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STABILITY OF THIN-WALLED BARS WITH VARIABLE CROSS-SECTION

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A solution of the problem of stability of a thin-walled bar with a variable section for any boundary conditions has been presented in the paper. The algorithm of calculations makes use of a method of a transfer matrix and of the Laplace operational calculus. The solutions relate to thin-walled bars with open sections under assumptions of Vlasov.

1. Introduction

The problem of stability of thin-walled bars with a variable cross section is in general case involved in a solution of a system of ordinary differential equations with functional coefficients. It is possible to obtain such a solution due to the application of one of the numerical methods (cf Szmelter, 1980; Mottersheat, 1988; Grunder and Witt, 1989). As a rule the calculations are laborious and burdened with a numerical error. In view of the above it has been decided to solve the above problem using the method of a transfer matrix.

The method enables to determine eigenvalues of a bar consisting of sections having constant cross-section. It is also possible to find lower and upper limit of every eigenvalue for a bar which cross-section varies (continually) along the length. The essence of the method lies in determining the matrix called a transfer matrix which is obtained as a result of a product of a span matrix and a section (node) matrix. The transfer matrix determines the relationship between values of a displacement function and their derivatives at the beginning and at the end of a bar, respectively. A form of the span matrix is determined on the basis of a solution of a given problem in general for a bar with a constant cross section. A section matrix is calculated from the conditions of equilibrium and inseparability of displacements in the given cross-section.

2. Determination of a span matrix

Differential equations describing the problem of stability of a thin-walled bar with an open section and constant cross section under a central force have the form (Vlasov, 1959)

$$
\begin{aligned}
&EJ_z\eta'''' + P\eta'' + Pz_\alpha\varphi'' = 0\\
&EJ_y\xi'''' + P\xi'' - Py_\alpha\varphi'' = 0\\
&EJ_\omega\varphi'''' + Pz_\alpha\eta'' - Py_\alpha\xi'' + (r^2P - GJ_x)\varphi'' = 0
\end{aligned}
\tag{2.1}
$$

The following variables have been introduced to these equations

η, ξ, φ – displacements of the shear centre and the torsion angle of the beam

x, y, z – principal central axes of inertia of cross section

y_α, z_α – coordinates of a shear centre towards axes y and z

E, G – Young's modulus and shear modulus

J_y, J_z, J_ω – moments of inertia of the cross section in relation to axes y and z as well as sectorial moment of inertia

$$
r^2 = \frac{J_0}{A} + y_\alpha^2 + z_\alpha^2 \qquad\qquad J_0 = J_x + J_y
$$

J_x – moment of inertia at pure torsion.

Eq (2.1) have been derived on the assumption of the so called technical theory of thin-walled bars developed by Kappus and generalized by Vlasov. The solution of the system of equations (2.2) has been obtained under the application of the Laplace transform (Doetsch, 1964). Let us denote the Laplace transform of particular functions as follows

$$
\begin{aligned}
&\int\limits_0^\infty \eta(x)\mathrm{e}^{-sx}\,dx = \overline{\eta}(s)\\
&\int\limits_0^\infty \xi(x)\mathrm{e}^{-sx}\,dx = \overline{\xi}(s)\\
&\int\limits_0^\infty \varphi(x)\mathrm{e}^{-sx}\,dx = \overline{\varphi}(s)
\end{aligned}
\tag{2.2}
$$

where

$s = \alpha + \mathrm{i}\beta$ – parameter of the transform

$\overline{\eta}, \overline{\xi}, \overline{\varphi}$ – transforms of the functions.

When introducing a symbolic notation in a form $L[\eta] = \overline{\eta}$, $L[\xi] = \overline{\xi}$ and $L[\varphi] = \overline{\varphi}$, to the expression (2.2), then on the basis of the theorem of Laplace transform of a derivative of function η, ξ and φ we shall write down the following relationships

$$
\begin{aligned}
&L[\eta''] = s^2\overline{\eta} - s\eta_0 - \eta_0' \\
&L[\eta''''] = s^4\overline{\eta} - s^3\eta_0 - s^2\eta_0' - s\eta_0'' - \eta_0''' \\
&L[\xi''] = s^2\overline{\xi} - s\xi_0 - \xi_0' \\
&L[\xi''''] = s^4\overline{\xi} - s^3\xi_0 - s^2\xi_0' - s\xi_0'' - \xi_0''' \\
&L[\varphi''] = s^2\overline{\varphi} - s\varphi_0 - \varphi_0' \\
&L[\varphi''''] = s^4\overline{\varphi} - s^3\varphi_0 - s^2\varphi_0' - s\varphi_0'' - \varphi_0'''
\end{aligned}
\tag{2.3}
$$

