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DYNAMICS PROBLEM OF MECHANICAL SYSTEMS WITH UNILATERAL CONSTRAINTS

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Two examples of systems with unilateral constraints are studied in this work. First system concerns a movement of the table tennis ball between rackets with consideration of the air resistance whereas the second one determines a motion of the rolling mill coupling as the result of a clearance existing in one of the universal joints. An elastoplastic type of impact has been considered in both examples what corresponds to the coefficient of restitution (during impact) ranging from zero to one. The initial-value problems for motion of the systems has been formulated and numerically solved. For solving the problem the Runge-Kutta method of numerical integration of equations of motion has been used.

1. Introduction

In solving problems of analytical mechanics the consideration of constraints, which influence equations of systems motion being formulated, is a very important problem. At the end of XVIII century the problem of holonomic constraints has been formulated and solved by Lagrange [1,2,4]. In the subsequent years various forms of anholonomic constraints and other two-sided constraints have been studied.

Amongst all the types of constraints, bounding motion of the system, the unilateral constraints which express relationships between coordinates of the system in form of inequality one may regard as a kind of curiosity in mechanics. Generally, in this case only a surface or a curve limits the range of space the system can not leave beyond. In spite of the fact that this type of constraints has been known for a long time, process of taking into consideration the constraints equations and formulation differential equations of motion in the regular form (without singularities) within infinite time interval have been just solved in the seventieth [3,5,6].

To the range of practical problems for which the description of the system motion requires introducing the unilateral constrains, belongs the phenomenon of impact.

For the systems with unilateral constraints the problems to be solved may be divided into two cathegories: kinematic and dynamic.

Kinematic problem is when on the grounds of the known state of the system before impact its after-impact state should be determined. Problems of this type constitute known part of the theoretical mechanics called theory of impacts. They are reduced to the solving of the algebraic equations system by using commonly known methods.

Dynamic problems allow one to investigate deeper the heart of the impact phenomenon. They also make it possible to take into consideration and to determine forces and motion of the system in terms of various models of interaction. In the theory of vibration the equations of motion describing different systems with impacts are still under consideration, because of difficulties which arise when solving them. This state of things is caused by two reasons.

Firstly, in equations describing constraints very inconvenient for transformation elements arise. The idea of taking them into consideration (the elimination of unilateral constraints equations) in the way like for systems with the simple constraints proves to be unrealizable.

Secondly, this operation is considered to be unnecessary because constraints of the system between two impacts may be regarded as linear. Then in the time interval corresponding to the constraints, equation of motion of the system may be found. State of the system at the end of the time interval for one form of motion is used for establishing initial conditions in the next time interval. Such method of assignment practically leads to the labor-consuming transformations in order to formulate equations of motion in form which is simpler to solve.

Among systems with unilateral constraints the significant group constitute the systems with the clearance. In equations of motion describing dynamics of the systems with clearance, constraints are considered through the appropriate characteristic of reactive forces. Generally, in the range of clearance the force equalls to zero and in the case when displacement of the system is greater than the clearance dimension, the reactive force is determined on the grounds of a certain characteristic related to the material properties of objects taking part in the impact process. Respective characteristic of forces are analytically approximated with the use of the Heaviside function. The characteristic assumed in such a way is a continuous function but non-differentiable in limited number of points.

This problem is connected with the necessity of solving highly nonlinear systems. The approximate methods of equation linearization, applied in the vibration theory simplify the solving procedure of equations of motion, but usually cause the large errors in obtained results in the problem in question. Polynominal approximation of the non–linear force characteristic, because of its non–differentiation in certain points, is also burdened with errors in model of the system itself and generally, does not allow to avoid numerical method of solving the problem.

The method of special transformation of coordinates applied by Ivanov and Markeev as well as by Zhuravlev [3,5,6], in order to obtain equations of motion (corresponding to the system with unilateral constraints), that are determined in the whole time interval of motion, generally, has an application for systems with coefficient of restitution equal to one.

Moreover, all this methods, with the exception of the method of linearization and systems with their characteristics being of the intervals–linear type, practically for the systems with multidegrees of freedom require application of the numerical methods for equations of motion integration.

Nowadays, when application of computers for investigation of practical dynamics problems of mechanic systems with the clearance is of comon use, it is advisible to employ the contemporary methods of numerical integration of ordinary differential equation systems. Peculiarity of the problem in question, with unilateral constraints, is expressed by the properly formulated procedure which determines numerically the force arising in the system with clearance. With such a procedure and properly chosen integration step, the accuracy of calculations depends on the applied method of the equations of motion integration.

2. Clearance characteristic approximation

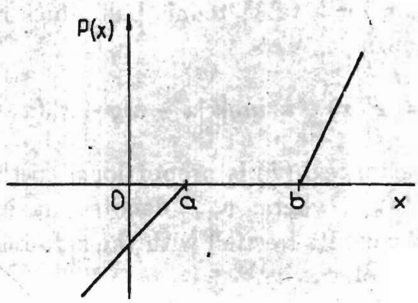


Fig. 1.

