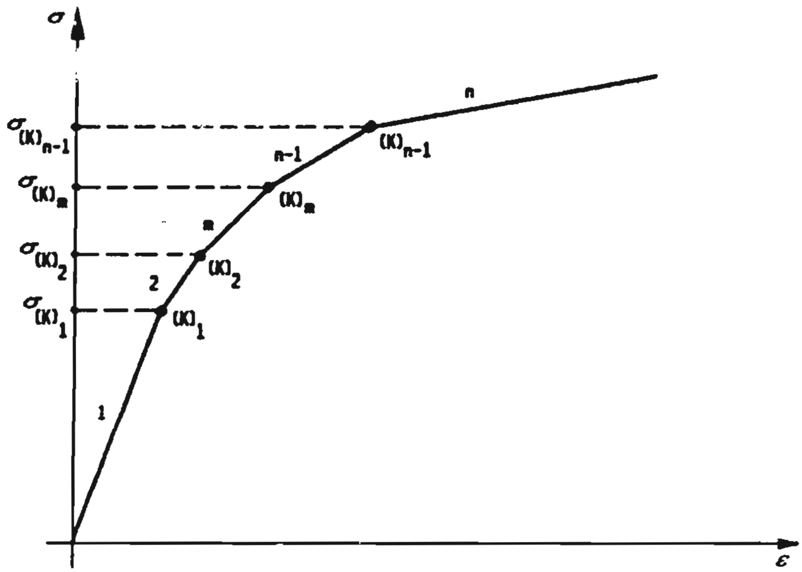

Fig. 2.

The multisectional schematization of the material characteristics requires making the assumptions refering to the course of the deformation process after crossing the yield point. They are as follows

- The $\sigma(\varepsilon)$ relationship has the same character for the both principal directions. At the succesive stages of the nonelas tic state of the material, the modulus of elasticity and the Poisson ratio for both stress components have the same value as under the uniaxial tension. It means, that the succesive seg ments of the characteristics $\sigma(\varepsilon)$, $\sigma_1(\varepsilon_1)$, $\sigma_2(\varepsilon_2)$ are pa rallel (Fig.3).
- The strain hardening of the material has the isotropic cha racter and the change of the material state takes place at the constant ratio σ_2/σ_1. It corresponds with the rectilinear way of turning into the plastic state illustrated on the (σ_1, σ_2) plane (Fig.4).

It appears, that the consequence of these assumptions is the characteristics of the plastifying process shown in Fig.3. It is provable, that three points describing the succeeding changes of the material state (e.g. $(K_1)_2$, $(K)_2$, $(K_2)_2$) lie on half lines starting from the coordinate system origin.

Similarly as for the bilinear model, the coefficients $(k_1)_i$, $(k_2)_i$ characterizing the values of the principal stress components at the moment of change of the material state are introduced. Taking into consideration these designations, the

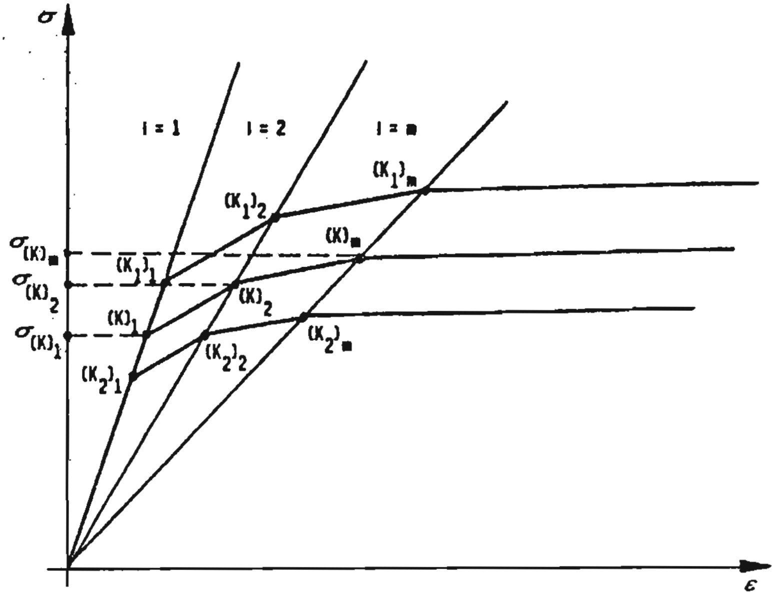
σ
ε
i = 1
i = 2
i = m
(K1)m
(K1)2
(K)m
σ(K)m
(K1)1
σ(K)2
(K)2
(K)1
σ(K)1
(K2)1
(K2)2
(K2)m

Fig. 3.

