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BIHARMONIC REPRESENTATION OF THE SOLUTION TO EQUILIBRIUM PROBLEM OF A PLATE MADE OF A COSSERAT MATERIAL

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The paper deals with a generalized plane stress problem in a Cosserat plate. There are given representations of the displacement and infinitesimal rotation fields that reduce the equilibrium problem to a single biharmonic equation involving a deflection function.

1. Introduction

In the previous paper [1] Author has reported a biharmonic representation of the displacement vector concerning a generalized plane state of stress (GPSS) in an elastic layer made of a Grioli–Toupin material. Such a representation of the displacement vector reduces the problem of bending of an elastic layer to one biharmonic equation

$$\nabla^4 v(x^\alpha) = 0 \tag{1.1}$$

Here $v(x^\alpha)$ stands for the layer deflection.

Similar representations for the displacement vector and vector of an infinitesimal rotation in the case of GPSS of a Cosserat medium have not been reported in the hitherto existing literature. In the present paper a generalization of the solution given in by the Author [1] to the Cosserat medium case will be put forward.

The summation convention is adopted. The Latin indices run over 1,2,3 and Greek ones – over 1,2. Comma implies partial differentiation.

2. Fundamental equations of the Cosserat medium

Following the notation of Nowacki [2,3] and Palmov [4] one can write down the constitutive relationships of an isotropic homogeneous and centrosymmetric medium in the form

$$\sigma_{ij} = (\mu + \alpha)\gamma_{ij} + (\mu - \alpha)\gamma_{ji} + \lambda\gamma^{k}{}_{k}\delta_{ij}$$
$$\mu_{ij} = (\gamma + \varepsilon)K_{ij} + (\gamma - \varepsilon)K_{ji} + \beta K^{k}{}_{k}\delta_{ij} \tag{2.1}$$

The stress tensor is denoted by (σ_{ij}) and the couple–stress tensor by (μ_{ij}); (δ_{ij}) is Kronecker delta; (γ_{ij}) and (K_{ij}) are components of the deformation tensors. They are defined by

$$\gamma_{ij} = u_{j,i} - \epsilon^{k}{}_{ij}\phi_k \qquad K_{ij} = \phi_{j,i} \tag{2.2}$$

(ϵ_{ijk}) represents the Levi–Civita permutation symbol; (u_i) are components of the displacement vector; (ϕ_i) represents components of the vector of infinitesimal rotations. Symbols μ, λ, α, γ, ε and β stand for the material constants of the Cosserat medium. The constants μ and λ can be viewed as Lamé moduli. Relations between the Nowacki–Palmov constants and the constants used by other authors are set up in Table 1.

Table 1

Authors	Material constants					
Nowacki [2] Palmov [4]	μ	λ	α	γ	ε	β
Aéro [5] Kuvshinskiĭ	μ	λ	$\hat{\gamma}$	$\tau + \theta$	$\tau - \theta$	$2\hat{\eta}$
Neuber [6]	G	$\frac{2\nu G}{1-2\nu}$	Ga	$2Gl^2(1+b)$	$2Gl^2(1-b)$	$4Gl^2c$
Kessel [7]	G	$\frac{2\nu G}{1-2\nu}$	$\frac{G}{2}c_1$	$G\tilde{L}^2$	$G\tilde{L}^2c_2$	$2G\tilde{L}^2c_3$
Koiter [8]	G	$\frac{2\nu G}{1-2\nu}$		$2Gl^2(1+\eta)$	$2Gl^2(1-\eta)$	
Eringen [9,10]	$\tilde{\mu} + \frac{K}{2}$	λ	$\frac{1}{2}K$	$\frac{1}{2}(\tilde{\gamma} + \tilde{\beta})$	$\frac{1}{2}(\tilde{\gamma} - \tilde{\beta})$	$\tilde{\alpha}$
Schaefer [11]	G	$\frac{2\nu G}{1-2\nu}$	$G\eta_1$	$L^2\frac{G}{12}$	$L^2\frac{G}{12}\eta_2$	$L^2\frac{G}{6}\eta_3$