Let us consider a mechanical system with the clearance affected by a certain force system and assume that the clearance lies within an interval (a, b) on the $0x$ axis only. Then within this interval on the element of the system do not act all the forces but a part of them appears at the end points (Fig.1). These forces may be expressed in the form

$$P(x) = \kappa_1(x-a)H(a-x) + \kappa_2(x-b)H(x-b) \tag{2.1}$$

where $H(\cdot)$ represents the Heaviside function, κ_1 and κ_2 are coefficients of elasticity. Thus, the differential equation, which containts the force $P(x)$, may be written in the form

$$m\ddot{x} = P(x) + F(x, \dot{x}, t) \tag{2.2}$$

where $F(x, \dot{x}, t)$ denotes the remaining forces acting on the element under consideration, with the mass m. Let us assume that

$$\kappa_1 = \kappa_2 = -\kappa \tag{2.3}$$

Now, if we put $a = -b$, what means that the origin of the reference system is assumed in the middle of the clearance, then

$$P(x) = -\kappa[(x-a)H(x-a) + (x+a)H(-x-a)] \tag{2.4}$$

There exist many real systems with the clearance, motion of which may be described by the system of equations being in the form of (2.2) where $P(x)$ is expressed by relationship (2.4). Two examples of such systems are given below.

3. Motion of the table–tennis ball between two rackets

The equation of motion for a table–tennis ball which is moving between two rackets (Fig.2) is the following

$$m\ddot{x} = P(x) - k^2 mg|\dot{x}|\dot{x} - mg - F_r(x, \dot{x}) \tag{3.1}$$

assuming that the air resistance $R(\dot{x})$ is proportional to the second power of the velocity, and that $k = 1/v_{\max}$, where: v_{max} denotes maximal velocity developed by the table–tennis ball during its free fall with the zero initial velocity, m – mass of the ball, g – acceleration of gravity.

The $F_r(x, \dot{x})$ force, which characterizes the dumping when the ball rebounds from the racket, is assumed in the form

$$F_r(x, \dot{x}) = k_r\dot{x}[H(x-a) + H(-x-a)] \tag{3.2}$$

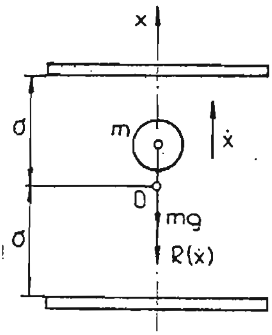


Fig. 2.

where k_r is the coefficient describing losses of energy in the time interval when the ball rebounds. $P(x)$ force, according to the suggestions established in the previous paragraph, is described by the relationship (2.4). Introducing coefficients κ and k_r one describes conditions of the elastoplastic type of impact. After introducing the relationships (2.4) and (3.2) into Eq (3.1), the initial conditions have been assumed in the form

$$x(t)\Big|_{t=0} = x_0 \qquad \dot{x}(t)\Big|_{t=0} = \dot{x}_0 \tag{3.3}$$

and in that way the initial–value problem was obtained. The problem has been solved by means of the numerical integration with the use of the Runge–Kutta method of the fourth order. The results of the numerical calculation are shown below.

4. Rolling mill coupling dynamics with consideration of the clearance in the universal joint

Rolling mill coupling with its length and mass being l and m, respectively, undergoes vibrations in two perpendicular planes: $\eta = 0$ and $\xi = 0$. The system of orthogonal coordinates $0\xi\eta\zeta$ is a moveable system rigidly fixed to the rotating rolling mill coupling, with its 0ξ axis overlying the axis of the rolling mill coupling (Fig.3).

Angles of rotation, which determine an instantaneous position of the rolling mill coupling within the clearance range, will be described as: ψ_ξ and ψ_η – about the 0ξ and 0η axes, respectively. On the rolling mill coupling acts constant force P, directed vertically in the upward direction. The problem will be formulated on

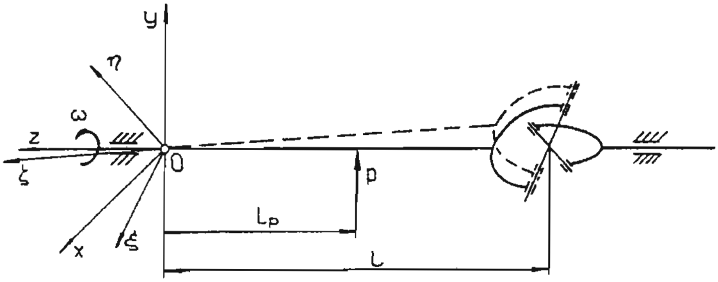


Fig. 3.

the grounds of the angular momentum variation law, from which results that

$$\frac{d\boldsymbol{K}}{dt} = \boldsymbol{M} \tag{4.1}$$

Derivative of the angular momentum $d\boldsymbol{K}/dt$ calculated in the stationary reference system $0xyz$ is equal to

$$\frac{d\boldsymbol{K}}{dt} = \frac{d\check{\boldsymbol{K}}}{dt} + \boldsymbol{\omega} \times \boldsymbol{K} \tag{4.2}$$

where $d\check{\boldsymbol{K}}/dt$ is derivative of the angular momentum in the movable system. Thus the vector equation (4.1) may be written as

$$\frac{d\check{\boldsymbol{K}}}{dt} + \boldsymbol{\omega} \times \boldsymbol{K} = \boldsymbol{M} \tag{4.1a}$$

