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DYNAMIC TORSION OF AN ORTHOTROPIC HALF-SPACE

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The exact formulae for displacement and stresses in an orthotropic half-space under a concentrated dynamic torque are obtained by means of Hankel and Laplace transforms.

1. Introduction

Solutions to problems of an elastic solid acted upon by single concentrated force have a property of Green function in the elastic problems, and can be applied to practical problems of elasticity by superpositions or integrations of the solutions. The notion of a concentrated force in elasticity was introduced by Kelvin, Theoretical idealization adopted to an analysis of the problems encountered in geomechanics, fibre-reinforced compsites and in micro-mechanics defects in solids often reduces to the solution of a boundary - value problem involving an orthotropic semi-infinite or infinite elastic medium.

Static problem of a concentrated force solution for anisotropic media has been reported by many authors, e.g. Pan and Chou [1] and Chow and Yang [2]. Several researchers have studied the response of an elastic medium twisted statically or dynamically by an attached rigid disc or annulus (cf Gladwell [3], Tang [4], Rogowski [5]) are can find there an analysis of the torsion of an isotropic, orthotropic and layered transversely isotropic structure. The torsional problem of an orthotropic half-space subject to a concentrated twisting moment, which acts in the interior or on a surface and varies in time is considered in the present contribution. The incorporation of both anisotropy and vibration into the load transfer analysis would enhance the applicability of the solution and its usefulness to engineering practice.

2. Basic equations

Relative to cylindrical coordinate system (r, Θ, z) the stress - displacement relations for axisymmetrical torsion problem are

$$\sigma_{r\Theta} = G_r\Big(\frac{\partial v}{\partial r} - \frac{v}{r}\Big)$$
$$\sigma_{\Theta z} = G_z\frac{\partial v}{\partial z} \tag{2.1}$$

where $\sigma_{r\Theta}$, $\sigma_{\Theta z}$ are the stress components, v is the displacement and G_r, G_z are the material shear moduli for the planes parallel and perpendicular to the r–axis, respectively. Substituting Eqs (2.1) into equation of motion

$$\frac{\partial \sigma_{r\Theta}}{\partial r} + \frac{\partial \sigma_{\Theta z}}{\partial z} + \frac{2}{r}\sigma_{r\Theta} = \rho\frac{\partial^2 v}{\partial t^2} \tag{2.2}$$

one obtains

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{1}{\mu^2}\frac{\partial^2 v}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} \tag{2.3}$$

where

$$\mu^2 = \frac{G_r}{G_z} \qquad\qquad c^2 = \frac{G_r}{\rho} \tag{2.4}$$

and ρ is the mass density.

The boundary and continuity (or discontinuity) conditions are

$$\sigma_{z\Theta}(r, 0, t) = 0 \tag{2.5}$$
$$[v(r, z', t)] = 0 \tag{2.6}$$
$$[\sigma_{z\Theta}(r, z', t)] = -T_0 f(t) g(r) \tag{2.7}$$

where T_0 is a constant couple, $f(t)$ is a function of time, $g(r)$ is a function of radial coordinate, which is axisymmetric, and the symbol $[\cdot]$ denotes the jump of the function, defined as follows

$$[h(z)]\Big|_{z=z'} = \lim_{\Delta\to 0}[h(z'+\Delta) - h(z'-\Delta)] \tag{2.8}$$

It is assumed that the solid body is initially at rest. With these assumptions the problem is solved by the use of integral transforms.

3. Application of integral transforms

To solve the partial differential Eq (2.3) under boundary conditions (2.5) to (2.7) we use the Laplace (with respect to time) and Hankel (with respect to radial coordinate) transforms defined as follows

$$\hat{v} = \int_0^\infty v \exp(-st)\, dt$$
$$v^1 = \int_0^\infty v r J_1(r\xi)\, dr \tag{3.1}$$

where s and ξ are the Laplace and Hankel parameters and $J_1(r\xi)$ is the Bessel function of the first kind and order one. Applying the foregoing transforms to Eqs (2.3) and Eqs (2.5) to (2.7) (with the use of (2.1) in (2.5) and (2.7)) we have

$$\frac{1}{\mu^2}\frac{\partial^2 \hat{v}^1}{\partial z^2} - \Big(\xi^2 + \frac{s^2}{c^2}\Big)\hat{v}^1 = 0 \tag{3.2}$$

$$\frac{\partial \hat{v}^1}{\partial z}\Big|_{z=0} = 0 \tag{3.3}$$

$$[\hat{v}^1]\Big|_{z=z'} = 0 \tag{3.4}$$

$$[\frac{\partial \hat{v}^1}{\partial z}]\Big|_{z=z'} = -\frac{T_0}{G_z}\hat{f}(s)g^1(\xi) \tag{3.5}$$