Note that the relations between the moduli γ, ε and Koiter constants l^2 and η assume the following form, cf Table 1

$$\gamma = 2Gl^2(1+\eta) = 2\mu l^2(1+\eta)$$
$$\varepsilon = 2Gl^2(1-\eta) = 2\mu l^2(1-\eta)$$

and hence we obtain

$$l^2 = \frac{\gamma+\varepsilon}{4\mu} \qquad \eta = \frac{\gamma-\varepsilon}{\gamma+\varepsilon} \tag{2.3}$$

Futher on we shall omit the body forces. The equilibrium equations assume the form

$$\sigma^{ji}{}_{,j} = 0 \qquad \epsilon^{ijk}\sigma_{jk} + \mu^{ji}{}_{,j} = 0 \tag{2.4}$$

The same equations expressed in terms of displacement and rotation fields read

$$\begin{aligned}
&(1+\kappa)\tilde{\nabla}^2\mathbf{u} + \Big(\frac{1}{1-2\nu} - \kappa\Big)\text{grad div}\mathbf{u} + 2\kappa\,\text{rot}\boldsymbol{\phi} = \mathbf{0}\\
&(\gamma+\varepsilon)\tilde{\nabla}^2\boldsymbol{\phi} + (\beta+\gamma-\varepsilon)\text{grad div}\boldsymbol{\phi} + 2\alpha(\text{rot}\mathbf{u} - 2\boldsymbol{\phi}) = \mathbf{0}
\end{aligned} \tag{2.5}$$

Here ν represents Poisson's ratio. The κ constant is given by

$$\kappa = \frac{\alpha}{\mu} \tag{2.6}$$

The Laplace operator in R^3 has been denoted by $\tilde{\nabla}^2$.

3. Biharmonic representation

Let us consider an elastic layer of thickness h, freed from loads on the faces $x^3 = z = \pm h/2$. To find the solution to Eqs (2.5) that fulfils the homogeneous boundary conditions on the faces

$$\begin{aligned}
\sigma_{3i}\Big(x^\beta, \pm\frac{h}{2}\Big) &= 0\\
\mu_{3i}\Big(x^\beta, \pm\frac{h}{2}\Big) &= 0
\end{aligned} \tag{3.1}$$

we adopt a semi-inverse method.

Let us represent the components of the displacement vector (u_i) and the vector of infinitesimal rotation (ϕ_i) in the form

$$\begin{aligned}
u_\alpha(x^\beta, z) &= t(z)v(x^\beta)_{,\alpha} + s(z)\nabla^2 v(x^\beta)_{,\alpha}\\
u_3(x^\beta, z) &= g(z)v(x^\beta) + f(z)\nabla^2 v(x^\beta)
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
\phi_\alpha(x^\gamma, z) &= \epsilon_\alpha{}^\beta\Big(R_1(z)v(x^\gamma)_{,\beta} + R_2(z)\nabla^2 v(x^\gamma)_{,\beta}\Big)\\
\phi_3 &= 0
\end{aligned} \tag{3.3}$$

where $t(z)$, $s(z)$, $g(z)$, $f(z)$, $R_1(z)$ and $R_2(z)$ are unknown functions which satisfy the following conditions

$$\begin{aligned} t(z) &= -t(-z) & s(z) &= -s(-z) \\ g(z) &= g(-z) & f(z) &= f(-z) \\ R_1(-z) &= R_1(z) & R_2(-z) &= R_2(z) \end{aligned} \tag{3.4}$$

On inserting (3.2) and (3.3) into the set of equations (2.5) one concludes that there exist non-trivial solutions to this system, provided the function $v(x^\alpha)$ satisfies Eq (1.1) and the unknown functions in z satisfy the following system of ordinary differential equations

$$\begin{aligned} &g'' = 0 \\ &(1+\kappa)g + \frac{2(1-\nu)}{1-2\nu}f'' + \Big(\frac{1}{1-2\nu} - \kappa\Big)t' - 2\kappa R_1 = 0 \\ &(1+\kappa)t'' + \Big(\frac{1}{1-2\nu} - \kappa\Big)g' + 2\kappa R_1' = 0 \\ &(1+\kappa)s'' + \frac{2(1-\nu)}{1-2\nu}t + \Big(\frac{1}{1-2\nu} - \kappa\Big)f' + 2\kappa R_2' = 0 \\ &R_1'' - (1+\kappa)k^2 R_1 + \frac{1}{2}(1+\kappa)k^2\Big(g - t'\Big) = 0 \\ &R_1 + R_2'' - (1+\kappa)k^2 R_2 - \frac{1}{2}(1+\kappa)k^2\Big(s' - f\Big) = 0 \end{aligned} \tag{3.5}$$

where $(\cdot)' = \dfrac{d(\cdot)}{dz}$ and the k constant is defined by

$$k^2 = \frac{N^2}{l^2} \qquad\qquad N^2 = \frac{\kappa}{1+\kappa} \tag{3.6}$$

The coefficient N is nondimensional with the value lying within the interval [0,1].