Vector $\boldsymbol{K}$ in the $0\xi\eta\zeta$ system has the following components

$$K_\xi = J_\xi\omega_\xi \qquad K_\eta = J_\eta\omega_\eta \qquad K_\zeta = J_\zeta\omega_\zeta \tag{4.3}$$

if axes 0ξ, 0η and 0ζ are principal axes of inertia. Components of the moment $\boldsymbol{M}$ will be described as: M_ξ, M_η and M_ζ. If we take into consideration the values of angular momentum and moment components we will get then from Eq (4.1a) the Euler equations system

$$\begin{aligned} J_\xi\dot{\omega}_\xi + (J_\zeta - J_\eta)\omega_\eta\omega_\zeta &= M_\xi \\ J_\eta\dot{\omega}_\eta + (J_\xi - J_\zeta)\omega_\zeta\omega_\xi &= M_\eta \\ J_\zeta\dot{\omega}_\zeta + (J_\eta - J_\xi)\omega_\xi\omega_\eta &= M_\zeta \end{aligned} \tag{4.4}$$

Let us assume that the velocity change in the Cardan universal joint is negligible. Moreover, assuming that angular velocity of the shaft rotation is $\omega_\xi = \omega$, we can rewrite Euler equations in the following form

$$\begin{aligned} J_\xi\dot{\omega}_\xi + (J_\zeta - J_\eta)\omega_\eta\omega &= M_\xi \\ J_\eta\dot{\omega}_\eta + (J_\xi - J_\zeta)\omega_\xi\omega &= M_\eta \end{aligned} \tag{4.5}$$

The latter equation from the system of equations (4.4) will not be used because the velocity changes will be imposed in advance e.g. in the form

$$\omega = b(c - e^{-at}) \tag{4.6}$$

where: a, b and c are the constants.

Moments M_ξ and M_η are determined taking into account the following factors: the fixed force P, the clearance in the universal joints and the friction in the kinematic pair. After some transformations we will get

$$\begin{aligned} M_\xi &= Pl_p \cos\varphi - T_\eta l \\ M_\eta &= -Pl_p \sin\varphi - T_\xi l \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} T_\xi &= \kappa[(\psi_\eta - \delta_\eta)H(\psi_\eta - \delta_\eta) + (\psi_\eta + \delta_\eta)H(-\psi_\eta - \delta_\eta)] + \\ &+ h\dot{\psi}_\eta + k_r\dot{\psi}_\eta[H(\psi_\eta - \delta_\eta) + H(-\psi_\eta - \delta_\eta)] \\ T_\eta &= \kappa[(\psi_\xi - \delta_\xi)H(\psi_\xi - \delta_\xi) + (\psi_\xi + \delta_\xi)H(-\psi_\xi - \delta_\xi)] + \\ &+ h\dot{\psi}_\xi + k_r\dot{\psi}_\xi[H(\psi_\xi - \delta_\xi) + H(-\psi_\xi - \delta_\xi)] \end{aligned} \tag{4.8}$$

in which φ represents the angle of shaft rotation about the 0ζ axis, κ is the coefficient of elasticity, h stands for the dumping coefficient, δ_ξ and δ_η are the parameters of the clearance in the direction of 0η and 0ξ axes, respectively, and k_r denotes the coefficient which characterizes energy losses during impact. As in the previous case, coefficients κ and k_r describe conditions of the elastoplastic type of impact. Initial conditions are assumed in the form

$$\begin{aligned} &\varphi(t)\Big|_{t=0} = \varphi_0 \\ &\psi_\xi(t)\Big|_{t=0} = \psi_{\xi 0} \qquad \dot{\psi}_\xi(t)\Big|_{t=0} = \dot{\psi}_{\xi 0} \\ &\psi_\eta(t)\Big|_{t=0} = \psi_{\eta 0} \qquad \dot{\psi}_\eta(t)\Big|_{t=0} = \dot{\psi}_{\eta 0} \end{aligned} \tag{4.9}$$

Like before, the formulated initial problem has been solved numerically with the use of Runge–Kutta method. Results of the numerical calculations are also specified below.

5. Results of the numerical calculation

5.1. Computer simulation of the table tennis ball motion

Differential equation of the table–tennis ball motion (3.1) with consideration of the initial conditions in the form (3.3), has been solved numerically with the use of Runge–Kutta method of the fourth order. Computations were performed assuming that the distance between rackets was $2a = 0.8$ [m] and the ball mass $m = 2.37 \cdot 10^{-3}$ [kg]. Coefficient of the air resistance has been estabished on the base of time measurements for ball falling down from the known height. An averaged value of it, determined after series of measurements, was $k = 0.122$ [s/m]. Moreover, for all the shown results the following initial conditions were assumed

$$x(t)\Big|_{t=0} = -a \qquad \dot{x}(t)\Big|_{t=0} = -\frac{1}{k}$$

It means that at the initial instant the ball has, when contacting with the lower racket, the non - zero initial velocity. Additionally, it should be noted, that this velocity is equal to the maximum value of velocity which may be obtained by the ball, during its free fall down, without any initial velocity.

The results which are presented in this work have been obtained for various values of the parameters κ and k_r, introduced in the force definitions (relationships (2.4) and (3.2), respectively). These relationships describe conditions of the elastoplastic type of impact. They, as it is known, corresponds to the coefficient of restitution value from the (0,1) range. Values of κ and k_r, which have been assumed for the individual examples, are listed in Table 1. In the enclosed figures was assumed that the curve numbers correspond to the number of examles from Table 1.