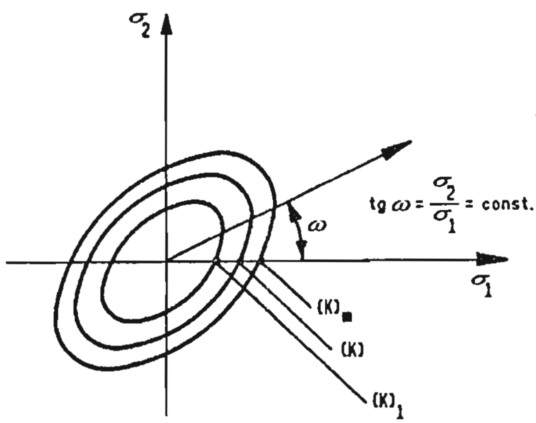
σ2
σ1
ω
tg ω = σ2/σ1 = const.
(K)m
(K)
(K)1

Fig. 4.

principal stress components at the moment of the conversion from the state i to the state $i+1$ (the points $(K_1)_i$ and $(K_2)_i$) may be written as

$$
\begin{aligned}
(\sigma_1)_{(K_1)_i} &= (k_1)_i \sigma_{(K)_i} \\
(\sigma_2)_{(K_2)_i} &= (k_2)_i \sigma_{(K)_i}
\end{aligned}
\tag{2.7}
$$

Then, the assumption II may be presented in the following form

$$
\frac{(k_1)_1}{(k_2)_1} = \frac{(k_1)_2}{(k_2)_2} = \dots = \frac{(k_1)_i}{(k_2)_i} = \dots = \frac{(k_1)_{n-1}}{(k_2)_{n-1}}
\tag{2.8}
$$

Making the appropriate transformations, it may be proved, that (when the principal stress components at the points $(K)_i$ are defined by the relationship (2.7)), the coefficients $(k_1)_i$, $(k_2)_i$ satisfy the relationships

$$
\begin{aligned}
(k_1)_1 &= (k_1)_2 = \dots = (k_1)_i = \dots = (k_1)_{n-1} = k_1 \\
(k_2)_1 &= (k_2)_2 = \dots = (k_2)_i = \dots = (k_2)_{n-1} = k_2
\end{aligned}
\tag{2.9}
$$

That is a very important conclusion for the further application of the formulas (2.7). It means, that for determining parameters at the particular stages of the change of material state, the knowledge of only one pair of the coefficients k_1 and k_2, is required. These coefficients determine the stress state at the moment of apperance of the first overelastic strains. Because these coefficients are determined experimentally, it considerably simplifies the experimental procedure.

The possibility of replacing the real material characteristics by the multisectional mathematical model, gives it the universal character.

As it appears from the relationship (2.9), increasing of the real curve $\sigma(\varepsilon)$ approximation accuracy in terms of the greater number of segments does not cause the necessity for determination of the additional coefficients k_1 and k_2 values.

3. Physical relations

In the chapter below, the relationships between stress and strain components called the physical relations are discussed. They are necessary to analyse the results of the experiment realized by the method of photoelastic coating.

3.1. Physical relations within the elastic range

For the elastic range of the material work (the first segment of the characteristic, $i = 1$, Fig.3) the relationships resulted from the Hooke's law for two-dimensional state of stress may be formulated as

$$
\begin{aligned}
(\varepsilon_1)_1 &= \frac{1}{E_1}\Big[(\sigma_1)_1 - \nu_1(\sigma_2)_1\Big] \\
(\varepsilon_2)_1 &= \frac{1}{E_1}\Big[(\sigma_2)_1 - \nu_1(\sigma_1)_1\Big]
\end{aligned}
\tag{3.1}
$$

At the moment of the material passing into the nonelastic state (points $(K_1)_1$, $(K_2)_1$ in Fig.3), the stress components are defined by the formulas (2.7) for $i = 1$. In that case the strain components may be expressed by the relationship

$$
\begin{aligned}
(\varepsilon_1)_{(K_1)_1} &= \frac{\sigma_{(K)_1}}{E_1}\Big[(k_1)_1 - \nu_1(k_2)_1\Big] \\
(\varepsilon_2)_{(K_2)_1} &= \frac{\sigma_{(K)_1}}{E_1}\Big[(k_2)_1 - \nu_1(k_1)_1\Big]
\end{aligned}
\tag{3.2}
$$

3.2. Physical relations in the nonelastic range

After the material passing into the nonelastic state (the second segment of the characteristics, $i = 2$, Fig.3) the relationship between strain and stress components remains the straight line, but of different slope, corresponding with the modulus E_2. The nonelastic part of the components of the strain tensor is determined similarly like in the elastic range, considering separately the stress effect in the directions 1 and 2 and using the principle of superposition.