In the relationships (2.3) values $\eta_0, \ldots, \eta_0'''$, $\xi_0, \ldots, \xi_0'''$, $\varphi_0, \ldots, \varphi_0'''$ get the corresponding values of the functions η, ξ and φ and their derivatives for $x = 0$.

Through introducing the relationships (2.3) into Eqs (2.1) the system of differential equations is converted into a system of three non-homogeneous algebraic equations considering the transforms $\overline{\eta}$, $\overline{\xi}$, and $\overline{\varphi}$.

Algebraic equations will take the form

$$
\begin{aligned}
&EJ_z(s^4\overline{\eta} - s^3\eta_0 - s^2\eta_0' - s\eta_0'' - \eta_0''') + P(s^2\overline{\eta} - s\eta_0 - \eta_0') + \\
&+Pz_\alpha(s^2\overline{\varphi} - s\varphi_0 - \varphi_0') = 0 \\[2ex]
&EJ_y(s^4\overline{\xi} - s^3\xi_0 - s^2\xi_0' - s\xi_0'' - \xi_0''') + P(s^2\overline{\xi} - s\xi_0 - \xi_0') + \\
&+Py_\alpha(s^2\overline{\varphi} - s\varphi_0 - \varphi_0') = 0 \\[2ex]
&EJ_\omega(s^4\overline{\varphi}_0 - s^3\varphi_0 - s^2\varphi_0' - s\varphi_0'' - \varphi_0''') + (s^2\overline{\varphi} - s\varphi_0 - \varphi_0')(r^2P - GJ_x) + \\
&+Pz_\alpha(s^2\overline{\eta} - s\eta_0 - \eta_0') - Py_\alpha(s^2\overline{\xi} - s\xi_0 - \xi_0') = 0
\end{aligned}
\tag{2.4}
$$

When solving the system of equations (2.4) considering the transforms $\overline{\eta}, \overline{\xi}, \overline{\varphi}$ we obtain the following equations

$$
\begin{aligned}
\overline{\eta} &= \eta_0\frac{1}{s} + \eta_0'\frac{1}{s^2} + \frac{1}{s(as^6 + bs^4 + cs^2 + d)}\Big[(as^4 + es^2 + f)\Big(\eta_0'' + \eta_0'''\frac{1}{s}\Big) + \\
&+ g\Big(\xi_0'' + \xi_0'''\frac{1}{s}\Big) + (s^2h + j)\Big(\varphi_0'' + \varphi_0'''\frac{1}{s}\Big)\Big] \\
\overline{\xi} &= \xi_0\frac{1}{s} + \xi_0'\frac{1}{s^2} + \frac{1}{s(as^6 + bs^4 + cs^2 + d)}\Big[k\Big(\eta_0'' + \eta_0'''\frac{1}{s}\Big) + \\
&+ (as^4 + ls^2 + o)\Big(\xi_0'' + \xi_0'''\frac{1}{s}\Big) + (ms^2 + n)\Big(\varphi_0'' + \varphi_0'''\frac{1}{s}\Big)\Big] \\
\overline{\varphi} &= \varphi_0\frac{1}{s} + \varphi_0'\frac{1}{s^2} + \frac{1}{s(as^6 + bs^4 + cs^2 + d)}\Big[(vs^2 + P)\Big(\eta_0'' + \eta_0'''\frac{1}{s}\Big)\Big] +
\end{aligned}
\tag{2.5}
$$

$$+ \quad (qs^2 + y)\Big(\xi_0'' + \xi_0'''\frac{1}{s}\Big) + (as^2 + ts^2 + w)\Big(\varphi_0'' + \varphi_0'''\frac{1}{s}\Big)\Big]$$