For a torsional load with the resultant couple T_0 which acts along the circumference of a circle with radius equal to a the function $g(r)$ takes the form

$$g(r) = \frac{\delta(r-a)}{2\pi r^2} \tag{3.6}$$

where $\delta(r-a)$ is the Dirac delta function. The Hankel transform of it is

$$g^1(\xi) = \frac{J_1(\xi a)}{2\pi a} \tag{3.7}$$

For concentrated point torque we obtain

$$g^1(\xi) = \frac{\xi}{4\pi} \tag{3.8}$$

Applying the foregoing transforms to Eqs (2.1) we obtain

$$\hat{\sigma}^2_{r\Theta} = -G_r \xi \hat{v}^1$$
$$\hat{\sigma}^1_{\Theta z} = G_z \frac{\partial \hat{v}^1}{\partial z} \tag{3.9}$$

where the superscript 2 denotes the second–order Hankel transform of $\sigma_{r\Theta}$.

The solution to Eq (3.2) is

$$\hat{v}^1 = A(s,\xi)\,\mathrm{e}^{-\beta\mu z} + B(s,\xi)\,\mathrm{e}^{\beta\mu z} \qquad 0 \le z \le z' \tag{3.10}$$

$$\hat{v}^1 = C(s,\xi)\,\mathrm{e}^{-\beta\mu z} \qquad z \ge z' \tag{3.11}$$

where

$$\beta = \left(\xi^2 + \frac{s^2}{c^2}\right)^{\frac{1}{2}} \tag{3.12}$$

In the domain $z > z'$ the solution has the form (3.11) to ensure regularity of displacement and stresses at infinity. The transforms $\hat{\sigma}^2_{r\Theta}$ and $\hat{\sigma}^1_{\Theta z}$ may be written in terms of the three unknown functions $A(s,\xi)$, $B(s,\xi)$ and $C(s,\xi)$ by substituting Eqs (3.10) and (3.11) into Eqs (3.9). The boundary conditions (3.3) to (3.5) yield these three functions. Finally the Laplace and Hankel transforms of displacement and stresses are

$$\hat{v}^1 = \frac{T_0}{8\pi G_z \mu}\,\frac{\xi}{\beta}\left[\mathrm{e}^{-\beta\mu(z+z')} + \mathrm{e}^{-\beta\mu|z-z'|}\right]\hat{f}. \tag{3.13}$$

$$\hat{\sigma}^1_{z\Theta} = -\frac{T_0}{8\pi}\,\xi\left[\mathrm{e}^{-\beta\mu(z+z')} + \mathrm{sgn}(z-z')\mathrm{e}^{-\beta\mu|z-z'|}\right]\hat{f} \tag{3.14}$$

$$\hat{\sigma}^2_{r\Theta} = -\frac{T_0\mu}{8\pi}\,\frac{\xi^2}{\beta}\left[\mathrm{e}^{-\beta\mu(z+z')} + \mathrm{e}^{-\beta\mu|z-z'|}\right]\hat{f} \tag{3.15}$$

4. Exact closed form solution

Taking, for the inverse Hankel transform of Eqs (3.13) to (3.15), the following integral

$$\int_0^\infty J_0(\xi r)\,\frac{\mathrm{e}^{-z\sqrt{\xi^2+(s/c)^2}}}{\sqrt{\xi^2+(s/c)^2}}\,\xi\,d\xi = \frac{\mathrm{e}^{(s/c)\sqrt{z^2+r^2}}}{\sqrt{z^2+r^2}} \tag{4.1}$$

and applying for inverse Laplace transform

$$\mathcal{L}^{-1}\left[e^{-qs}\hat{f}(s)\right] = \begin{cases} f(t-q) & t \geq q \\ 0 & 0 \leq t < q \end{cases} \tag{4.2}$$

the original solution of vibrations problem about an equilibrium state may be expresses by

