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Journal of Theoretical
and Applied Mechanics
4, 30, 1992

OPTIMAL FIBRE ORIENTATION IN COMPOSITE ELEMENT

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The paper deals with the composite materials in which matrix is reinforced with the two systems of parallel continuous fibres. The optimal directions of the systems of fibres are obtained from the criterion of the minimum of strain energy. The numerical solutions are obtained for different normal and tangent loads.

1. Introduction

Optimization of the high strength composites internal structure was considered in many papers. The angle of the orientation of the parallel fibres system was there often the only design variable. The optimal direction of one system of fibres obtained for example from the criterium of maximum strength (cf [1,6]) and minimum weight (cf [3]).

Brandt [2] deals with the internal structure optimization can be for different kind of materials composed of brittle matrix and ductile fibres. The optimum fibre orientation is determined there using the maximum fracture energy as a criterion.

In the present paper like in the previous ones (cf [4] and [5]) the optimal directions of two systems of fibres are obtained using the criterion of the minimum of strain energy. The numerical solutions are obtained for different normal and tangent loads in the two cases: 1 – a composite element in which polyamide matrix is reinforced with carbon fibres, 2 – composite element in which epoxide matrix is reinforced with boron fibres.

2. Strains in a composite element

Let us consider the composite element in the shape of a plate of thickness $2h$, in which the matrix is reinforced with two families of parallel fibres of the slope

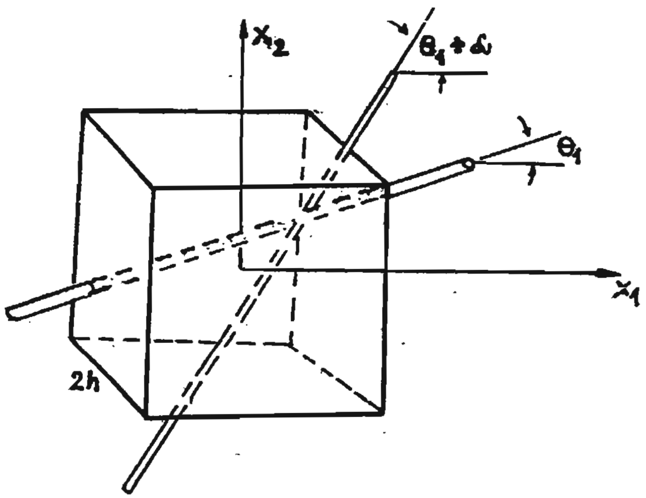

Fig. 1.

Θ_1 and $\Theta_1+\alpha$, respectively (Fig.1). The composite element is under the plane state of stress which can be described by three stress components of a generalized plane stress state $\sigma^{(c)}_{\alpha\beta}$, $\alpha,\beta=1,2$. The stress components $\sigma^{(c)}_{\alpha\beta}$ are corresponding to certain mean values over the thickness of the plate. From the definition of a generalized plane stress state the following relation may be found for the mean composite stress $\sigma^{(c)}_{\alpha\beta}$, the mean matrix $\sigma^{(m)}_{\alpha\beta}$ and the stresse in both fibre systems $\sigma^{(f)}_a$, $\sigma^{(f)}_b$, respectively (cf [4]).

$$\sigma^{(c)}_{\alpha\beta}=\frac{h_m}{h}\sigma^{(m)}_{\alpha\beta}+\frac{h_a}{h}\sigma^{(f)}_a a_\alpha a_\beta+\frac{h_b}{h}\sigma^{(f)}_b b_\alpha b_\beta \tag{2.1}$$

here h_m defines the total thickness of the matrix, h_a and h_b define the thicknesses of both layers of fibres $(h=h_m+h_a+h_b)$. The stresses $\sigma^{(f)}_a$ and $\sigma^{(f)}_b$ are oriented along the fibres, it means according to tangents to the both fibre systems – vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Assuming, that the matrix and the thin fibres are build of linear – elastic, isotropic and homogeneous materials, the corresponding constitutive equations have the form

$$\begin{gathered}\sigma^{(m)}_{\alpha\beta}=\frac{E^{(m)}}{1+\nu}\varepsilon^{(m)}_{\alpha\beta}+\frac{\nu E^{(m)}}{(1+\nu)(1-\nu)}\delta_{\alpha\beta}\varepsilon^{(m)}_{\delta\delta}\\ \sigma^{(f)}_a=E^{(f)}\varepsilon^{(f)}_a \qquad \sigma^{(f)}_b=E^{(f)}\varepsilon^{(f)}_b\end{gathered} \tag{2.2}$$

here $E^{(m)}$ and $E^{(f)}$ are the Young moduli of matrix and fibres, respectively and ν is the Poisson ratio of the matrix.