The following denotations have been used in these relationships

$$\begin{aligned}
a &= E^3 J_y J_z J_\omega \\
b &= D(E^2 J_0 J_\omega + E^2 r^2 J_y J_z) - E^2 G J_y J_x J_z \\
c &= P^2(E J_\omega + E r^2 J_0 - E z_\alpha^2 J_y - E y_\alpha^2 J_z) - PEGJ_x J_0 \\
d &= P^3 \frac{J_0}{A} - P^2 G J_x \\
e &= P(r^2 E J_z J_y + E^2 J_z J_\omega) - E^2 G J_x J_z J_y \\
f &= -P^2 y_\alpha^2 E J_z + P(r^2 E J_z - EGJ_x J_z) \\
g &= -P^2 y_\alpha z_\alpha E J_y \\
h &= -PE^2 J_y J_\omega z_\alpha \\
j &= -P^2 E J_\omega z_\alpha \\
k &= -P^2 y_\alpha z_\alpha E J_z \\
l &= P(r^2 E^2 J_y J_z + E^2 J_y J_\omega) - E^2 G J_x J_y J_z \\
o &= P^2(r^2 E J_y - z_\alpha^2 E J_y) - PEGJ_x J_y \\
m &= D y_\alpha E J_\omega \\
n &= P^2 y_\alpha E J_\omega \\
v &= -P z_\alpha E^2 J_y J_z \\
p &= -P^2 z_\alpha E J_z \\
q &= P y_\alpha E^2 J_y J_z \\
u &= P^2 y_\alpha E J_y \\
t &= P(E^2 J_y J_\omega + E^2 J_z J_\omega) \\
w &= P^2 E J_\omega
\end{aligned}$$

Functions of displacements $\eta(x)$, $\xi(x)$ and of the angle of rotation $\varphi(x)$ will be determined due to an inverse Laplace transform according to the relationship

$$\begin{aligned}
\eta(x) &= \frac{1}{2\pi \mathrm{i}} \int\limits_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} \overline{\eta}(s) \mathrm{e}^{sx}\, ds \\
\xi(x) &= \frac{1}{2\pi \mathrm{i}} \int\limits_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} \overline{\xi}(s) \mathrm{e}^{sx}\, ds \\
\varphi(x) &= \frac{1}{2\pi \mathrm{i}} \int\limits_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} \overline{\varphi}(s) \mathrm{e}^{sx}\, ds
\end{aligned} \tag{2.6}$$

Let us denote the inverse Laplace transform of particular functions as follows

$$
\begin{aligned}
L^{-1}\Big[\frac{1}{s^2(as^6+bs^4+cs^2+d)}\Big] &= \overline{s}_8 \\
L^{-1}\Big[\frac{1}{s(as^6+bs^4+cs^2+d)}\Big] &= \overline{s}_7 \\
L^{-1}\Big[\frac{1}{as^6+bs^4+cs^2+d}\Big] &= \overline{s}_6 \\
L^{-1}\Big[\frac{s}{as^6+bs^4+cs^2+d}\Big] &= \overline{s}_5 \\
L^{-1}\Big[\frac{s^2}{as^6+bs^4+cs^2+d}\Big] &= \overline{s}_4 \\
L^{-1}\Big[\frac{s^3}{as^6+bs^4+cs^2+d}\Big] &= \overline{s}_3 \\
L^{-1}\Big[\frac{1}{s^2}\Big] &= \overline{s}_2 \\
L^{-1}\Big[\frac{1}{s}\Big] &= \overline{s}_1
\end{aligned}
\tag{2.7}
$$

Then the relationships for the functions $\eta(x)$, $\xi(x)$ and $\varphi(x)$ take the form