In the case of the effect of stress component σ_1, nonelastic parts of the components of strain tensor are expressed by the formulas (according to designations in Fig.5)

$$
\begin{aligned}
(\varepsilon_1)'_{ps2} &= (\varepsilon_1)'_2 - (\varepsilon_1)'_{(K_1)_1} = \frac{(\sigma_1)_2}{E_2} - \frac{(k_1)_1\sigma_{(K)_1}}{E_2} \\
(\varepsilon_2)'_{ps2} &= -\nu_2(\varepsilon_1)'_{ps2} = -\nu_2\Big[\frac{(\sigma_1)_2}{E_2} - \frac{(k_1)_1\sigma_{(K)_1}}{E_2}\Big]
\end{aligned}
\tag{3.3}
$$

In analogous way may be formulated nonelastic parts of the components of strain tensor connected with the effect of stress component σ_2

$$\begin{aligned}(\varepsilon_2)''_{ps2} &= (\varepsilon_2)''_2 - (\varepsilon_2)''_{(K_2)_1} = \frac{(\sigma_2)_2}{E_2} - \frac{(k_2)_1\sigma_{(K)_1}}{E_2} \\ (\varepsilon_1)''_{ps2} &= -\nu_2(\varepsilon_2)''_{ps2} = -\nu_2\Big[\frac{(\sigma_2)_2}{E_2} - \frac{(k_2)_1\sigma_{(K)_1}}{E_2}\Big]\end{aligned} \tag{3.4}$$

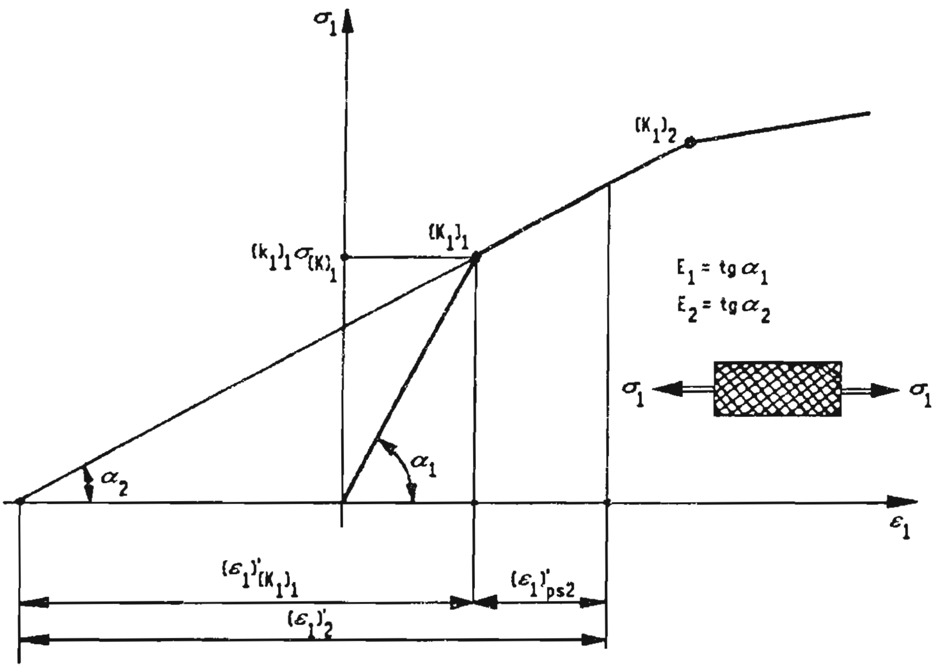

Fig. 5.

Summing up the corresponding expressions (3.3) and (3.4), the formulas describing nonelastic parts of the components of strain tensor in the two-dimensional stress state σ_1, σ_2 are obtained

$$\begin{aligned}(\varepsilon_1)_{ps2} &= (\varepsilon_1)'_{ps2} + (\varepsilon_1)''_{ps2} = \\ &= \frac{1}{E_2}\Big\{\Big[(\sigma_1)_2 - \nu_2(\sigma_2)_2\Big] - \sigma_{(K)_1}\Big[(k_1)_1 - \nu_2(k_2)_1\Big]\Big\} \\ (\varepsilon_2)_{ps2} &= (\varepsilon_2)'_{ps2} + (\varepsilon_2)''_{ps2} = \\ &= \frac{1}{E_2}\Big\{\Big[(\sigma_2)_2 - \nu_2(\sigma_1)_2\Big] - \sigma_{(K)_1}\Big[(k_2)_1 - \nu_2(k_1)_1\Big]\Big\}\end{aligned} \tag{3.5}$$

To get the physical relations between the total strain $(\varepsilon_1, \varepsilon_2)$ and stress (σ_1, σ_2), the superposition of the strain states described by the formulas (3.1) and (3.5) should be carried out. After realizing a number of transformations the following expressions are obtained

$$
\begin{aligned}
(\varepsilon_1)_2 &= \frac{1}{E_2}\Big[(\sigma_1)_2 - \nu_2(\sigma_2)_2\Big] + \\
&+ \sigma_{(K)_1}\Big\{\frac{1}{E_1}\Big[(k_1)_1 - \nu_1(k_2)_1\Big] - \frac{1}{E_2}\Big[(k_1)_1 - \nu_2(k_2)_1\Big]\Big\} \\
(\varepsilon_2)_2 &= \frac{1}{E_2}\Big[(\sigma_2)_2 - \nu_2(\sigma_1)_2\Big] + \\
&+ \sigma_{(K)_1}\Big\{\frac{1}{E_1}\Big[(k_2)_1 - \nu_1(k_1)_1\Big] - \frac{1}{E_2}\Big[(k_2)_1 - \nu_2(k_1)_1\Big]\Big\}
\end{aligned}
\tag{3.6}
$$

It has to be noticed that the above formulas are valid within the first overelastic range $(i = 2)$, what is marked by 2 index at the values ε_1, ε_2, σ_1, σ_2.