Next, the continuity of strain of the matrix and fibres is assumed

$$\varepsilon_a^{(f)} = \varepsilon_a^{(m)} \qquad \varepsilon_b^{(f)} = \varepsilon_b^{(m)} \tag{2.3}$$

Substituting Eqs (2.2) and (2.3) into Eq (2.1) and using the following relations

$$\varepsilon_a^{(m)} = \varepsilon_{\gamma\delta}^{(m)} a_\gamma a_\delta$$

$$\varepsilon_b^{(m)} = \varepsilon_{\gamma\delta}^{(m)} b_\gamma b_\delta$$

results in the expression for the mean stress

$$\begin{aligned}\sigma_{\alpha\beta}^{(c)} &= \frac{E^{(m)}}{1+\nu}\Big(\varepsilon_{\alpha\beta}^{(m)} + \frac{\nu}{1-\nu}\delta_{\alpha\beta}\varepsilon_{\delta\delta}^{(m)}\Big)\frac{h_m}{h} + E^{(f)}\frac{h_a}{h}\varepsilon_{\gamma\delta}^{(m)} a_\gamma a_\delta a_\alpha a_\beta + \\ &+ E^{(f)}\frac{h_b}{h}\varepsilon_{\gamma\delta}^{(m)} b_\gamma b_\delta b_\alpha b_\beta\end{aligned} \tag{2.4}$$

From this equation the strain components $\varepsilon_{11}^{(m)}, \varepsilon_{12}^{(m)}, \varepsilon_{22}^{(m)}$ can be uniquely derived.

Let us suppose that the element is subjected to the two tensile loads: p along the axis x_2 and q along the axis x_1 and to a tangent load τ. If the fibres of both systems are uniformly distributed, that is if

$$\frac{h_a}{h} = \frac{h_b}{h} = \frac{1}{2}\frac{h-h_m}{h}$$

and if for compactness the substitution are made for

$$\gamma = E^{(m)}\frac{h_m}{h} \qquad \beta = E^{(f)}\frac{h-h_m}{h}$$

the strain components are given in the following form

$$\begin{aligned}\varepsilon_{11}^{(m)} &= \frac{1}{D}\Big\{q\Big[\frac{\gamma^2}{(1+\nu)^2(1-\nu)} + \frac{\gamma\beta}{4(1+\nu)(1-\nu)}\Big(1-\cos 2(2\Theta_1+\alpha)\cos 2\alpha\Big) + \\ &+\frac{\gamma\beta}{2(1+\nu)}\Big(\frac{3}{4} - \cos(2\Theta_1+\alpha)\cos\alpha + \frac{1}{4}\cos 2(2\Theta_1+\alpha)\cos 2\alpha\Big) + \\ &+\frac{\beta^2}{8}\Big(\cos\alpha - \cos(2\Theta_1+\alpha)\Big)^2\sin^2\alpha\Big] + p\Big[-\frac{\nu\gamma^2}{(1+\nu)^2(1-\nu)} - \\ &-\frac{\gamma\beta}{8(1-\nu)}\Big(1-\cos 2(2\Theta_1+\alpha)\cos 2\alpha\Big) + \frac{\beta^2}{16}\Big(\cos 2\alpha - \cos 2(2\Theta_1+\alpha)\Big)\cdot \\ &\cdot\sin^2\alpha\Big] + \tau\Big[-\frac{\gamma\beta}{2(1+\nu)}\sin(2\Theta_1+\alpha)\cos\alpha - \frac{\gamma\beta}{4(1-\nu)}\sin 2(2\Theta_1+\alpha)\cdot\end{aligned}$$

$$\cdot \cos 2\alpha - \frac{\beta^2}{4}\Big(\cos\alpha - \cos(2\Theta_1+\alpha)\Big)\sin(2\Theta_1+\alpha)\sin^2\alpha\Big]\Big\}$$

$$\begin{aligned}
\varepsilon_{22}^{(m)} = \frac{1}{D}\Big\{q\Big[&-\frac{\nu\gamma^2}{(1+\nu)^2(1-\nu)} - \frac{\gamma\beta}{8(1-\nu)}\Big(1-\cos 2(2\Theta_1+\alpha)\cos 2\alpha\Big) + \\
&+\frac{\beta^2}{16}\Big(\cos 2\alpha - \cos 2(2\Theta_1+\alpha)\Big)\sin^2\alpha\Big] + p\Big[\frac{\gamma^2}{(1+\nu)^2(1-\nu)} + \\
&+\frac{\gamma\beta}{4(1+\nu)(1-\nu)}\Big(1-\cos 2(2\Theta_1+\alpha)\cos 2\alpha\Big) + \\
&+\frac{\gamma\beta}{8(1+\nu)}\Big(3+\cos 2(2\Theta_1+\alpha)\cos 2\alpha + 4\cos(2\Theta_1+\alpha)\cos\alpha\Big) + \\
&+\frac{\beta^2}{8}\Big(\cos(2\Theta_1+\alpha)+\cos\alpha\Big)^2\sin^2\alpha\Big] + \\
&+\tau\Big[-\frac{\gamma\beta}{2(1+\nu)}\sin(2\Theta_1+\alpha)\cos\alpha + \frac{\gamma\beta}{4(1-\nu)}\sin 2(2\Theta_1+\alpha)\cos 2\alpha - \\
&-\frac{\beta^2}{4}\Big(\cos\alpha + \cos(2\Theta_1+\alpha)\Big)\sin(2\Theta_1+\alpha)\sin^2\alpha\Big]\Big\}
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
\varepsilon_{12}^{(m)} = \frac{1}{D}\Big\{q\Big[&-\frac{\gamma\beta}{4(1+\nu)}\sin(2\Theta_1+\alpha)\cos\alpha - \frac{\gamma\beta}{8(1-\nu)}\sin 2(2\Theta_1+\alpha)\cdot \\
&\cdot\cos 2\alpha - \frac{\beta^2}{8}\Big(\cos\alpha - \cos(2\Theta_1+\alpha)\Big)\sin(2\Theta_1+\alpha)\sin^2\alpha\Big] + \\
&+p\Big[-\frac{\gamma\beta}{4(1+\nu)}\sin(2\Theta_1+\alpha)\cos\alpha + \frac{\gamma\beta}{8(1-\nu)}\sin 2(2\Theta_1+\alpha)\cos 2\alpha - \\
&-\frac{\beta^2}{8}\Big(\cos(2\Theta_1+\alpha)+\cos\alpha\Big)\sin(2\Theta_1+\alpha)\sin^2\alpha\Big] + \\
&+\tau\Big[\frac{\gamma^2}{(1+\nu)(1-\nu)} + \frac{\gamma\beta}{(1+\nu)(1-\nu)} - \frac{\gamma\beta}{4(1-\nu)}\cdot \\
&\cdot\Big(1-\cos 2(2\Theta_1+\alpha)\cos 2\alpha\Big) + \frac{\beta^2}{4}\sin^2(2\Theta_1+\alpha)\sin^2\alpha\Big]\Big\}.
\end{aligned}$$