$$
\begin{aligned}
\eta(x) &= \eta_0\overline{s}_1 + \eta_0'\overline{s}_2 + \eta_0''(a\overline{s}_3 + e\overline{s}_5 + f\overline{s}_7) + \eta_0'''(as_4 + es_6 + fs_8) + \\
&+ \xi_0''g\overline{s}_7 + \xi_0'''g\overline{s}_8 + \varphi_0''(h\overline{s}_5 + f\overline{s}_7) + \varphi_0'''(h\overline{s}_6 + j\overline{s}_8) \\
\xi(x) &= \eta_0''k\overline{s}_7 + \eta_0'''k\overline{s}_8 + \xi_0\overline{s}_1 + \xi_0'\overline{s}_2 + \xi_0''(as_3 + ls_5 + rs_7) + \\
&+ \xi_0'''(as_4 + l\overline{s}_6 + r\overline{s}_8) + \varphi_0''(m\overline{s}_5 + n\overline{s}_7) + \varphi_0'''(m\overline{s}_6 + n\overline{s}_8) \\
\varphi(x) &= \eta_0''(v\overline{s}_5 + p\overline{s}_7) + \eta_0'''(v\overline{s}_6 + p\overline{s}_8) + \xi_0''(g\overline{s}_5 + u\overline{s}_7) + +\xi_0'''(g\overline{s}_6 + u\overline{s}_8) + \\
&+ \varphi_0\overline{s}_1 + \varphi_0'\overline{s}_2 + \varphi_0''(a\overline{s}_3 + t\overline{s}_5 + w\overline{s}_7)\varphi_0'''(a\overline{s}_4 + t\overline{s}_6 + w\overline{s}_8)
\end{aligned}
\tag{2.8}
$$

Let us introduce the following denotations into the expressions (2.8)

$$
\begin{aligned}
s_1 &= \overline{s}_1 & s_{11} &= a\overline{s}_3 + l\overline{s}_5 + f\overline{s}_7 \\
s_2 &= \overline{s}_2 & s_{12} &= a\overline{s}_4 + l\overline{s}_6 + f\overline{s}_8 \\
s_3 &= g\overline{s}_7 & s_{13} &= m\overline{s}_5 + n\overline{s}_7 \\
s_4 &= g\overline{s}_8 & s_{14} &= m\overline{s}_6 + n\overline{s}_8 \\
s_5 &= k\overline{s}_7 & s_{15} &= v\overline{s}_5 + p\overline{s}_7 \\
s_6 &= k\overline{s}_8 & s_{16} &= v\overline{s}_6 + p\overline{s}_8 \\
s_7 &= a\overline{s}_3 + e\overline{s}_5 + f\overline{s}_7 & s_{17} &= q\overline{s}_5 + u\overline{s}_7 \\
s_8 &= a\overline{s}_4 + e\overline{s}_6 + f\overline{s}_8 & s_{18} &= q\overline{s}_6 + u\overline{s}_8 \\
s_9 &= h\overline{s}_5 + j\overline{s}_7 & s_{19} &= a\overline{s}_3 + t\overline{s}_5 + w\overline{s}_7 \\
s_{10} &= h\overline{s}_6 + j\overline{s}_8 & s_{20} &= a\overline{s}_4 + t\overline{s}_6 + w\overline{s}_8
\end{aligned}
$$

Then the expressions (2.8) determining the functions of displacements η, ξ and φ as well as their derivatives written down in a matrix form are as follows

$$
\begin{bmatrix} \eta \\ \eta' \\ \eta'' \\ \eta''' \\ \xi \\ \xi' \\ \xi'' \\ \xi''' \\ \varphi \\ \varphi' \\ \varphi'' \\ \varphi''' \end{bmatrix} =
\begin{bmatrix}
s_1 & s_2 & s_7 & s_8 & 0 & 0 & s_3 & s_4 & 0 & 0 & s_9 & s_{10} \\
s_1 & s_2 & s_7 & s_8 & 0 & 0 & s_3 & s_4 & 0 & 0 & s_9 & s_{10} \\
s_1 & s_2 & s_7 & s_8 & 0 & 0 & s_3 & s_4 & 0 & 0 & s_9 & s_{10} \\
s_1 & s_2 & s_7 & s_8 & 0 & 0 & s_3 & s_4 & 0 & 0 & s_9 & s_{10} \\
0 & 0 & s_5 & s_6 & s_1 & s_2 & s_{11} & s_{12} & 0 & 0 & s_{13} & s_{14} \\
0 & 0 & s_5 & s_6 & s_1 & s_2 & s_{11} & s_{12} & 0 & 0 & s_{13} & s_{14} \\
0 & 0 & s_5 & s_6 & s_1 & s_2 & s_{11} & s_{12} & 0 & 0 & s_{13} & s_{14} \\
0 & 0 & s_5 & s_6 & s_1 & s_2 & s_{11} & s_{12} & 0 & 0 & s_{13} & s_{14} \\
0 & 0 & s_{15} & s_{16} & 0 & 0 & s_{17} & s_{18} & s_1 & s_2 & s_{19} & s_{20} \\
0 & 0 & s_{15} & s_{16} & 0 & 0 & s_{17} & s_{18} & s_1 & s_2 & s_{19} & s_{20} \\
0 & 0 & s_{15} & s_{16} & 0 & 0 & s_{17} & s_{18} & s_1 & s_2 & s_{19} & s_{20} \\
0 & 0 & s_{15} & s_{16} & 0 & 0 & s_{17} & s_{18} & s_1 & s_2 & s_{19} & s_{20}
\end{bmatrix}
\begin{bmatrix} \eta_0 \\ \eta_0' \\ \eta_0'' \\ \eta_0''' \\ \xi_0 \\ \xi_0' \\ \xi_0'' \\ \xi_0''' \\ \varphi_0 \\ \varphi_0' \\ \varphi_0'' \\ \varphi_0''' \end{bmatrix}
\tag{2.9}
$$