In the analogous way the total strain components may be calculated for the succeeding stages of the material overelastic state which correspond to particular segments of the characteristics $(i = 3, 4, ..., m, ..., n-1)$ (Fig.2). The generalized relationships between the total strain and stress components in the m segment of the characteristics are as follows

$$
\begin{aligned}
(\varepsilon_1)_m &= \frac{1}{E_m}\Big[(\sigma_1)_m - \nu_m(\sigma_2)_m\Big] + \\
&+ \sum_{i=2}^{m}\sigma_{(K)_{i-1}}\Big\{\frac{1}{E_{i-1}}\Big[(k_1)_{i-1} - \nu_{i-1}(k_2)_{i-1}\Big] - \frac{1}{E_i}\Big[(k_1)_{i-1} - \nu_i(k_2)_{i-1}\Big]\Big\} \\
(\varepsilon_2)_m &= \frac{1}{E_m}\Big[(\sigma_2)_m - \nu_m(\sigma_1)_m\Big] + \\
&+ \sum_{i=2}^{m}\sigma_{(K)_{i-1}}\Big\{\frac{1}{E_{i-1}}\Big[(k_2)_{i-1} - \nu_{i-1}(k_1)_{i-1}\Big] - \frac{1}{E_i}\Big[(k_2)_{i-1} - \nu_i(k_1)_{i-1}\Big]\Big\}
\end{aligned}
\tag{3.7}
$$

To analyze the results obtained by the method of photoelastic coating it is more convenient to use the sum and the difference of the principal strain or stress components. These relations calculated on the base of the formulas (3.7) (taking into account (2.9) and (2.4)) have the following form

$$
\begin{aligned}
(\varepsilon_1 + \varepsilon_2)_m &= (1 - \nu_m)\Big[\frac{1}{E_m}(\sigma_1 + \sigma_2)_m + 4\sqrt{1 - 3k_{pl}^2}\,(w_S)_m f_\varepsilon\Big] \\
(\varepsilon_1 - \varepsilon_2)_m &= (1 + \nu_m)\Big[\frac{1}{E_m}(\sigma_1 - \sigma_2)_m + 4k_{pl}(w_R)_m f_\varepsilon\Big]
\end{aligned}
\tag{3.8}
$$

where the new designation have been introduced

$$
\begin{aligned}
(w_S)_m &= \frac{1}{2f_e(1-\nu_m)}\sum_{i=2}^{m}\sigma_{(K)_{i-1}}\Big[\frac{1-\nu_{i-1}}{E_{i-1}}-\frac{1-\nu_i}{E_i}\Big]\\
(w_R)_m &= \frac{1}{2f_e(1+\nu_m)}\sum_{i=2}^{m}\sigma_{(K)_{i-1}}\Big[\frac{1+\nu_{i-1}}{E_{i-1}}-\frac{1+\nu_i}{E_i}\Big]
\end{aligned}
\tag{3.9}
$$

The inverse to (3.8) relations can be written in the form

$$
\begin{aligned}
(\sigma_1+\sigma_2)_m &= E_m\Big[\frac{1}{1-\nu_m}(\varepsilon_1+\varepsilon_2)_m+4\sqrt{1-3k_{pl}^2}\,(w_S)_m f_e\Big]\\
(\sigma_1-\sigma_2)_m &= E_m\Big[\frac{1}{1+\nu_m}(\varepsilon_1-\varepsilon_2)-4k_{pl}(w_R)_m f_e\Big]
\end{aligned}
\tag{3.10}
$$

It has to be noticed that the relationships (3.7), (3.87) and (3.10) have the universal character. If in Eq (3.10) the coefficients $(w_R)_m$ and $(w_S)_m$ are equal to zero, the physical relations proper for the elastic range are obtained (compare with the formulas (3.1)). These terms are then equal to zero, because the summation starts from the segment $i = 2$ of the characteristics, while the elastic segment is designated by the number $i = 1$.

It should be noticed also, that the coefficients of Eq (3.9) depend on the constants of the material only. They can be calculated "a priori" before the separation of strains.

4. Application of the obtained physical relations – the method of the strain separation on the axis of symmetry

In many practical cases the region tested by the method of photoelastic coating has the geometric and loading plane of symmetry (Fig.6). In this case the calculation of the strain components on the axis of symmetry (which is the line of intersection of the plane of symmetry and the plane of photoelastic coating) is simplyfied. To illustrate the making use of the previously derived physical relations, they were applied to the strain separation in this simplyfied case.

4.1. Basic equations

The coordinate system, in which the axis x stands for the axis of symmetry is accepted.