$$\begin{aligned} v(r,z,t) = \frac{T_0}{8\pi G_z \mu} r \Bigg\{ & \frac{1}{\left[\mu^2(z+z')^2+r^2\right]^{3/2}} \left[1 + \frac{\sqrt{\mu^2(z+z')^2+r^2}}{c}\frac{\partial}{\partial t}\right] \cdot \\ & \cdot f\left(t - \frac{\sqrt{\mu^2(z+z')^2+r^2}}{c}\right) + \frac{1}{\left[\mu^2(z-z')^2+r^2\right]^{3/2}} \cdot \\ & \cdot \left[1 + \frac{\sqrt{\mu^2(z-z')^2+r^2}}{c}\frac{\partial}{\partial t}\right] f\left(t - \frac{\sqrt{\mu^2(z-z')^2+r^2}}{c}\right) \Bigg\} \end{aligned} \tag{4.3}$$

$$\begin{aligned} \sigma_{z\Theta}(r,z,t) = -\frac{3T_0\mu}{8\pi} r \Bigg\{ & \frac{z+z'}{\left[\mu^2(z+z')^2+r^2\right]^{5/2}} \left[1 + \frac{\sqrt{\mu^2(z+z')^2+r^2}}{c}\frac{\partial}{\partial t} + \right. \\ & \left. + \frac{1}{3}\frac{\mu^2(z+z')^2+r^2}{c^2}\frac{\partial^2}{\partial t^2}\right] f\left(t - \frac{\sqrt{\mu^2(z+z')^2+r^2}}{c}\right) + \frac{z-z'}{\left[\mu^2(z-z')^2+r^2\right]^{5/2}} \cdot \\ & \cdot \left[1 + \frac{\sqrt{\mu^2(z-z')^2+r^2}}{c}\frac{\partial}{\partial t} + \frac{1}{3}\frac{\mu^2(z-z')^2+r^2}{c^2}\frac{\partial^2}{\partial t^2}\right] \cdot \\ & \cdot f\left(t - \frac{\sqrt{\mu^2(z-z')^2+r^2}}{c}\right) \Bigg\} \end{aligned} \tag{4.4}$$

$$\begin{aligned} \sigma_{r\Theta}(r,z,t) = -\frac{3T_0\mu}{8\pi} r^2 \Bigg\{ & \frac{1}{\left[\mu^2(z+z')^2+r^2\right]^{5/2}} \left[1 + \frac{\sqrt{\mu^2(z+z')^2+r^2}}{c}\frac{\partial}{\partial t} + \right. \\ & \left. + \frac{1}{3}\frac{\mu^2(z+z')^2+r^2}{c^2}\frac{\partial^2}{\partial t^2}\right] f\left(t - \frac{\sqrt{\mu^2(z+z')^2+r^2}}{c}\right) + \frac{1}{\left[\mu^2(z-z')^2+r^2\right]^{5/2}} \cdot \\ & \cdot \left[1 + \frac{\sqrt{\mu^2(z-z')^2+r^2}}{c}\frac{\partial}{\partial t} + \frac{1}{3}\frac{\mu^2(z-z')^2+r^2}{c^2}\frac{\partial^2}{\partial t^2}\right] \cdot \\ & \cdot f\left(t - \frac{\sqrt{\mu^2(z-z')^2+r^2}}{c}\right) \Bigg\} \end{aligned} \tag{4.5}$$

where $f(t-q)$ equals zero for negative argument. The disturbance of displacement and stress fields vanishes if $t < \sqrt{\mu^2(z-z')^2+r^2}/c$, in other words, the displacement equals zero ahead of the wave–front. If

$$\frac{\sqrt{\mu^2(z-z')^2+r^2}}{c} \le t < \frac{\sqrt{\mu^2(z+z')^2+r^2}}{c}$$

the second summands are the only non–zero in the solution (4.3) to (4.5), while for $t \ge \sqrt{\mu^2(z+z')^2+r^2}/c$ the solution has both summands. The latter is the superposition case of moving wave with the one reflected at the boundary surface. In the special cases we observe the following significant results.