A square matrix formed together with expression $s_1 - s_{20}$ will be called a span matrix.

Calculations of the inverse transformation made on the basis of the expression (2.6) result in a positive effect, if functions of the parameter s are relatively simple.

In the case of composite functions it is more convenient to calculate the inverse Laplace transform by the residua method when applying the Jordan theorem (Smirnov, 1966).

The inverse Laplace tramsforms are then determined from the expression

$$
\begin{aligned}
\eta(x) &= \sum_{k-1}^{n} \operatorname{res}_{s=s_k}\Big[\overline{\eta}(s)e^{sx}\Big] \qquad & (x>0) \\
\xi(x) &= \sum_{k-1}^{n} \operatorname{res}_{s=s_k}\Big[\overline{\xi}(s)e^{sx}\Big] \qquad & (x>0) \\
\varphi(x) &= \sum_{k-1}^{n} \operatorname{res}_{s=s_k}\Big[\overline{\varphi}(s)e^{sx}\Big] \qquad & (x>0)
\end{aligned}
\tag{2.10}
$$

where

n – number of poles of the function being transformed

s_k – singular point of a given function.

Residua of the function are calculated from from the following relationships

a) for the function $f(s)$ having a simple pole at the point $s = s_k$

$$
\operatorname{res}_{s=s_k} f(s) = \lim_{s\to s_k} f(s)(s-s_k)
$$

b) for the function $f(s)$ having a multiple pole

$$
\operatorname{res} f(s) = \frac{1}{(k-1)!}\lim_{s\to s_k}\frac{d^{(k-1)}}{ds^{(k-1)}}\Big[f(s)(s-s_k)^k\Big]
\tag{2.11}
$$

The values of the function $F_k(x)$ calculated on the basis of relationships (2.10) and (2.11) and constituting the inverse Laplace transform $\sum\limits_k F_k(x) = L^{-1}[f(s)]$ are specified by Świtoński (1974).

3. Determination of a section matrix

A section matrix is determined in the cross section of a bar in which a change of dynamic features of the bar or another cause of discontinuity of functions of displacements and their derivatives occurs.

Elements of the section matrix are determined from conditions of the equality of internal forces and from geometric conditions formulated for a left and right side of the cross section under consideration (cf Świtoński, 1978; Świtoński and Bizoń, 1993), Fig.1.

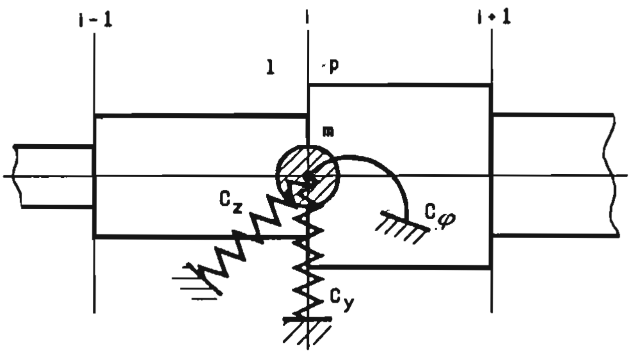

Fig. 1.