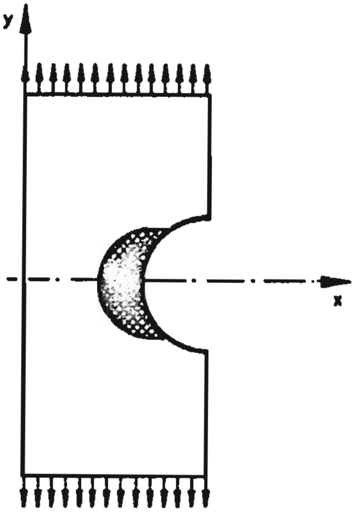

Fig. 6.

In this case for the strain separation only the isochromatic pattern is used, since the directions of the principal strain components in the points of the axis of symmetry are known.

The method of the strain separation on the axis of symmetry in the elastic range of the material work was described in the paper by Kapkowski (1977).

Below, the case in which some points on the axis of symmetry may lay in the elastic region and others in the nonelastic zone, is considered. This is schematically shown in Fig.6 (the nonelastic zone is marked by hatching).

Therefore the equations used for the strain separation were derived for the general case – the multisectional model of the material (Fig.2). The method of deriving of these equations is analogous to the one presented by Kapkowski (1977). The physical relations (3.10) are introduced to the element equilibrium equations. After making adequate transformations, these equations assume the form

$$\begin{aligned}
&\frac{1}{1-\nu_m}\frac{\partial}{\partial x}\Big(\frac{e}{f_e}\Big) + \frac{1}{2(1+\nu_m)}\Big[\frac{\partial}{\partial x}(N\cos 2\alpha_n) + \frac{\partial}{\partial y}(N\sin 2\alpha_n)\Big] = \\
&= 2(w_S)_m\frac{\partial}{\partial x}\Big(\sqrt{1-3k_{pl}^2}\Big) + 2(w_R)_m\Big[\frac{\partial}{\partial x}(k_{pl}\cos 2\alpha_n) + \frac{\partial}{\partial y}(k_{pl}\sin 2\alpha_n)\Big] \\
&\frac{1}{1-\nu_m}\frac{\partial}{\partial y}\Big(\frac{e}{f_e}\Big) + \frac{1}{2(1+\nu_m)}\Big[\frac{\partial}{\partial x}(N\sin 2\alpha_n) - \frac{\partial}{\partial y}(N\cos 2\alpha_n)\Big] = \\
&= 2(w_S)_m\frac{\partial}{\partial y}\Big(\sqrt{1-3k_{pl}^2}\Big) + 2(w_R)_m\Big[\frac{\partial}{\partial x}(k_{pl}\sin 2\alpha_n) - \frac{\partial}{\partial y}(k_{pl}\cos 2\alpha_n)\Big]
\end{aligned} \tag{4.1}$$

Now, the new variable is established

$$b(x) = \frac{\partial}{\partial y}(M \sin 2\alpha_n) \tag{4.2}$$

together with the designations

$$\begin{aligned} M &= \frac{N}{2} - (1 - \nu_m^2)(w_R)_m k_{pl} \\ K &= (1 - \nu_m^2)(w_S)_m \sqrt{1 - 3k_{pl}^2} \end{aligned} \tag{4.3}$$

The quantities ε_x and ε_y are the principal strain components on the axis x, so along this axis: $\partial \alpha_n/\partial x = 0$. Besides, because of the symmetry, $\partial N/\partial y = 0$.

Taking the above properties into account, the quality (e/f_e) may be eliminated from Eq (4.1) by means of the strain compatibility condition.

Then, the only one equation in the unknown function $b(x)$ is obtained

$$\begin{aligned} \frac{db}{dx} &= \frac{1}{2}\Big(\frac{\partial^2 M}{\partial y^2} - \frac{\partial^2 M}{\partial x^2}\Big) \cos 2\alpha_n - \frac{b^2}{2M \cos 2\alpha_n} - \\ &- M \sin 2\alpha_n \frac{\partial^2 \alpha_n}{\partial y^2} + \frac{1}{2}\Big(\frac{\partial^2 K}{\partial x^2} + \frac{\partial^2 K}{\partial y^2}\Big) \end{aligned} \tag{4.4}$$

After determining from the above equation the function $b(x)$, the function (e/f_e) may be calculated from one of the equilibrium equations (4.1) (e.g. from the first one). After making adequate transformations and taking the strain field properties on the axis of symmetry into account, the equilibrium equation may be transformed into the form

$$\frac{d}{dx}\Big(\frac{e}{f_e}\Big) = \frac{2}{1+\nu_m}\Big[\frac{\partial K}{\partial x} - \frac{\partial}{\partial x}\Big(M - \frac{1+\nu_m}{4}N\Big)\cos 2\alpha_n - \frac{b}{M}\Big(M - \frac{1+\nu_m}{4}N\Big)\Big] \tag{4.5}$$