here

$$D = \frac{\gamma}{(1+\nu)(1-\nu)}\Big[\frac{\gamma^2}{1+\nu} + \frac{\gamma\beta}{1+\nu} + \frac{\beta^2}{4}\Big(1-\cos^4\alpha - \nu\sin^4\alpha\Big)\Big]$$

3. Optimal directions of fibre systems

3.1. Optimization criterion and the necessary conditions for the minimum

The minimum of the strain energy is chosen as the optimization criterion to determine the fibres directions (cf [4]). The energy may be expressed by a following relation

$$U = \frac{1}{2}\iiint\limits_V \sigma_{ij}^{(c)}\varepsilon_{ij}^{(c)}\,dV = h\iint\limits_\Omega \frac{1}{2}\sigma_{\alpha\beta}^{(c)}\varepsilon_{\alpha\beta}^{(m)}\,d\Omega$$

The angle Θ_1 directing the first system of fibres and the angle α,between the two systems are independent variables.

Because the functional U does not depend on the derivatives of Θ_1 and α, therefore the necessary conditions for the minimum are

$$\frac{\partial W}{\partial \Theta_1} = 0 \qquad \frac{\partial W}{\partial \alpha} = 0 \tag{3.1}$$

here

$$W = \frac{1}{2}\sigma_{\alpha\beta}^{(c)}\varepsilon_{\alpha\beta}^{(m)}$$

The sufficient conditions for the minimum of the functional have the following form

$$\begin{gathered}\frac{\partial^2 W}{\partial \Theta_1^2}\frac{\partial^2 W}{\partial \alpha^2} - \Big(\frac{\partial^2 W}{\partial \Theta_1 \partial \alpha}\Big)^2 > 0 \\ \frac{\partial^2 W}{\partial \Theta_1^2} > 0 \qquad \frac{\partial^2 W}{\partial \alpha^2} > 0\end{gathered} \tag{3.2}$$

Let us suppose that the element is subjected to the two tensile loads p along the axis x_2 and $q = kp$ along the axis x_1 and to a tangent load $\tau = lp$. The strain energy in the form obtained after substituting for the strain components from Eq (2.5) is

$$\begin{aligned}W = \frac{p^2}{2D}\Big\{&K_1 - \Big((1-k)^2 - 2l^2\Big)\Big(\frac{\gamma\beta}{8(1-\nu)}\cos 2(2\Theta_1+\alpha)\cos 2\alpha - \\ &- \frac{\beta^2}{8}\cos^2(2\Theta_1+\alpha)\sin^2\alpha\Big) - 3l(1-k)\Big[\Big(\frac{\gamma\beta}{4(1-\nu)} + \frac{\beta^2}{16}\Big)\sin^2\alpha - \\ &- \frac{\gamma\beta}{8(1-\nu)}\Big]\sin 2(2\Theta_1+\alpha) + (1-k)^2\Big[\frac{\gamma\beta}{2(1+\nu)} + \frac{\beta^2}{4}\sin^2\alpha\Big]\cdot \\ &\cdot\cos(2\Theta_1+\alpha)\cos\alpha - 3l(1+k)\Big[\frac{\gamma\beta}{4(1+\nu)} + \frac{\beta^2}{8}\sin^2\alpha\Big]\cdot \\ &\cdot\sin(2\Theta_1+\alpha)\cos\alpha + (1+k)^2\frac{\beta^2}{8}\sin^2\alpha\cos^2\alpha + l^2\frac{\beta^2}{4}\sin^2\alpha\Big\}\end{aligned} \tag{3.3}$$

here

$$K_1 = \Big(k^2 - 2\nu k + 1 + (1+\nu)l^2\Big)\frac{\gamma^2}{(1+\nu)^2(1-\nu)} +$$
$$+\Big(k^2 - (1+\nu)k + 1 + 4l^2 - (1+\nu)l^2\Big)\frac{\gamma\beta}{4(1+\nu)(1-\nu)} + (1+k^2)\frac{3\gamma\beta}{8(1+\nu)}$$