On the ground of the conditions of inability of distribution of displacements and of the conditions of equilibrium we shall write down the relationships existing between functions of displacements and their derivatives on both sides of the cross section in which a sudden change of geometric features and an elastic support take place. When denoting constant elastic supports with C_y, C_z and C_φ towards the axes y, z and around the axis x, respectively, we obtain the following relationships

$$\begin{aligned}
\eta_p &= \eta_l - (z_{\alpha_p} - z_{\alpha_l})\varphi_l \\
\xi_p &= \xi_l + (z_{\alpha_p} - z_{\alpha_l})\varphi_l \\
\eta'_p &= \eta'_l - (z_{\alpha_p} - z_{\alpha_l})\varphi'_l \\
\xi'_p &= \xi'_l + (z_{\alpha_p} - z_{\alpha_l})\varphi'_l \\
\varphi^*_p &= \varphi_l
\end{aligned}$$

$$
\begin{aligned}
u_p &= u_l \\
M_{sp} &= M_{sl} + C_\varphi \varphi_0 - M_B + Q_{yl}(z_{\alpha_p} - z_{\alpha_l}) - Q_{zl}(y_{\alpha_p} - y_{\alpha_l}) \\
M_{yp} &= M_{yl} \\
M_{zp} &= M_{zl} \\
B_p &= B_l \\
Q_{yp} &= Q_{yl} + C_y \eta_0 \\
Q_{zp} &= Q_{zl} + C_z \xi_0
\end{aligned}
\tag{3.1}
$$

where

$$
\begin{aligned}
\eta_0 &= \eta_l + z_{\alpha_i} \varphi_l \\
\xi_0 &= \xi_l - y_{\alpha_i} \varphi_l \\
\varphi_0 &= \varphi_l
\end{aligned}
\tag{3.2}
$$

These relationships do not take into account elastic supports around the axes y and z.

The condition $U_p = U_l$ which we use to determine the relationships between φ_p^I and φ_l^I is fulfilled for common points of outlines on both sides of the cross section being considered only. When having regard to the constraint of free warping of the cross section we obtain, in general, different relationships between φ_p^I and φ_l^I for particular common points.

From the condition of equilibrium of displacements U for the kth point of contact of both cross sections we obtain

$$
\varphi'_{kp} = \frac{(z_{\alpha_p} - z_{\alpha_l})y_k - (y_{\alpha_p} - y_{\alpha_l})z_k + \omega_l^k}{\omega_p^k} \varphi'_l
\tag{3.3}
$$

or

$$
\varphi'_{kp} = \omega_k \varphi'_l
\tag{3.4}
$$

where

$$
\omega_k = \frac{(z_{\alpha_p} - z_{\alpha_l})y_k - (y_{\alpha_p} - y_{\alpha_l})z_k + \omega_l^k}{\omega_p^k}
$$

The latter relationship between the quantities φ_p^I and φ_l^I for a given cross section is obtained when applying one of the approximation methods.

Then the relationships will take the form

$$
\varphi'_p = \omega_k \varphi'_k
\tag{3.5}
$$

If we express internal forces in the relationships (3.1) as derivatives of the functions η, ξ and φ and if we take into regard the expressions (3.2) and (3.3) we shall receive

$$
\begin{aligned}
\eta_p &= \eta l - \Delta z \varphi_l \\
\xi_p &= \xi l - \Delta z \varphi_l \\
\eta_p' &= \eta l' - \Delta z \varphi_l' \\
\xi_p' &= \xi l' - \Delta z \varphi_l' \\
\varphi_p &= \varphi_l \\
\varphi_p' &= w_1 \varphi_l' \\
\xi_p'' &= \frac{J_{yl}}{J_{yp}} \xi_l'' \\
\eta_p'' &= \frac{J_{zl}}{J_{zp}} \eta_l'' \\
\varphi_p'' &= \frac{J_{\omega l}}{J_{\omega p}} \varphi_l'' \\
\eta_p''' &= \frac{-C_y}{EJ_{zp}} \eta_l + \frac{J_{zl}}{J_{zp}} \eta_l''' - \frac{C_y z_{\alpha_l}}{EJ_{zp}} \varphi_l \\
\xi_p''' &= \frac{-C_z}{EJ_{up}} \xi_l + \frac{J_{yl}}{J_{yp}} \xi_l''' - \frac{C_z z_{\alpha_l}}{EJ_{yp}} \varphi_l \\
\varphi_p''' &= \frac{-C_\varphi}{EJ_{\omega p}} \varphi_l + \frac{GJ_{xp} w_1 - GJ_{xl}}{EJ\omega_p} \varphi_l' + \frac{J_{\omega l}}{J_{\omega p}} \varphi''' + \frac{J_{zl}}{J_{\omega p}} \Delta z \eta_l''' - \frac{J_{yl}}{J_{\omega p}} \Delta z \xi_l''' \\
\Delta z &= z_{\alpha_p} - z_{\alpha_l}
\end{aligned}
\tag{3.6}
$$