Further, taking under consideration the fact, that on the axis of symmetry the directions of the principal strain components are known $(\alpha_n = 0$ or $\alpha_n = \pi/2)$ Eqs (4.4) and (4.5) may be rewritten as

$$\frac{db}{dx} = \pm\Big(\frac{\partial^2 M}{\partial y^2} - \frac{\partial^2 M}{\partial x^2}\Big) \mp \frac{b^2}{2M} + \frac{1}{2}\Big(\frac{\partial^2 K}{\partial x^2} + \frac{\partial^2 K}{\partial y^2}\Big) \tag{4.6}$$

$$\frac{d}{dx}\Big(\frac{e}{f_e}\Big) = \frac{2}{1+\nu_m}\Big[\frac{\partial K}{\partial x} \mp \frac{\partial}{\partial x}\Big(M - \frac{1+\nu_m}{4}N\Big) - \frac{b}{M}\Big(M - \frac{1+\nu_m}{4}N\Big)\Big] \tag{4.7}$$

In the above equations superscripts refer to the case, when the greater principal strain component overlaps the direction of axis of symmetry $(\alpha_n = 0)$, while the lower ones refer to the case when $\alpha_n = \pi/2$.

Substituting in Eqs (4.6) and (4.7) for $(w_R)_m = (w_S)_m = 0$, the relationships adequate for the elastic region are obtained (see Kapkowski, 1977).

So the strain separation at the points of the axis of symmetry, both in the elastic region and in the plastified one, resolves itself into successive solving Eqs (4.6) and (4.7) with the appropriate boundary conditions.

4.2. Boundary conditions

For the equations (4.6) and (4.7) the boundary conditions at the point, from which starts the solution should be formulated. Analogously as Kapkowski et al. (1987), we establish that, the stress state in the boundary element lying on the axis of symmetry has to be considered (Fig.7 – the axis x is the axis of symmetry).

It is assumed, that the shape of the border is described by the function $y = f(x)$ and on the border may act the normal loading σ_b and the tangent one τ_b.

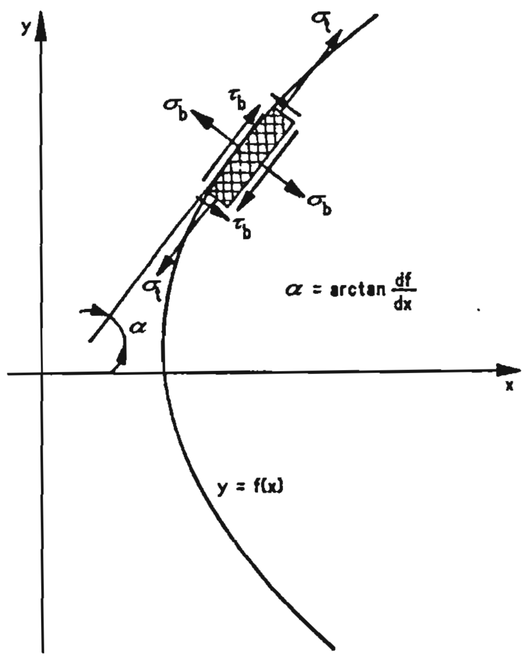

Fig. 7.

The boundary loading is connected with the principal stress components at this point: $(\sigma_1)_b$ and $(\sigma_2)_b$ by the relationships

$$\left.\begin{array}{l}\sigma_t\\ \sigma_b\end{array}\right\} = \frac{(\sigma_1+\sigma_2)_b}{2} \pm \frac{(\sigma_1-\sigma_2)_b}{2}\cos 2[(\alpha_n)_b - \alpha]$$

$$\tau_b = -\frac{(\sigma_1-\sigma_2)_b}{2}\sin 2[(\alpha_n)_b - \alpha] \tag{4.8}$$

After introduction the physical relations (3.10) and taking (2.4) into consideration, it is possible to calculate from the relationship (4.8) at the boundary point (with the known boundary loading) the following quantities

— the angle between the direction of the greater principal strain component and the axis x

$$(\alpha_n)_b = \alpha - \frac{1}{2}\arcsin\Big[\frac{\tau_b}{\frac{E_m}{2(1+\nu_m)}N_b f_\varepsilon - 2E_m f_\varepsilon (w_R)_m (k_{pl})_b}\Big] \tag{4.9}$$

— half of the sum of the principal strain components

$$\Big(\frac{e}{f_\varepsilon}\Big)_b = \frac{1-\nu_m}{E_m f_\varepsilon}\Big\{\sigma_b + 2E_m f_\varepsilon (w_S)_m\sqrt{1-3(k_{pl})_b^2} + \sqrt{\Big[\frac{E_m f_\varepsilon}{2(1+\nu_m)}N_b - 2E_m f_\varepsilon (w_R)_m (k_{pl})_b\Big]^2 - \tau_b^2}\Big\} \tag{4.10}$$

where N_b is the isochromatic order at the boundary point.

Above relationships enable us to calculate the boundary quantities for Eqs (4.6) and (4.7).