The condition $(3.1)_1$ has the following form

$$\begin{aligned}\frac{\partial W}{\partial \Theta_1} = \frac{p^2}{2D}\Big\{&\Big[(1-k)^2 - 2l^2\Big]\Big(\frac{\gamma\beta}{2(1-\nu)}\cos 2\alpha - \frac{\beta^2}{4}\sin^2\alpha\Big)\sin 2(2\Theta_1+\alpha) - \\ &-3l(1-k)\Big[\Big(\frac{\gamma\beta}{1-\nu} + \frac{\beta^2}{4}\Big)\sin^2\alpha - \frac{\gamma\beta}{2(1-\nu)}\Big]\cos 2(2\Theta_1+\alpha) - \\ &-(1-k)^2\Big[\frac{\gamma\beta}{1+\nu} + \frac{\beta^2}{2}\sin^2\alpha\Big]\sin(2\Theta_1+\alpha)\cos\alpha - \\ &-3l(1+k)\Big[\frac{\gamma\beta}{2(1+\nu)} + \frac{\beta^2}{4}\sin^2\alpha\Big]\cos(2\Theta_1+\alpha)\cos\alpha\Big\} = 0\end{aligned} \tag{3.4}$$

The condition $(3.1)_2$ is expressed by the following equation

$$\begin{aligned}\frac{\partial W}{\partial \alpha} = \frac{p^2}{2D^2}\Big\{&\Big[(1-k)^2 - 2l^2\Big]\Big[L_1\Big(\cos 2(2\Theta_1+\alpha)\sin 2\alpha + \\ &+\sin 2(2\Theta_1+\alpha)\cos 2\alpha\Big) - L_2\cos 2(2\Theta_1+\alpha)\sin 2\alpha\sin^2\alpha\cos^2\alpha - \\ &-\frac{L_2}{1+\nu}\Big(\cos^4\alpha + \nu\sin^4\alpha\Big)\sin 2(2\Theta_1+\alpha)\cos 2\alpha - (L_3+L_4)\cdot \\ &\cdot\Big(\sin 2(2\Theta_1+\alpha)\sin^2\alpha - \cos^2(2\Theta_1+\alpha)\sin 2\alpha\Big) - \frac{L_5}{1+\nu}\Big(1 - \\ &-\cos^4\alpha - \nu\sin^4\alpha\Big)\sin 2(2\Theta_1+\alpha)\sin^2\alpha + L_5\cos^2(2\Theta_1+\alpha)\sin 2\alpha\cdot \\ &\cdot\sin^4\alpha\Big] - \frac{\gamma\beta^2}{2(1+\nu)(1-\nu)}K_1\Big(\cos^2\alpha - \nu\sin^2\alpha\Big)\sin 2\alpha + 3l(1-k)\cdot \\ &\cdot\Big[-L_1\Big(\sin 2(2\Theta_1+\alpha)\sin 2\alpha - \cos 2(2\Theta_1+\alpha)\cos 2\alpha\Big) - (L_3+L_4)\cdot \\ &\cdot\Big(\cos 2(2\Theta_1+\alpha)\sin^2\alpha + \frac{1}{2}\sin 2(2\Theta_1+\alpha)\Big) + L_2\sin 2(2\Theta_1+\alpha)\cdot \\ &\cdot\sin 2\alpha\sin^2\alpha\cos^2\alpha - \frac{L_2}{1+\nu}\Big(\cos^4\alpha + \nu\sin^4\alpha\Big)\cos 2(2\Theta_1+\alpha)\cos 2\alpha - \\ &-\frac{2L_5}{1+\nu}\cos 2(2\Theta_1+\alpha)\sin^4\alpha - L_5\Big(\sin 2(2\Theta_1+\alpha)\cos\alpha - \cos 2(2\Theta_1+\alpha)\cdot \\ &\cdot\sin\alpha\Big)\sin^5\alpha\Big] + (1-k^2)\Big[-2(L_3+2L_4)\sin(2\Theta_1+\alpha)\cos\alpha\sin^2\alpha -\end{aligned} \tag{3.5}$$

$$-2(L_3 - 2\nu L_4)\cos(2\Theta_1 + \alpha)\sin^3\alpha - L_6\Big(\sin(2\Theta_1 + \alpha)\cos\alpha +$$

$$+\cos(2\Theta_1 + \alpha)\sin\alpha\Big) + (1+\nu)\Big(L_4 - \frac{4}{(1+\nu)^2}L_5\Big)\sin(2\Theta_1 + \alpha)\sin^4\alpha\,\cdot$$

$$\cdot\cos\alpha + 2L_3\cos(2\Theta_1 + \alpha)\sin 2\alpha\cos\alpha + (1+\nu)\Big(3L_4 - \frac{4\nu}{(1+\nu)^2}L_5\Big)\,\cdot$$

$$\cdot\cos(2\Theta_1 + \alpha)\sin^5\alpha + 2L_5\sin^6\alpha\Big(\sin(2\Theta_1 + \alpha)\cos\alpha - \cos(2\Theta_1 + \alpha)\,\cdot$$