The relationships (3.6) written down in a matrix form are

$$
\begin{bmatrix} \eta_p \\ \eta_p' \\ \eta_p'' \\ \eta_p''' \\ \xi_p \\ \xi_p' \\ \xi_p'' \\ \xi_p''' \\ \varphi_p \\ \varphi_p' \\ \varphi_p'' \\ \varphi_p''' \end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta z & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta z & 0 & 0 \\
0 & 0 & \frac{J_{zl}}{J_{zp}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
W_3 & 0 & 0 & \frac{J_{zl}}{J_{zp}} & 0 & 0 & 0 & 0 & W_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \Delta z & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \Delta z & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{J_{zl}}{J_{zp}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & W_5 & 0 & 0 & \frac{J_{zl}}{J_{zp}} & W_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W_1 & 0 & 0 \\
0 & 0 & 0 & W_8 & 0 & 0 & 0 & W_9 & 0 & 0 & \frac{J_{zl}}{J_{zp}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W_7 & W_2 & 0 & \frac{J_{zl}}{J_{zp}}
\end{bmatrix}
\begin{bmatrix} \eta_l \\ \eta_l' \\ \eta_l'' \\ \eta_l''' \\ \xi_l \\ \xi_l' \\ \xi_l'' \\ \xi_l''' \\ \varphi_l \\ \varphi_l' \\ \varphi_l'' \\ \varphi_l''' \end{bmatrix}
\tag{3.7}
$$

where

$$
W_2 = \frac{GJ_{xp} W_1 - GJ_{xl}}{EJ_{wp}} \qquad W_3 = -\frac{C_y}{EJ_{zp}}
$$

$$W_4 = -\frac{C_y Z_{\alpha_l}}{EJ_{zp}} \qquad W_5 = -\frac{C_z}{EJ_{yp}}$$

$$W_6 = -\frac{C_z Y_{\alpha_l}}{EJ_{yp}} \qquad W_7 = -\frac{C_\varphi}{EJ_{wp}}$$

$$W_8 = \frac{J_{zl}}{J_{wp}}\Delta z \qquad W_9 = \frac{J_{yl}}{J_{wp}}(y_{\alpha_p} - y_{\alpha_l})$$

A square matrix of the expression (3.7) is called a section matrix or a kinematic pair matrix.

If we denote the matrix of the ith span with H_i and the matrix of the ith cross section with F_i the transfer matrix H for a bar divided into n segments takes the form

$$[H] = H_m \prod_{l=1}^{m-1} F_i H_i \tag{3.8}$$

4. Determination of eigenvalues and eigenfunctions

Eigenvalues are determined by equating the appropriate minor of the transform matrix $\mathbf{H}$ (3.8) – the so called characteristic determinant – with zero.

The form of the characteristic determinant depends on the boundary conditions of the bar under consideration. There are six from among twelve boundary values of functions of displacements $\eta(x)$, $\xi(x)$ and $\varphi(x)$ and their derivatives forming the so called vector of state in the initial cross section which are determined on the ground of the boundary conditions for $x = 0$. The remaining six values form a system of homogeneous equations determining the boundary conditions for $x = l$.

Eigenfunctions for the i-segment of the bar are determined from relationships

$$\mathbf{Y}_i(x) = H_i(x)\Big[\prod_{j=1}^{i-1} F_j H_j\Big]\mathbf{Y}_0 \tag{4.1}$$

where

$\mathbf{Y}_i(x)$ – matrix of the eigenfunction for the i-segment of the bar

$\mathbf{Y}_0$ – column matrix of boundary values for $x = 0$.

5. Example expressed in numbers

In order to illustrate the presented solution a critical force for a bar consisting of two segments with a constant cross section (Fig.2) has been calculated.