4.1. Time – harmonic excitation

With the $\exp(\mathrm{i}\omega t)$ time–dependence (steady – state disturbance of frequency ω, $\mathrm{i}^2=-1$), the solutions of vibrations about an equilibrium state are

$$\begin{aligned} v(r,z,t) = \frac{T_0\exp(\mathrm{i}\omega t)}{8\pi G_z\mu}\, r \Bigg\{ & \frac{1}{\big[\mu^2(z+z')^2+r^2\big]^{3/2}}\Big[1+\sqrt{\mu^2(z+z')^2+r^2}\frac{\mathrm{i}\omega}{c}\Big]\cdot \\ & \cdot \mathrm{e}^{-\frac{\mathrm{i}\omega}{c}\sqrt{\mu^2(z+z')^2+r^2}} + \frac{1}{\big[\mu^2(z-z')^2+r^2\big]^{3/2}}\cdot \\ & \cdot\Big[1+\sqrt{\mu^2(z-z')^2+r^2}\frac{\mathrm{i}\omega}{c}\Big]\mathrm{e}^{-\frac{\mathrm{i}\omega}{c}\sqrt{\mu^2(z-z')^2+r^2}}\Bigg\} \end{aligned} \tag{4.6}$$

$$\begin{aligned} \sigma_{z\Theta}(r,z,t) = -\frac{3T_0\exp(\mathrm{i}\omega t)}{8\pi}\,\mu r \Bigg\{ & \frac{z+z'}{\big[\mu^2(z+z')^2+r^2\big]^{5/2}}\cdot \\ & \cdot\Big[1+\sqrt{\mu^2(z+z')^2+r^2}\frac{\mathrm{i}\omega}{c}-\frac{1}{3}\big(\mu^2(z+z')^2+r^2\big)\frac{\omega^2}{c^2}\Big]\cdot \\ & \cdot \mathrm{e}^{-\frac{\mathrm{i}\omega}{c}\sqrt{\mu^2(z+z')^2+r^2}} + \frac{z-z'}{\big[\mu^2(z-z')^2+r^2\big]^{5/2}}\cdot \\ & \cdot\Big[1+\sqrt{\mu^2(z-z')^2+r^2}\frac{\mathrm{i}\omega}{c}-\frac{1}{3}\big(\mu^2(z-z')^2+r^2\big)\frac{\omega^2}{c^2}\Big]\cdot \\ & \cdot \mathrm{e}^{-\frac{\mathrm{i}\omega}{c}\sqrt{\mu^2(z-z')^2+r^2}}\Bigg\} \end{aligned} \tag{4.7}$$

$$\begin{aligned} \sigma_{r\Theta}(r,z,t) = -\frac{3T_0\exp(\mathrm{i}\omega t)}{8\pi}\,\mu r^2 \Bigg\{ & \frac{1}{\big[\mu^2(z+z')^2+r^2\big]^{5/2}}\cdot \\ & \cdot\Big[1+\sqrt{\mu^2(z+z')^2+r^2}\frac{\mathrm{i}\omega}{c}-\frac{1}{3}\big(\mu^2(z+z')^2+r^2\big)\frac{\omega^2}{c^2}\Big]\cdot \end{aligned}$$

$$\cdot \mathrm{e}^{-\frac{\mathrm{i}\omega}{c}\sqrt{\mu^2(z+z')^2+r^2}} + \frac{1}{\left[\mu^2(z-z')^2+r^2\right]^{5/2}}\Big[1+ \tag{4.8}$$

$$+\sqrt{\mu^2(z-z')^2+r^2}\frac{\mathrm{i}\omega}{c} - \frac{1}{3}\left(\mu^2(z-z')^2+r^2\right)\frac{\omega^2}{c^2}\Big]\mathrm{e}^{-\frac{\mathrm{i}\omega}{c}\sqrt{\mu^2(z-z')^2+r^2}}\Bigg\}$$

4.2. Static solution

The stress and displacement equations derived above can be readily reduced to the static case if $\frac{\partial}{\partial t}(\cdot) = 0$ and $f(t) = 1$. These are

$$v_S(r,z) = \frac{T_0}{8\pi G_z \mu}\left\{\frac{r}{\left[\mu^2(z+z')^2+r^2\right]^{3/2}} + \frac{r}{\left[\mu^2(z-z')^2+r^2\right]^{3/2}}\right\} \tag{4.9}$$

$$\sigma_{\Theta z,S}(r,z) = -\frac{3T_0\mu}{8\pi}\left\{\frac{(z+z')r}{\left[\mu^2(z+z')^2+r^2\right]^{5/2}} + \frac{(z-z')r}{\left[\mu^2(z-z')^2+r^2\right]^{5/2}}\right\} \tag{4.10}$$

$$\sigma_{r\Theta,S}(r,z) = -\frac{3T_0\mu}{8\pi}\left\{\frac{r^2}{\left[\mu^2(z+z')^2+r^2\right]^{5/2}} + \frac{r^2}{\left[\mu^2(z-z')^2+r^2\right]^{5/2}}\right\} \tag{4.11}$$