$$\cdot\sin\alpha\Big)\Big] + 3l(1+k)\Big[-\Big(L_3 + L_4 + \frac{L_5}{1+\nu}\Big)\cos(2\Theta_1 + \alpha)\sin^2\alpha\cos\alpha +$$

$$+\frac{L_5}{1+\nu}\Big(\cos^4\alpha + \nu\sin^4\alpha\Big)\cos(2\Theta_1 + \alpha)\sin^2\alpha\cos\alpha -$$

$$-\frac{1}{2}L_6\Big(\cos(2\Theta_1 + \alpha)\cos\alpha - \sin(2\Theta_1 + \alpha)\sin\alpha\Big) - \frac{1}{2}L_4\cos(2\Theta_1 + \alpha)\,\cdot$$

$$\cdot\cos\alpha\Big(1 - \cos^4\alpha - \nu\sin^4\alpha\Big) - L_3\sin(2\Theta_1 + \alpha)\sin 2\alpha\cos\alpha -$$

$$-L_5\sin(2\Theta_1 + \alpha)\sin 2\alpha\sin^4\alpha\cos\alpha - \frac{1+\nu}{2}L_4\sin(2\Theta_1 + \alpha)\sin^3\alpha\,\cdot$$

$$\cdot\Big(1 + 3\cos^2\alpha\Big) + L_4\sin(2\Theta_1 + \alpha)\sin^3\alpha\Big] + (1+k^2)\Big[(L_3 + L_4)\cos 2\alpha -$$

$$-\frac{1-\nu}{1+\nu}L_5\sin^4\alpha\Big]\sin 2\alpha + 2l^2\Big(L_3 + L_4 + L_5\sin^4\alpha\Big)\sin 2\alpha\Big\} = 0$$

here

$$L_1 = \frac{\gamma^4\beta}{4(1+\nu)^2(1-\nu)^2} + \frac{\gamma^3\beta^2}{4(1+\nu)^2(1-\nu)^2} + \frac{\gamma^2\beta^3}{16(1+\nu)(1-\nu)^2}$$

$$L_2 = \frac{\gamma^2\beta^3}{16(1-\nu)^2} \qquad L_3 = \frac{\gamma^3\beta^2}{8(1+\nu)^2(1-\nu)}$$

$$L_4 = \frac{2(1-\nu)}{(1+\nu)^2}L_2 \qquad L_5 = \frac{\gamma\beta^4}{32(1-\nu)}$$

$$L_6 = \frac{\gamma^4\beta}{2(1+\nu)^3(1-\nu)} + \frac{\gamma^3\beta^2}{2(1+\nu)^3(1-\nu)}$$

Eqs (3.4) and (3.5) determine the necessary conditions for the minimum of the strain energy for different normal and tangent loads.

3.2. Solutions for special cases of loads

The solutions to Eqs (3.4) and (3.5) for different normal and tangent loads can't be found in the analytical form. The analytical solutions can be found in the cases when the composite element is subjected to normal loads only $(p = q, \tau = 0)$ or when the element is subjected to tangent loads only $(p = q = 0, \tau \neq 0)$ (cf [4]). In the first case the strain energy attains to minimum for two ortogonal systems of fibres, with the slope of any angle Θ_1. If the element is subjected to tangent loads only, then the optimal orientation of fibres corresponds to a single fibre system, which makes with axis x_1 angle $45°$ as well as to the orthogonal systems of fibres, with the incline of $45°$.

We consider the composite element, which is subjected to the normal load p along the axis x_2 and $q = kp$ along the axis x_1 $(\tau = 0)$. Eqs (3.4) asnd (3.5) are expressed in the form