The solution to the system (3.5) which satisfies the boundary conditions (3.1) can be cast in the following form

$$\begin{aligned} t(z) &= -zC_1 \\ s(z) &= -\frac{(2-\nu)h^2}{24(1-\nu)}z\Big(C_2 - C_1\frac{4z^2}{h^2}\Big) - l^2hC_1\frac{\sinh kz}{\sinh\frac{kh}{2}} \\ g(z) &= R_1(z) = C_1 \\ f(z) &= -\frac{h^2}{24(1-\nu)}\Big[6\Big(1 - \frac{2\nu z^2}{h^2}\Big)C_1 - (2-\nu)C_2\Big] \\ R_2(z) &= -\frac{h^2}{24(1-\nu)}\Big(3C_1 - (2-\nu)C_2 + 12(1-\nu)\frac{z^2}{h^2}C_1 - \frac{12(1-\nu)\cosh kz}{kh\sinh\frac{kh}{2}}C_1\Big) \end{aligned} \tag{3.7}$$

The constants C_1 and C_2 determine the physical meaning of $v(x^\alpha)$. Without any loss of generality the constant C_1 can be fixed as equal to unity, cf [1].

Let as substitute (3.7) into (3.2) and (3.3) and assume $C_1 = 1$. Then one arrives at the following representation for displacement and rotation fields

$$\begin{aligned} u_\alpha(x^\beta, z) &= -\Big\{zv(x^\beta)_{,\alpha} + \Big[\frac{(2-\nu)h^2}{24(1-\nu)} z\Big(C_2 - \frac{4z^2}{h^2}\Big) + \\ &\quad + l^2 h \frac{\sinh kz}{\sinh \frac{kh}{2}}\Big] \nabla^2 v(x^\beta)_{,\alpha}\Big\} \\ u_3(x^\beta, z) &= v(x^\beta) - \frac{h^2}{24(1-\nu)}\Big[6\Big(1 - \frac{2\nu z^2}{h^2}\Big) - (2-\nu)C_2\Big] \nabla^2 v(x^\beta) \end{aligned} \tag{3.8}$$

$$\begin{aligned} \phi_\alpha(x^\gamma, z) &= \epsilon_\alpha{}^\beta \Big[v(x^\gamma)_{,\alpha} + \frac{h^2}{24(1-\nu)}\Big((2-\nu)C_2 - 3 - 12(1-\nu)\frac{z^2}{h^2} + \\ &\quad + \frac{12(1-\nu)\cosh kz}{kh \sinh \frac{kh}{2}}\Big) \nabla^2 v(x^\gamma)_{,\alpha}\Big] \\ \phi_3 &= 0 \end{aligned} \tag{3.9}$$

On using formulae (2.1), (2.2), (3.8) and (3.9) one obtains expressions defining the stress-and couple–stress components

$$\begin{aligned} \sigma_{\alpha\beta}(x^\gamma, z) &= -\frac{2\mu}{1-\nu}\Big\{z\Big((1-\nu)v_{,\alpha\beta} + \nu \nabla^2 v \delta_{\alpha\beta} + \\ &\quad + \Big[\frac{(2-\nu)h^2}{24} z\Big(C_2 - \frac{4z^2}{h^2}\Big) + (1-\nu) l^2 h \frac{\sinh kz}{\sinh \frac{kh}{2}}\Big] \nabla^2 v(x^\gamma)_{,\alpha\beta}\Big\} \\ \sigma_{\alpha 3}(x^\beta, z) &= -\frac{\mu h^2}{4(1-\nu)}\Big[\Big(1 - \frac{4z^2}{h^2}\Big) + 8(1-\nu)\frac{kl^2 \cosh kz}{h \sinh \frac{kh}{2}}\Big] \nabla^2 v(x^\beta)_{,\alpha} \end{aligned} \tag{3.10}$$