Table 1

Example number	κ [kg/s^2]	k_r [kg/s]
1	10000.0	0.1
2	10000.0	0.2
3	10000.0	0.4
4	5000.0	0.2
5	20000.0	0.2

Figs.4 and 5 show influence of the coefficient k_r on the displacement and velocity changes (in time) of the table–tennis ball during its motion between rackets. Curves 1,2 and 3 have been obtained for constant value of the coefficient κ. Increase in k_r coefficient value both causes the greater energy losses what results in lowering the displacement and velocity amplitudes, and as one can see in the pictures affects also the vibration frequency.

Curves shown in Figs.6 and 7 have been obtained for constant value of k_r in order to illustrate the influence of the coefficient κ on the ball motion. Within the assumed changes range of this coefficient, the effect of its value on amplitude magnitude and vibration frequency is also observed.

Phase diagram shown in Fig.8 illustrates motion of the ball for data from Table 1 (example 2). One can clearly distinguish in the picture between different phases

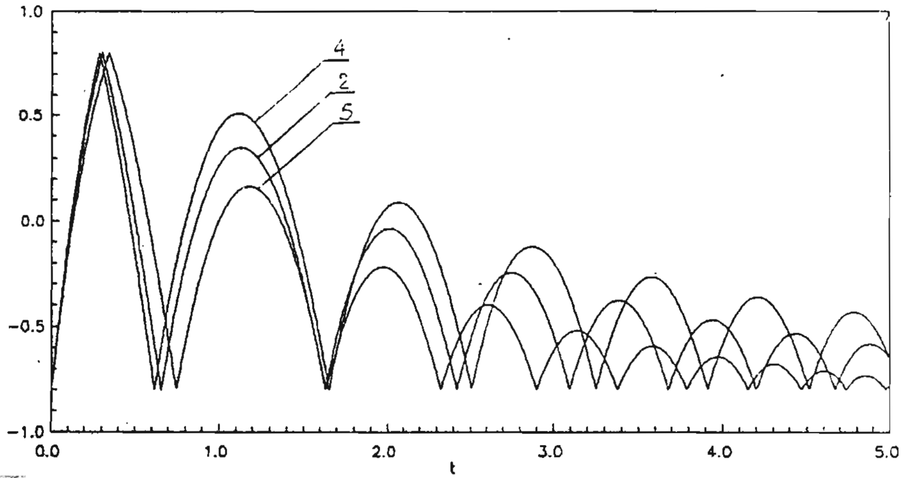

1.0
0.5
0.0
−0.5
−1.0
0.0
1.0
2.0
3.0
4.0
5.0
t
4
2
5

Fig. 4.

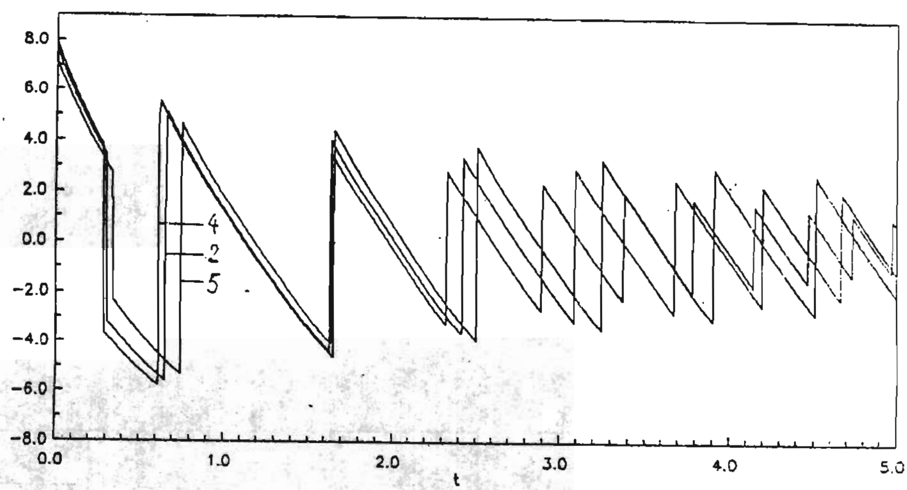

8.0
6.0
4.0
2.0
0.0
-2.0
-4.0
-6.0
-8.0
0.0
1.0
2.0
3.0
4.0
5.0
t
4
2
5

Fig. 5.