$$
\begin{aligned}
\frac{\partial W}{\partial \Theta_1} &= \frac{(1-k)p^2}{2D}\Big\{(1-k)\Big[\frac{\gamma\beta}{2(1-\nu)}\cos 2\alpha - \frac{\beta^2}{4}\sin^2\alpha\Big]\sin 2(2\Theta_1+\alpha) - \\
&-(1+k)\Big[\frac{\gamma\beta}{1+\nu} + \frac{\beta^2}{2}\sin^2\alpha\Big]\sin 2(2\Theta_1+\alpha)\cos\alpha\Big\} = 0 \\
\frac{\partial W}{\partial \alpha} &= \frac{p^2}{2D^2}\Big\{(1-k)^2\Big[L_1\Big(\cos 2(2\Theta_1+\alpha)\sin 2\alpha + \sin 2(2\Theta_1+\alpha)\cos 2\alpha\Big) - \\
&-L_2\cos 2(2\Theta_1+\alpha)\sin 2\alpha\sin^2\alpha\cos^2\alpha - \frac{L_2}{1+\nu}\Big(\cos^4\alpha + \nu\sin^4\alpha\Big)\cdot \\
&\cdot\sin 2(2\Theta_1+\alpha)\cos 2\alpha - (L_3+L_4)\Big(\sin 2(2\Theta_1+\alpha)\sin^2\alpha - \\
&-\cos^2(2\Theta_1+\alpha)\sin 2\alpha\Big) - \frac{L_5}{1+\nu}\Big(1-\cos^4\alpha - \nu\sin^4\alpha\Big)\cdot \\
&\cdot\sin 2(2\Theta_1+\alpha)\sin^2\alpha + L_5\cos^2(2\Theta_1+\alpha)\sin 2\alpha\sin^4\alpha\Big] - \\
&-K_2\Big(\cos^2\alpha - \nu\sin^2\alpha\Big)\sin 2\alpha + (1-k^2)\Big[-2(L_3+2L_4)\sin(2\Theta_1+\alpha)\cdot \\
&\cdot\cos\alpha\sin^2\alpha - 2(L_3-2\nu L_4)\cos(2\Theta_1+\alpha)\sin^3\alpha - \\
&-L_6\Big(\sin(2\Theta_1+\alpha)\cos\alpha + \cos(2\Theta_1+\alpha)\sin\alpha\Big) + \\
&+(1+\nu)\Big(L_4 - \frac{4}{(1+\nu)^2}L_5\Big)\sin(2\Theta_1+\alpha)\sin^4\alpha\cos\alpha + \\
&+2L_3\cos(2\Theta_1+\alpha)\sin 2\alpha\cos\alpha - (1+\nu)\Big(3L_4 - \frac{4\nu}{(1+\nu)^2}L_5\Big)\cdot \\
&\cdot\cos(2\Theta_1+\alpha)\sin^5\alpha + 2L_5\sin^6\alpha\Big(\sin(2\Theta_1+\alpha)\cos\alpha - \\
&-\cos(2\Theta_1+\alpha)\sin\alpha\Big)\Big] + (1+k)^2\Big[(L_3+L_4)\cos 2\alpha -
\end{aligned}
\tag{3.6}
$$

$$-\frac{1-\nu}{1+\nu}L_5\sin^4\alpha\Big]\sin 2\alpha\Big\} = 0$$

here

$$K_2 = \frac{\gamma^2\beta^2}{2(1+\nu)^2(1-\nu)}\Big[\Big(k^2-2k\nu+1\Big)\frac{\gamma}{(1+\nu)(1-\nu)} + \\ + \Big(k^2-(1+\nu)k+1\Big)\frac{\beta}{4(1-\nu)} + \Big(k^2+1\Big)\frac{3\beta}{8}\Big]$$

The foregoing equations are satisfied for

$$\begin{aligned} &\sin(2\Theta_1+\alpha)=0 \quad \text{and} \quad \sin\alpha = 0 \\ \text{or} \\ &\cos(2\Theta_1+\alpha)=0 \quad \text{and} \quad \cos\alpha = 0 \end{aligned} \tag{3.7}$$

therfore

$$\begin{aligned} &\alpha = 0^o \quad \text{and} \quad \Theta_1 = 0^o \quad \text{or} \quad \Theta_1 = 90^o \\ \text{or} \\ &\alpha = 90^o \quad \text{and} \quad \Theta_1 = 0^o \quad \text{or} \quad \Theta_1 = 90^o \end{aligned}$$

Eqs (3.6) are satisfied too for

$$\begin{aligned} &\sin(2\Theta_1+\alpha) = 0 \\ &(1-k)^2\Big[2L_1\cos\alpha - L_2\sin 2\alpha\sin\alpha\cos^2\alpha + L_5\sin 2\alpha + 2(L_3+L_4)\cos\alpha\Big] - \\ &-2K_2\Big(\cos^2\alpha-\nu\sin^2\alpha\Big)\cos\alpha + (1-k^2)\Big(2(L_3-2\nu L_4)\sin^2\alpha - 4L_3\cos^2\alpha + \\ &+(1+\nu)\Big(3L_4-\frac{4\nu}{(1+\nu)^2}L_5\Big)\sin^4\alpha + 2L_5\sin^6\alpha + L_6\Big] + \\ &+(1+k)^2\Big[-\frac{1-\nu}{1+\nu}L_5\sin 2\alpha\sin^3\alpha + 2(L_3+L_4)\cos 2\alpha\cos\alpha\Big] = 0 \end{aligned} \tag{3.8}$$

If the substition is made for $t = \cos\alpha$, the relation $(3.8)_2$ is given by the following equation of degree 6

$$\begin{aligned} &-2(1-k^2)L_5t^6 + 2\Big[(1-k)^2(L_2+L_5) - (1+k)^2\frac{1-\nu}{1+\nu}L_5\Big]t^5 + \\ &+(1-k^2)\Big[(1+\nu)\Big(3L_4-\frac{4\nu}{(1+\nu)^2}L_5\Big)+6L_5\Big]t^4 - \\ &-2\Big[(1-k)^2(L_2+2L_5)+(1+\nu)K_2 - 2(1+k)^2\Big(\frac{1-\nu}{1+\nu}L_5+L_3+L_4\Big)\Big]t^3 - \end{aligned} \tag{3.9}$$

$$-2(1-k^2)\Big[3L_3+(3+\nu)L_4+\Big(3-\frac{4\nu}{1+\nu}\Big)L_5\Big]t^2+$$

$$+2\Big[(1-k)^2(L_1+L_3+L_4+L_5)+\nu K_2-(1+k)^2\Big(\frac{1-\nu}{1+\nu}L_5+L_3+L_4\Big)\Big]t+$$

$$+(1-k^2)\Big[2L_3+(3-\nu)L_4-2\Big(1-\frac{2\nu}{1+\nu}\Big)L_5+L_6\Big]=0$$

Eq (3.9) can only be solved numerically. If the root t satisfies the condition $-1\le t\le 1$ then $\alpha=\arccos t$ and the angle Θ_1 is obtained from Eq (3.8) $\Theta_1=\frac{1}{2}(180^\circ-\alpha)$. The smallest value of the strain energy corresponds to the optimal orientation of the fibres.