According to (4.2) the value of b_b should be calculated at the starting point. After introduction of the expression (4.9), taking (4.3) into account and making some transformations it is obtained

$$(b)_b = -\Big[N_b - 2(1-\nu_m^2)(w_R)_m (k_{pl})_b\Big]\cdot \Big[\Big(\frac{d^2x}{dy^2}\Big)_b + \frac{1+\nu_m}{E_m}\frac{1}{N_b - 4(1-\nu_m)(w_R)_m (k_{pl})_b}\frac{\partial}{\partial y}\Big(\frac{\tau}{f_\varepsilon}\Big)_b\Big] \tag{4.11}$$

In the above equation $x = f(y)$ is the inverse function of the function describing the border shape.

The boundary condition for Eq (4.7) is obtained directly from Eq (4.10) by introducing $\tau_b = 0$. After transformations this condition assumes the form

$$\Big(\frac{e}{f_\varepsilon}\Big)_b = \frac{1-\nu_m}{E_m f_\varepsilon}\Big[\sigma_b + \frac{E_m f_\varepsilon}{2(1+\nu_m)}N_b\Big] + 2(1-\nu_m)\Big[(w_S)_m\sqrt{1-3(k_{pl})_b^2} - (w_R)_m (k_{pl})_b\Big] \tag{4.12}$$

It should be noticed, that the boundary conditions (4.11) and (4.12), similarly as Eqs (4.6) and (4.7) describe the elastic range, if only $(w_R)_m = (w_S)_m = 0$ is assumed.

4.3. Example

To illustrate above method, the strain separation along the horizontal axis of symmetry AB of the two-dimensional model showed in Fig.8.

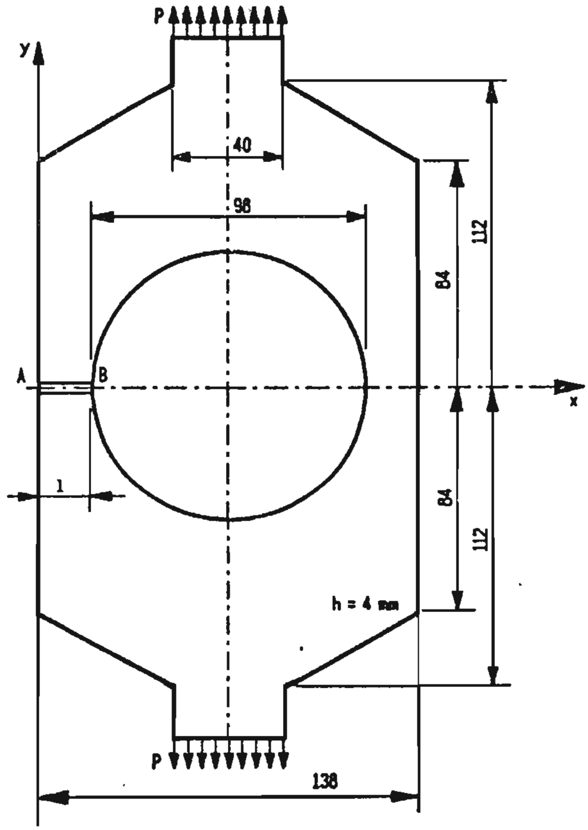

Fig. 8.

The model was made of duralumin sheet 4.0 mm thick. After polishing and etching the model was covered (to avoid bending effect) with the photoelastic coating 2 mm thick made of epoxy resin on both sides. The model was loaded within the overelastic range of material and under various levels of loading the photogra-

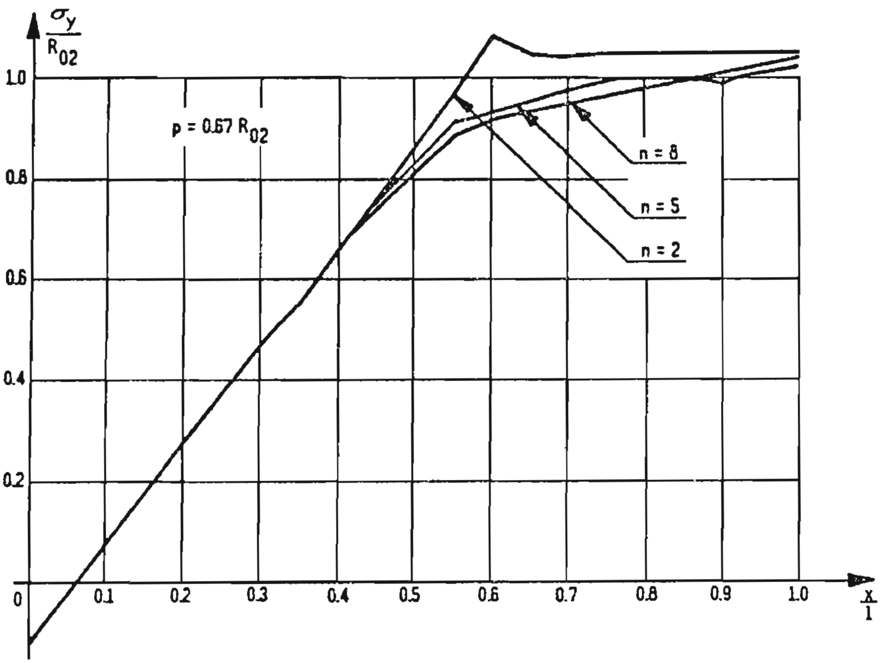
σy / R02
1.0
0.8
0.6
0.4
0.2
0
p = 0.67 R02
n = 8
n = 5
n = 2
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
x/l

Fig. 9.