$$\begin{aligned} \mu_{\alpha\beta}(x^\delta, z) &= 4\mu l^2 \epsilon_\beta{}^\gamma \Big\{v_{,\gamma\alpha} + \Big[\frac{h}{2k}\frac{\cosh kz}{\sinh \frac{kh}{2}} - \frac{h^2}{24(1-\nu)}\Big(3 + 12(1-\nu)\frac{z^2}{h^2} - \\ &\quad - (2-\nu)C_2\Big)\Big] \nabla^2 v_{,\gamma\alpha}\Big\} + 4\mu l^2 \eta \epsilon_\alpha{}^\gamma \Big\{v_{,\gamma\beta} + \\ &\quad + \Big[\frac{h}{2k}\frac{\cosh kz}{\sinh \frac{kh}{2}} - \frac{h^2}{24(1-\nu)}\Big(3 + 12(1-\nu)\frac{z^2}{h^2} - (2-\nu)C_2\Big)\Big] \nabla^2 v_{,\gamma\beta}\Big\} \end{aligned} \tag{3.11}$$

$$\sigma_{3a}(x^\beta, z) = -\frac{\mu h^2}{4(1-\nu)}\Big(1 - \frac{4z^2}{h^2}\Big) \nabla^2 v_{,\alpha} \tag{3.12}$$

$$\sigma_{33}(x^\alpha, z) = 0 \qquad \mu_{33}(x^\alpha, z) = 0 \tag{3.13}$$

$$\mu_{3\alpha} = -2\mu l^2 h\epsilon_\alpha{}^\beta\Big(2\frac{z}{h} - \frac{\sinh kz}{\sinh \frac{kh}{2}}\Big)\nabla^2 v_{,\beta}$$
$$\mu_{\alpha 3} = \eta\mu_{3\alpha} \tag{3.14}$$

It is readily seen that if the function v fulfils the Eq (1.1), then all differential equations (2.4), (2.5) and boundary conditions (3.1) are satisfied. In their general form (3.8) $\div$ (3.14) these equations have not been up till now reported in the literature.

The C_2 constant can be chosen so as to assign a clear physical meaning to the function $v(x^\alpha)$.

1. For $C_2 = 3$ the $v(x^\alpha)$ function represents deflection of the layer faces

$$\hat{w}(x^\alpha) \stackrel{\text{df}}{=} u_3\Big(x^\alpha, \pm\frac{h}{2}\Big)$$

2. for $C_2 = \frac{6}{2-\nu}$ this function stands for the mid-plane deflection

$$w(x^\alpha) \stackrel{\text{df}}{=} u_3(x^\alpha, 0)$$

3. at $C_2 = \frac{6-\nu}{2-\nu}$ this function represents a common mean value

$$\overset{*}{w}(x^\alpha) \stackrel{\text{df}}{=} \frac{1}{h}\int\limits_{-h/2}^{h/2} u_3(x^\alpha, z)dz$$

4. at $C_2 = \dfrac{3(10-\nu)}{5(2-\nu)}$ we obtain a weighted mean value (cf [1])

$$\tilde{w}(x^\alpha) \stackrel{\text{df}}{=} \frac{3}{2h}\int\limits_{-h/2}^{h/2}\Big(1 - 4\frac{z^2}{h^2}\Big)u_3(x^\alpha, z)dz$$

Thus the representations for displacements, stresses and stress resultants can be expressed in terms of different scalar functions ($\hat{w}$, w, $\overset{*}{w}$, $\tilde{w}$, etc.) standing for the deflection of the layer.

Let us compute the quantity $\phi = \frac{1}{2}\text{rot}\boldsymbol{u}$. On using (3.8) we obtain

$$\varphi_\alpha(x^\gamma, z) = \epsilon_\alpha{}^\beta\Big[v(x^\gamma)_{,\beta} + \frac{h^2}{24(1-\nu)}\Big((2-\nu)C_2 - 3 - \frac{12(1-\nu)z^2}{h^2} + \frac{12l^2(1-\nu)k\cosh kz}{h\sinh\frac{kh}{2}}\Big)\nabla^2 v(x^\gamma)_{,\beta}\Big] \tag{3.15}$$
$$\varphi_3 = 0$$

It is readily seen that the components of the averaged – rotation vector (φ_α) do not coincide with the components of the infinitesimal vector (ϕ_α).