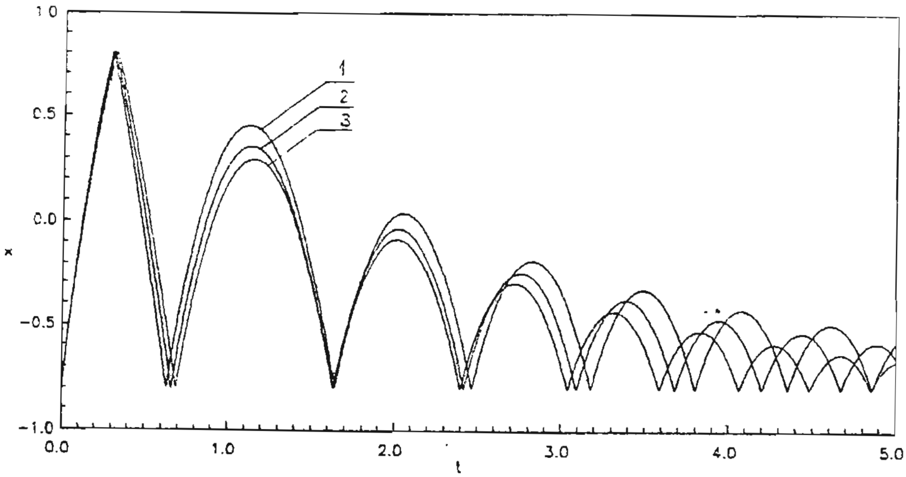
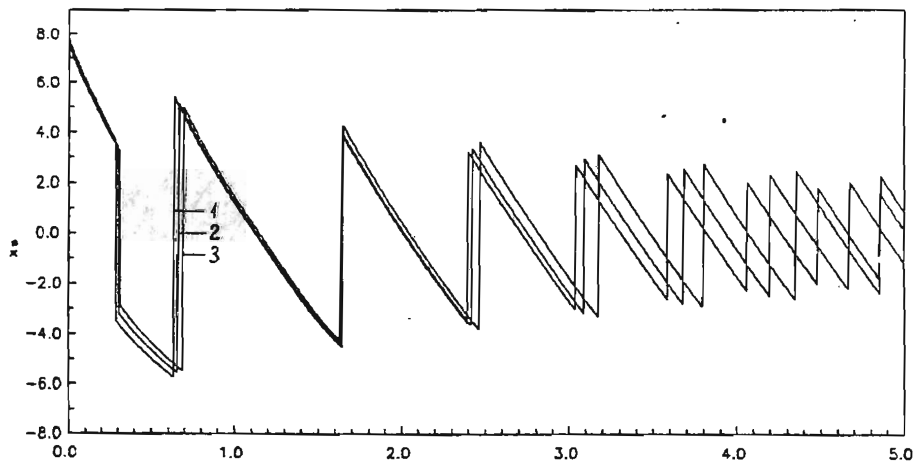

1.0
0.5
0.0
−0.5
−1.0
x
0.0
1.0
2.0
3.0
4.0
5.0
t
1
2
3

Fig. 6.


8.0
6.0
4.0
2.0
0.0
−2.0
−4.0
−6.0
−8.0
0.0
1.0
2.0
3.0
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1
2
3

Fig. 7.

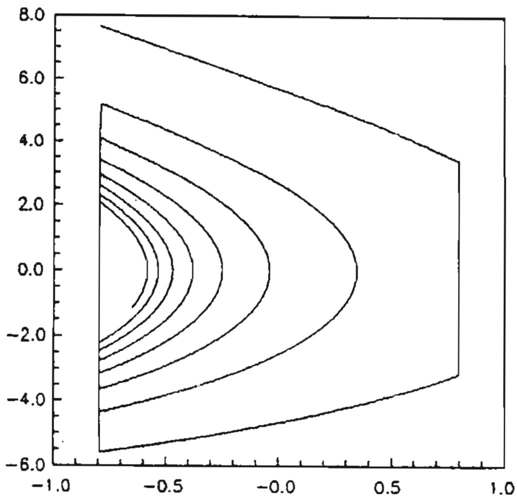


Fig. 8.

of the ball motion, i.e. its motion in the air and by contact with rackets. It should be noted, that for the assumed initial conditions only one contact with the upper racket follows.

5.2. Computer simulation of the rolling mill coupling motion

Like in the case of the table–tennis ball, Euler differential equations (4.5) describing motion of the rolling mill coupling together with the initial conditions (4.9) have been solved numerically with the use of the Runge–Kutta method of the fourth order. The computational results presented below refer to the run–up stage of the rolling mill coupling during the first five seconds. Mass of the coupling has been estimated $m = 24000$ [kg], its length $l = 13$ [m], diameter $d = 0.545$ [m] and the distance from the support to the point of application of the force P, $l_p = 6.5$ [m]. Accordingly, values of the moments of inertia have been calculated on the base of relationships: $J_\eta = J_\xi = ml^2/3$, $J_\zeta = md^2/8$. The run–up is executed from zero velocity. Increase of velocity has been assumed according to the relationship (4.6) and values of the constants are: $a = 0.768$ [1/s], $c = 1$ (b is given in Table 2). Moreover, the following initial conditions have been assumed

$$\varphi(t)\Big|_{t=0} = 0$$

$$\psi_\xi(t)\Big|_{t=0} = -\delta \qquad\qquad \dot{\psi}_\xi(t)\Big|_{t=0} = 0$$

$$\psi_\eta(t)\Big|_{t=0} = 0 \qquad \dot{\psi}_\eta(t)\Big|_{t=0} = 0$$

One may notice that the assumed form of the initial conditions corresponds to the shaft position at the starting moment in static equilibrium point (lowest possible static position of the shaft). The value of the parameter δ depends on the value of the clearance parameter Δ in Cardan universal joint and is determined by the relationship: $\delta = \arctan(\Delta/l)$. Parameters κ and k_r, characterizing the elastoplastic type of impact, remain the for same all examples and have the following values: $\kappa = 0.05J_\xi$, $k_r = 0.02J_\xi$. All other parameters which are different for each example have been listed in Table 2.

Table 2

Example number	$\alpha = \frac{P}{mg}$	Δ [m]	b [1/s]
1	- 0.500	0.010	12
2	- 0.500	0.005	12
3	- 0.500	0.001	12
4	- 0.100	0.005	12
5	- 0.001	0.005	12
6	- 0.100	0.005	6

Curves shown in Figs.9 and 10 are given for illustration influence of the parameters (listed in Table 2) on the value of impact forces in Cardan universal joints. Components of the analyzed forces are described as

$$T_\xi = \kappa[(\psi_\eta - \delta_\eta)H(\psi_\eta - \delta_\eta) + (\psi_\eta + \delta_\eta)H(-\psi_\eta - \delta_\eta)]$$
$$T_\eta = \kappa[(\psi_\xi - \delta_\xi)H(\psi_\xi - \delta_\xi) + (\psi_\xi + \delta_\xi)H(-\psi_\xi - \delta_\xi)]$$

It should be noted, that values of the impact forces increase with the increase of clearance, rotational speed and absolute value of coefficient α which charactrizes the supporting force.