4.3. External point torque

The solution for the external concentrated torque is obtained for $z' = 0$. As an example, we consider time–harmonic excitation by means of a surface torque. The dynamic displacement and stress components being complex quantities have the form

$$v_D(r,z,t) = v_S(r,z)\Big[1+\sqrt{\mu^2z^2+r^2}\frac{\mathrm{i}\omega}{c}\Big]\exp\Big[\mathrm{i}\omega\Big(t-\frac{\sqrt{\mu^2z^2+r^2}}{c}\Big)\Big] \tag{4.12}$$

$$\sigma_{\Theta z,D}(r,z,t) = \sigma_{\Theta z,S}(r,z)\Big[1-\frac{1}{3}(\mu^2z^2+r^2)\frac{\omega^2}{c^2}+\sqrt{\mu^2z^2+r^2}\frac{\mathrm{i}\omega}{c}\Big]\cdot$$

$$\cdot\exp\Big[\mathrm{i}\omega\Big(t-\frac{\sqrt{\mu^2z^2+r^2}}{c}\Big)\Big] \tag{4.13}$$

$$\sigma_{r\Theta,D}(r,z,t) = \sigma_{r\Theta,S}(r,z)\Big[1 - \frac{1}{3}(\mu^2z^2 + r^2)\frac{\omega^2}{c^2} + \sqrt{\mu^2z^2 + r^2}\frac{\mathrm{i}\omega}{c}\Big] \cdot$$
$$\cdot \exp\Big[\mathrm{i}\omega\Big(t - \frac{\sqrt{\mu^2z^2 + r^2}}{c}\Big)\Big] \tag{4.14}$$

where

$$v_S(r,z) = \frac{T_0}{4\pi G_z\mu}\frac{r}{\left(\mu^2z^2 + r^2\right)^{3/2}} \tag{4.15}$$

$$\sigma_{\Theta z,S}(r,z) = -\frac{3T_0\mu}{4\pi}\frac{rz}{\left(\mu^2z^2 + r^2\right)^{5/2}} \tag{4.16}$$

$$\sigma_{r\Theta,S}(r,z) = -\frac{3T_0\mu}{4\pi}\frac{r^2}{\left(\mu^2z^2 + r^2\right)^{5/2}} \tag{4.17}$$

are the static solutions, which agree with the solutions given by Chow and Yang [2].

The dynamic amplification factors, i.e. the ratio of the displacement and stress amplitudes $|v|$ and $|\sigma_{ij}|$ for dynamic problem to a static response may be written as

$$\frac{|v_D|}{v_S} = \Big[1 + (\mu^2z^2 + r^2)\frac{\omega^2}{c^2}\Big]^{\frac{1}{2}} \tag{4.18}$$

$$\frac{|\sigma_{r\Theta,D}|}{|\sigma_{r\Theta,S}|} = \Big[1 + \frac{1}{3}(\mu^2z^2 + r^2)\frac{\omega^2}{c^2}\Big(1 + \frac{1}{3}(\mu^2z^2 + r^2)\frac{\omega^2}{c^2}\Big)\Big]^{\frac{1}{2}} \tag{4.19}$$

and $|\sigma_{r\Theta,D}|/|\sigma_{\Theta z,S}|$ has the same value as (4.19) for $z \neq 0$.

The other cases of excitation being a function of time may be also easily considered. The isotropic counterpart of the solution is obtained for $\mu = 1$.

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Dynamiczne skręcanie ortotropowej półprzestrzeni

Streszczenie

Otrzymano ścisłe wzory dla przemieszczenia i naprężeń w ortotropowej półprzestrzeni skręcanej zmieniającym się w czasie momentem skupionym, wykorzystując metodę transformacji całkowych Hankela i Laplace'a.

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