4. Passages to the limits

The following passage to a limit transforms the displacement vector representation for the Cosserat layer to the representation for the layer made of the Grioli–Toupin (G–T) material [6,12,13]

(GPSS in the G–T material) = $\lim\limits_{\kappa\to\infty}$ (GPSS for the Cosserat material)

Passing in (3.6) with κ to infinity one finds

$$N^2 = 1 \qquad\qquad k^2 = \frac{1}{l^2} \tag{4.1}$$

Substituting equality (4.1) into (3.8) and (3.9) one obtains

$$\begin{aligned} u_\alpha(x^\beta, z) &= -\Big\{ z v(x^\beta)_{,\alpha} + \Big[\frac{(2-\nu)h^2}{24(1-\nu)} z\Big(C_2 - \frac{4z^2}{h^2}\Big) + \\ &\qquad + l^2 h \frac{\sinh\frac{z}{l}}{\sinh\frac{h}{2l}}\Big] \nabla^2 v(x^\beta)_{,\alpha}\Big\} \\ u_3(x^\beta, z) &= v(x^\beta) - \frac{h^2}{24(1-\nu)}\Big[\Big(6 - \frac{12\nu z^2}{h^2}\Big) - (2-\nu)C_2\Big]\nabla^2 v(x^\beta) \end{aligned} \tag{4.2}$$

$$\begin{aligned} \phi_\alpha(x^\gamma, z) &= \varphi_\alpha(x^\gamma, z) = \epsilon_\alpha{}^\beta\Big\{ v(x^\gamma)_{,\alpha} + \Big[\frac{h^2}{24(1-\nu)}\Big((2-\nu)C_2 - 3 - \\ &\qquad - 12(1-\nu)\frac{z^2}{h^2}\Big) + \frac{1}{2} l h \frac{\cosh\frac{z}{l}}{\sinh\frac{h}{2l}}\Big] \nabla^2 v(x^\gamma)_{,\alpha}\Big\} \\ \varphi_3 &= 0 \end{aligned} \tag{4.3}$$

The representation given above concerns the displacement field in a medium with constrained rotations [1]. Under the assumptions (4.1) the stresses given by (3.10) ÷ (3.14) describe the GPSS in the plate made of a Grioli–Toupin material (cf [1]).

Similar representations for the Hookean material can be arrived at by passing to zero with l in the formulae (3.8) ÷ (3.14) or (4.2). Such representations for displacements and stresses turn out to coincide with those found previously (cf [1]). The couple–stresses become zero and the infinitesimal rotation $\boldsymbol{\phi}$ becomes equal to the averaged rotation $\boldsymbol{\varphi}$.

Let us compute the limits of the expressions (3.8) ÷ (3.14) for $\alpha \to 0$, i.e. at $N = 0$. In the micropolar elasticity this case is viewed as a pathological one [14].

If $N = 0$, in many problems of micropolar elasticity the solutions do not tend to classical elasticity solutions. In the problem considered here we just face this situation. Obtained via passing to zero with N the expressions for the stresses satisfy the equilibrium equations $(2.4)_1$ (provided that Eq (1.1) is satisfied) but the couple–stresses $\mu_{\alpha\beta}$ become indeterminate and the equilibrium equations (2.4) turn out to be violated. This is a consequence of the fact that at $N = 0$ the components ϕ_α tend to infinity.

Finally let us note that in the GPSS considered the classical (i.e. symmetric) elasticity solution [1] can be obtained if one assumes $l \to 0$ ($k \to \infty$) in Eqs (3.8) $\div$ (3.14).

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Reprezentacja biharmoniczna w rozwiązywaniu problemów równowagi płyty wykonanej z materiału Cosseratów

Streszczenie

W pracy wyznaczono przedstawienie wektora przemieszczenia i infinitezymalnego obrotu, opisujące uogólniony płaski stan naprężenia w płycie Cosseratów. Przedstawiona reprezentacja wektora przemieszczenia prowadzi do rozwiązania równania biharmonicznego na funkcję przedstawiającą ugięcie.

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