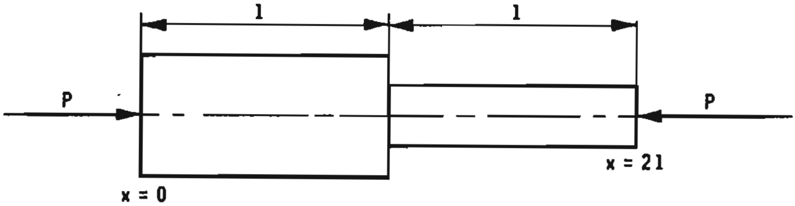

Fig. 2.

The segment 1 has the cross section shown in Fig.3a, and the segment 2 has the cross section shown in Fig.3b.

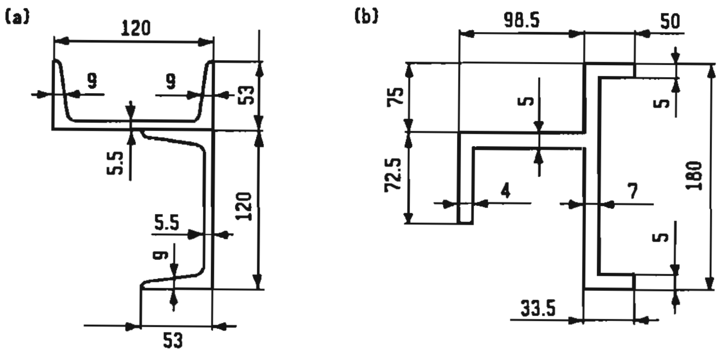

Fig. 3.

The following boundary conditions have been taken into consideration

$$
\begin{array}{llll}
x = 0 & \eta = 0 & \xi = 0 & \varphi = 0 \\
 & \eta' = 0 & \xi' = 0 & \varphi' = 0 \\
x = 2l & \eta = 0 & \xi = 0 & \varphi = 0 \\
 & \eta' = 0 & \xi' = 0 & \varphi' = 0
\end{array}
\tag{5.1}
$$

$$
\begin{array}{llll}
x = 0 & \eta = 0 & \xi = 0 & \varphi = 0 \\
 & \eta' = 0 & \xi' = 0 & \varphi' = 0 \\
x = 2l & \eta = 0 & \xi = 0 & \varphi = 0 \\
 & \eta'' = 0 & \xi'' = 0 & \varphi'' = 0
\end{array}
\tag{5.2}
$$

Results of numerical calculations of the minimum critical force are presented in diagrams in a function of the bar length

a) for boundary conditions (4.2) – Fig.4,

b) for boundary conditions (4.3) – Fig.5.

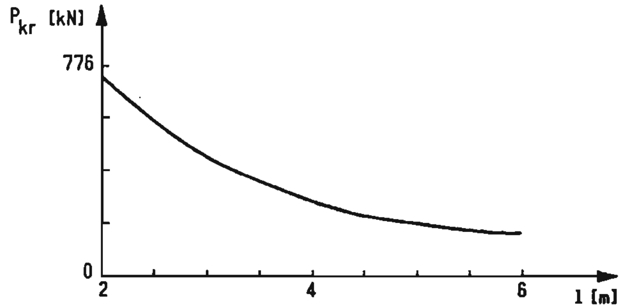

Fig. 4.

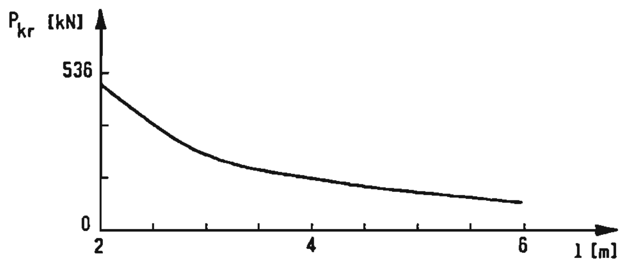

Fig. 5.

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Stateczność prętów cienkościennych o zmiennym przekroju

Streszczenie

W pracy przedstawiono rozwiązanie zagadnienia prętów cienkościennych o zmiennym przekroju, dla dowolnych warunków brzegowych. W algorytmie obliczeń wykorzystano metodę macierzy przeniesienia i rachunek operatorowy Laplace'a. Rozwiązania dotyczą prętów cienkościennych o profilu otwartym przy założeniach Własowa.

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