Let us suppose that the composite element is subjected to the normal loads $p=q$ and to a tangent load τ $(k=1, l\neq 0)$. Eqs (3.4) and (3.5) are expressed in the form

$$\frac{\partial W}{\partial\Theta_1}=\frac{p^2}{D}\Big\{-l^2\Big(\frac{\gamma\beta}{1-\nu}\cos 2\alpha-\frac{\beta^2}{2}\sin^2\alpha\Big)\sin(2\Theta_1+\alpha)-$$

$$-\frac{3}{2}l\Big(\frac{\gamma\beta}{1+\nu}+\frac{\beta^2}{2}\sin^2\alpha\Big)\cos\alpha\Big\}\cos(2\Theta_1+\alpha)=0$$

(3.10)

$$\frac{\partial W}{\partial\alpha}=\frac{p^2}{2D^2}\Big\{-2l^2\Big[L_1\Big(\cos 2(2\Theta_1+\alpha)\sin 2\alpha+\sin 2(2\Theta_1+\alpha)\cos 2\alpha\Big)-$$

$$-L_2\cos 2(2\Theta_1+\alpha)\sin 2\alpha\sin^2\alpha\cos^2\alpha-\frac{L_2}{1+\nu}\Big(\cos^4\alpha+\nu\sin^4\alpha\Big)\cdot$$

$$\cdot\sin 2(2\Theta_1+\alpha)\cos 2\alpha-(L_3+L_4)\Big(\sin(2\Theta_1+\alpha)\sin^2\alpha-$$

$$-\cos^2(2\Theta_1+\alpha)\sin 2\alpha\Big)-\frac{L_5}{1+\nu}\Big(1-\cos^4\alpha-\nu\sin^4\alpha\Big)\cdot$$

$$\cdot\sin 2(2\Theta_1+\alpha)\sin^2\alpha+L_5\cos^2(2\Theta_1+\alpha)\sin 2\alpha\sin^4\alpha\Big]-$$

$$-K_3\Big(\cos^2\alpha-\nu\sin^2\alpha\Big)\sin 2\alpha+6l\Big[-\Big(L_3+L_4+\frac{L_5}{1+\nu}\Big)\cos(2\Theta_1+\alpha)\cdot$$

$$\cdot\sin^2\alpha\cos\alpha+\frac{L_5}{1+\nu}\Big(\cos^4\alpha+\nu\sin^4\alpha\Big)\cos(2\Theta_1+\alpha)\sin^2\alpha\cos\alpha-$$

$$-\frac{1}{2}L_6\Big(\cos(2\Theta_1+\alpha)\cos\alpha-\sin(2\Theta_1+\alpha)\sin\alpha\Big)-\frac{1}{2}L_4\cos(2\Theta_1+\alpha)\cdot$$

$$\cdot\cos\alpha\Big(1-\cos^4\alpha-\nu\sin^4\alpha\Big)-L_3\sin(2\Theta_1+\alpha)\sin 2\alpha\cos\alpha-$$

$$-L_5\sin(2\Theta_1+\alpha)\sin 2\alpha\sin^4\alpha\cos\alpha-\frac{1+\nu}{2}L_4\sin(2\Theta_1+\alpha)\sin^3\alpha\cdot$$

$$\cdot\Big(1+3\cos^2\alpha\Big) + L_4\sin(2\Theta_1+\alpha)\sin^3\alpha\Big] + (1+k)^2\Big[(L_3+L_4)\cos 2\alpha -$$

$$-\frac{1-\nu}{1+\nu}L_5\sin^4\alpha\Big]\sin 2\alpha + 2l^2(L_3+L_4+L_5\sin^4\alpha\Big)\sin 2\alpha\Big\} = 0$$

here

$$\begin{aligned} K_3 &= \frac{\gamma\beta^2}{2(1+\nu)(1-\nu)}\Big[\Big(2(1-\nu)+(1+\nu)l^2\Big)\frac{\gamma^2}{(1+\nu)^2(1-\nu)} + \\ &+ \Big((1-\nu)+4l^2-(1+\nu)l^2\Big)\frac{\gamma\beta}{4(1+\nu)(1-\nu)} + \frac{3\gamma\beta}{4(1+\nu)}\Big] \end{aligned}$$

Eq $(3.10)_1$ is satisfied if

$$\begin{aligned} &\cos(2\Theta_1+\alpha) = 0 \\ &\sin(2\Theta_1+\alpha) = \frac{3\Big(\frac{\gamma\beta}{1+\nu} + \frac{\beta^2}{2}\sin^2\alpha\Big)\cos\alpha}{2l\Big(\frac{\gamma\beta}{1-\nu}\cos 2\alpha - \frac{\beta^2}{2}\sin^2\alpha\Big)} \end{aligned} \tag{3.11}$$