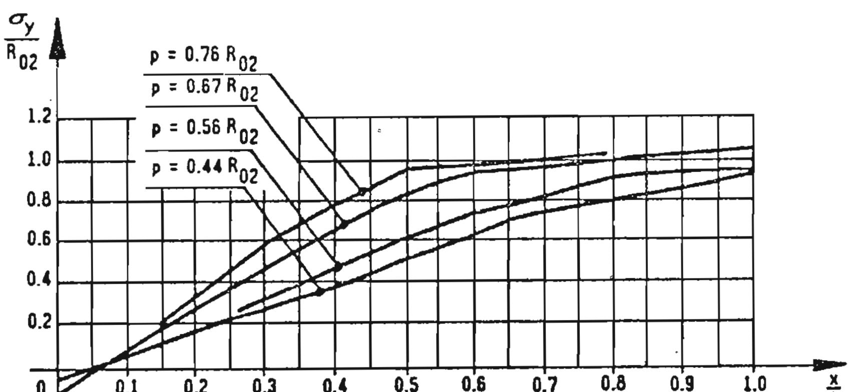
σy / R02
p = 0.76 R02
p = 0.67 R02
p = 0.56 R02
p = 0.44 R02
1.2
1.0
0.8
0.6
0.4
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
x/l

Fig. 10.

phs of isochromatic pattern were taken. On the base of these pictures the $b(x)$ function was determined enabling the $e/f_e(x)$ function along the axis of symmetrydetermination is then possible to calculate all the strain and stress components along the axis of symmetry. To determine necessary derivatives the finite-difference method was applied while to numerical integration the Runge-Kutta method was used. Because the loading of the model caused its transition into the overelastic state, it was necessary to define the multisectional model of the material characteristics. On the base of experimentally determined characteristics the adequate mathematical model was accepted. To prove the effect of approximation of the real material characteristics on the accuracy of results, three different mathematical models were accepted: 2-sectional (bilinear) model, 5-sectional model and 8-sectional model (see Fig.2).

To complete calculations it was also necessary to determine the values of the coefficients k_1 and k_2 along the axis of symmetry. In the first approximation it was assumed that: $k_1 = 1$ and $k_2 = 0$ (the direction 1 is perpendicular to the axis of symmetry x and the direction 2 – along the axis x). As it was shown after doing some calculations for different cases of increasing loading, real coefficients k_1 and k_2 are approximately constant along the axis of symmetry and their average values are very close to the assumed ones ($k_1 = 1$ and $k_2 = 0$).

The calculations for different cases (different mathematical models and different loading levels) were done using the computer program.

The distribution of stress component σ_y along the axis of symmetry for one of the loading levels ($P/R_{02} = 0.67$) and for three different mathematical models is shown in Fig.9.

The distribution of the stress component σ_y along the axis of symmetry for different loading levels calculated for one chosen mathematical model (the 5-sectional model) is shown in Fig.10.

It has to be mentioned that the participation of the photoelastic coating in the transmitting of the loading (the coefficient of correction) was neglected in the calculations because of its small value (less than 3%).

5. Conclusions

On the base of the results shown in Fig.9 it may be proved that increasing the number of segments in the mathematical model describing the real material characteristics gives the better accuracy of the values of strain and stress components in the most interesting range of the first overelastic deformation ($n = 2$, Fig.2). It gives the possibility of more real evaluation of the material effort in this range. Practically, however, there is no use for taking too many segments in the mathematical model. As it is shown in Fig.9 the difference of the calculation accuracy between 5-sectional model and 8-sectional model is inconsiderable. Further increasing the number of segments has no sense, though it requires only some more data input $(E_i, \nu_i, \sigma_{(K)_i})$ and does not cause noticable increase in the calculating time.

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Analiza odkształceń w obszarach uplastycznionych metodą elastooptycznej warstwy powierzchniowej

Streszczenie

W pracy przedstawiono metodę wyznaczania odkształceń i naprężeń w uplastycznionych obszarach konstrukcji na podstawie obrazu izochrom otrzymanego w badaniach metodą elastooptycznej warstwy powierzchniowej. Obliczenie składowych odkształcenia (rozdzielenie odkształceń) wymagało wprowadzenia schematyzacji charakterystyki materiału i związków fizycznych obowiązujących w obszarach, w których została przekroczona granica plastyczności. Wyprowadzone zależności zastosowano do rozdzielenia odkształceń wzdłuż osi symetrii. Przebieg obliczeń zilustrowano przykładem.

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