Subsequent Figs.11, 12, 13 and 14 illustrate motion of the rolling mill coupling for the Example 6. Fig.11 is showing the course of changes of the displacements ψ_ξ (—) and ψ_η (- - -). In Figs.12 and 13 one can see phase diagrams of the motion, in planes $\xi = 0$ and $\eta = 0$, respectively. Trajectory of the motion projected on to the plane $z = 0$ of the fixed system of coordinates is shown in Fig.14. The latter figures may suggest, that, for particular parametrs of the system, the motion of the rolling mill coupling may be a chaotic one. This question has to be studied separately.

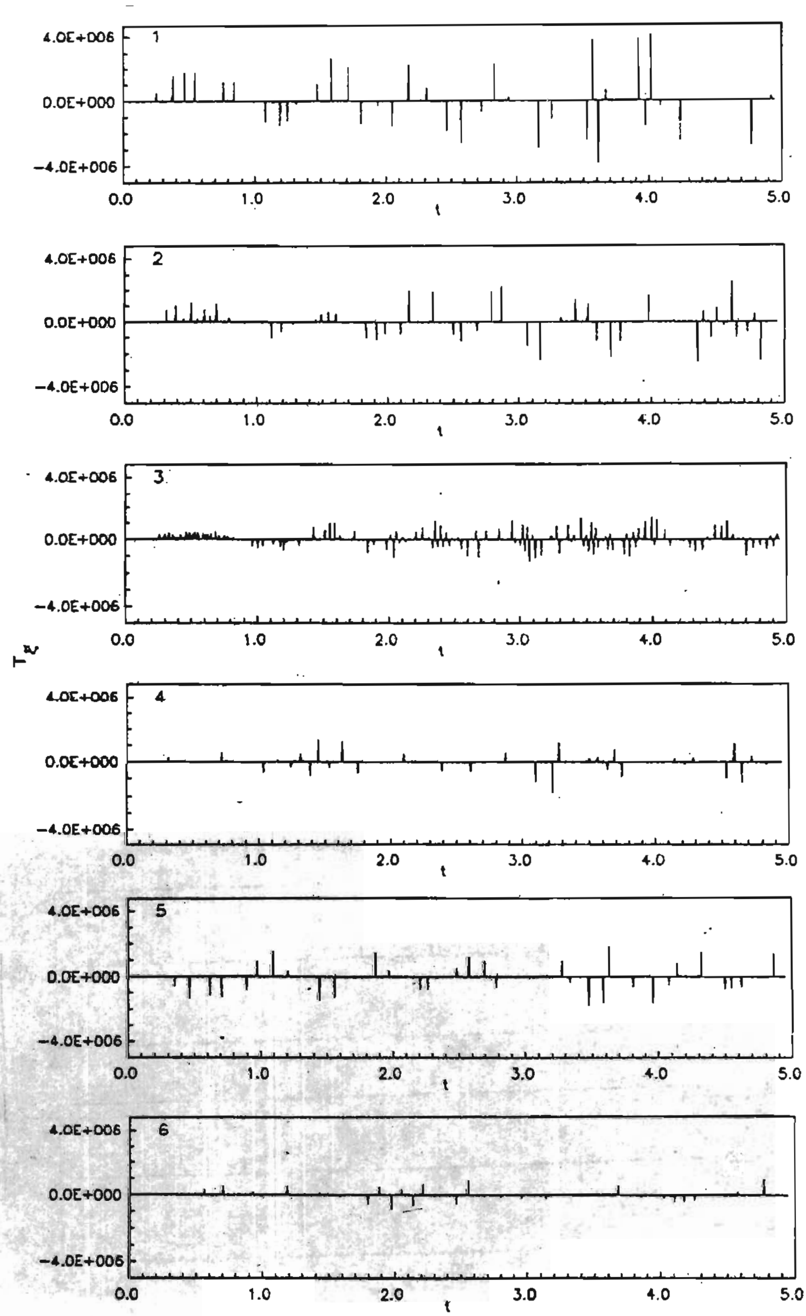


Fig. 9.

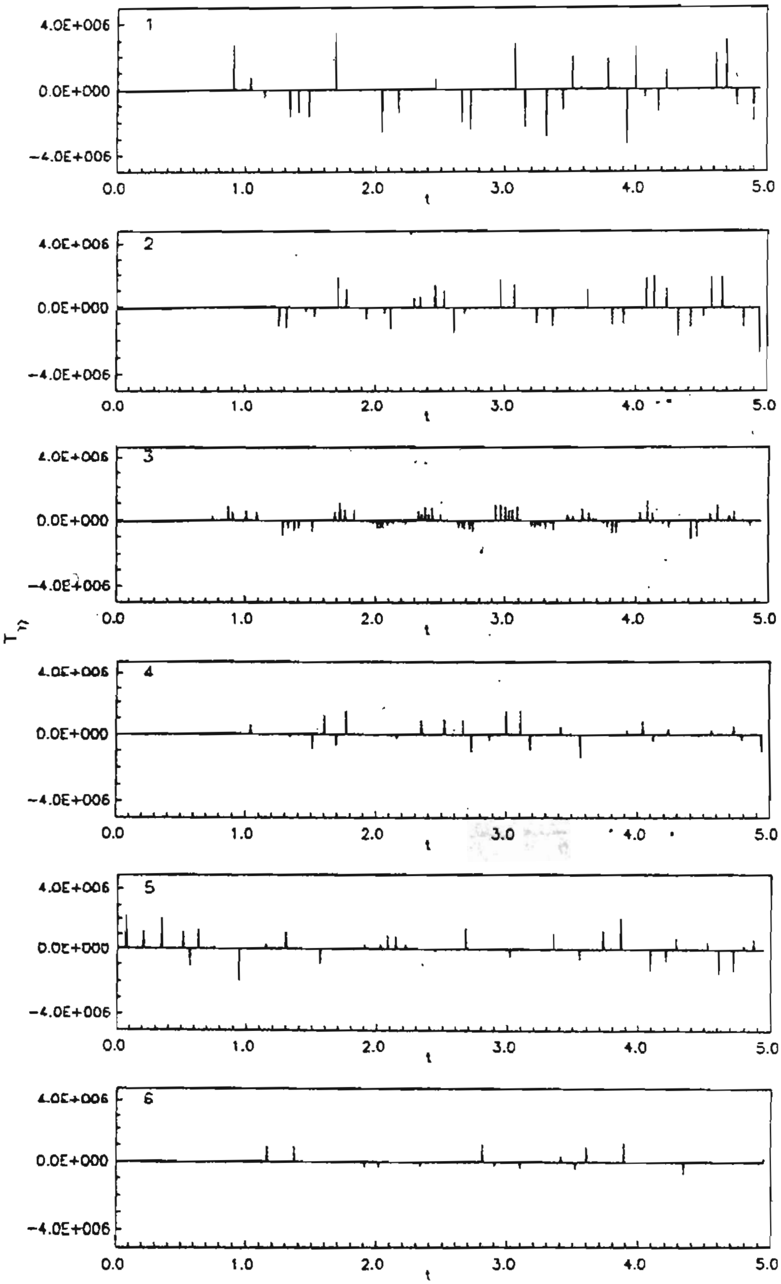
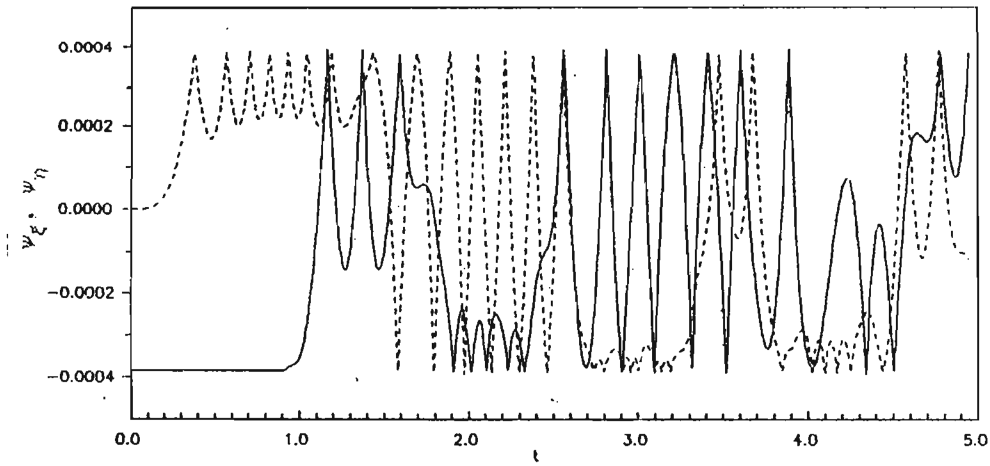


Fig. 10.


0.0004
0.0002
0.0000
−0.0002
−0.0004
0.0
1.0
2.0
3.0
4.0
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t

Fig. 11.

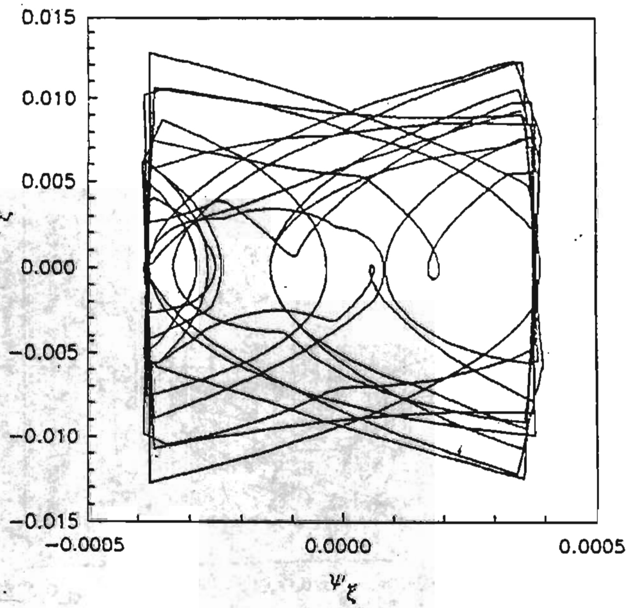
0.015
0.010
0.005
0.000
−0.005
−0.010
−0.015
−0.0005
0.0000
0.0005

Fig. 12.

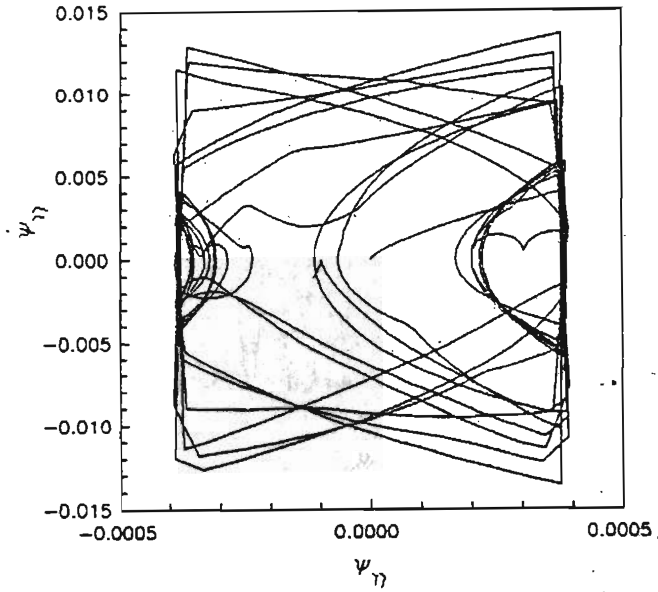
0.015
0.010
0.005
0.000
−0.005
−0.010
−0.015
−0.0005
0.0000
0.0005

Fig. 13.

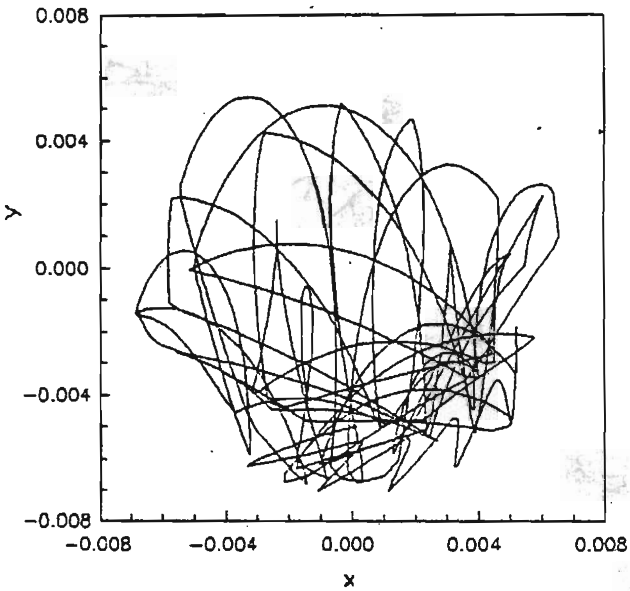
0.008
0.004
0.000
−0.004
−0.008
y
−0.008
−0.004
0.000
0.004
0.008
x

Fig. 14.

6. Final remarks

Examples of dynamics of the systems presented in this work give an illustration of the way how to solve practical problems of the mechanical system dynamics with one-sided constraints.

From the nature of the ball motion rebounding from the table-tennis racket one can evaluate the properties of the base material. Tests of the subsequent rebounds may be used as qualifying factor of the racket-lining with fixed properties of the ball. Similarly, for the fixed properties of base material – the quality of the table-tennis balls may be evaluated on the base of their motion. Results of this work may set up a base for establishing the rackets and balls characteristics identification.

Omitting the problem of table-tennis balls, similar way of approach may be applied to the other element identification e.g. for balls of the rolling bearings.

In the case of the rolling mill coupling the analysis method of system dynamics, which has been presented in this work, allows one to determine dynamic forces with which supporting elements of the coupling are loaded. Knowledge of this force magnitudes and determination of the factors which may allow us to diminish them is, from the construction durability point of view, the basic value problem. Solution to the problem of rolling mill coupling dynamics, which has been formulated in the work, fulfil exactly for the analysed system all the above mentioned requirements. It allows one to determine and to analyse the values of the dynamic reaction.

Basing on the results presented in the paragraph 5.2 of this work we can observe the dependence of dynamic forces on the system parameters under consideration. Moreover, we get the possibility to determine, by using the derived relationships, the values of dynamic forces for arbitrary taken (different) parameters of the system.

Computations, which have been done, give us the basis for the assumption, that for the particular parameters of the system – the motion of the system may be a chaotic motion. This problem, which is a very interesting one, requires additional studies which exceeds the scope of this work.

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Problem dynamiki układów mechanicznych z więzami jednostronnymi

Streszczenie

W pracy rozpatrzono dwa przykłady układów z więzami jednostronnymi. Pierwszy z nich dotyczy ruchu piłeczki pingpongowej pomiędzy rakietkami z uwzględnieniem oporu powietrza. Drugi zaś określa ruch łącznika walcarki w wyniku luzu powstałego w jednym z przegubów. W obu przykładach uwzględniono uderzenie sprężysto-plastyczne, co odpowiada współcznnikowi restytucji (uderzenia) z przedziału od zera do jeden. Sformułowano i rozwiązano numerycznie zagadnienia początkowe ruchu ukadów. Do rozwiązania wykorzystano metodę Rungego-Kutty numerycznego całkowania równań ruchu.

Praca wpłynęła do Redakcji dnia 29 lipca 1990 roku