Substituting relation $(3.11)_1$ into Eq $(3.10)_2$ we obtain the equation in α

$$-4l^2\Big[-L_1\cos\alpha + L_2\sin^2\alpha\cos^3\alpha\Big] - K_3\Big(\cos^3\alpha - \nu\sin^2\alpha\cos\alpha\Big) +$$

$$+6l\Big[\frac{1}{2}L_6 - 2L_3\cos^2\alpha - 2L_5\sin^4\alpha\cos^2\alpha -$$

$$-\frac{1+\nu}{2}L_4\sin^2\alpha\Big(1+3\cos^2\alpha) + L_4\sin^2\alpha\Big] + 8\Big[(L_3+L_4)\cos 2\alpha -$$

$$-\frac{1-\nu}{1+\nu}L_5\sin^4\alpha\Big]\cos\alpha + 4l^2\Big(L_3+L_4+L_5\sin^4\alpha\Big)\cos\alpha = 0$$

Next substituting for $t = \cos\alpha$ the foregoing equation can be expressed as the algebraic equation of degree 6

$$-12lL_5t^6 + 4\Big[l^2(L_2+L_5) - \frac{2(1-\nu)}{1+\nu}L_5\Big]t^5 + l\Big[24L_5 + 9(1+\nu)L_4\Big]t^4 +$$

$$+\Big[-4l^2(L_2+2L_5) - (1+\nu)K_3 + 16(L_3+L_4) + \frac{16(1-\nu)}{1+\nu}L_5\Big]t^3 -$$

$$-6l\Big[2(L_3+L_5) + (1+\nu)L_4 + L_4\Big]t^2 + \Big[4l^2(L_1+L_3+L_4+L_5) + \nu K_3 -$$

$$-8(L_3+L_4) - \frac{8(1-\nu)}{1+\nu}L_5\Big]t + 3l\Big[L_6 + (1-\nu)L_4\Big] = 0 \tag{3.12}$$

This equation can only be solved numerically. If the angles α and $\Theta_1 = \frac{1}{2}(90^\circ-\alpha)$ satisfy the relation (3.12) then it is necessary to calculate the strain energy value.

Next we calculate the strain energy for $\alpha = 0$, $\Theta_1 = 45°$ and for $\alpha = 90°$, $\Theta_1 = 0°$. The smallest value of the strain energy coresponds to the optimal orientation of the fibres.

The solution $(3.11)_2$ does not determine the minimum value of the strain energy. The values of the strain energy have calculated for the angles α for which the relation $(3.11)_2$ is determined using the following data: $l = q/p = 0.3,\ 0.5$; $M = \beta/\gamma = 25,\ 85.24$. The values of the strain energy for these angles were bigger than those obtained from the solution to Eq $(3.11)_1$.

3.3. Solutions for different normal and tangent loads

Table 1

k	$l=0$		$l=0.3$		$l=0.5$	
	Θ_1	α	Θ_1	α	Θ_1	α
−1	90	90	83.690	90	79.849	90
−0.9	90	90	83.375	90	79.374	90
−0.8	90	90	83.025	90	78.855	90
−0.7	90	90	82.638	90	78.286	90
−0.6	90	90	82.206	90	77.660	90
−0.5	90	90	81.722	90	76.968	90
−0.4	90	90	81.176	90	76.202	90
−0.3	90	90	80.557	90	75.348	90
−0.2	90	90	79.849	90	74.393	90
−0.1	90	0	78.990	0	73.208	0
0	90	0	78.024	0	71.959	0
0.1	78.380	23.240	76.883	0	70.536	0
0.2	69.240	41.520	75.520	0	68.909	0
0.3	63.605	52.790	49.835	44.250	67.040	0
0.4	59.400	61.200	40.851	53.729	40.497	41.530
0.5	55.990	68.020	31.641	61.068	31.554	50.038
0.6	53.155	73.690	23.393	66.689	24.899	55.765
0.7	50.716	78.568	17.436	70.478	20.275	59.641
0.8	48.600	82.800	13.208	72.948	16.919	62.367
0.9	46.700	86.600	9.956	74.552	14.291	64.350
1	$\forall\Theta_1$	90	7.230	75.540	12.097	65.806

In the case of different normal and tangent loads ($k \neq 0$ and $k \neq 1$ and $l \neq 0$) the solution to the problem is obtained from the Schittkowski program NLPQL. In the Schittkowski program the nonlinearity of the optimization problem is solved by means of the quadratic approximation methods of Wilson, Hand and Powell (cf [7]). In the tables 1 and 2 the solutions, for the following ratios of the loads: $k = q/p \in [-1, 1]$ for 0.1 value and for $l = \tau/p = 0,\ 0.3,\ 0.5$ are presented. In the table 1 the optimal directions of fibre systems for the material which is composed of polyamide matrix and of carbon fibres of volumetric content 20% are shown. In this case the solutions are calculated for the following data: $E^{(m)} = 2$ GPa, $E^{(f)} = 200$ GPa, $\nu = 0.35$ ($M = \beta/\gamma = 25$). In the table 2 the optimal solutions for the boron fiber–reinforced epoxide of volumetric content 48% using the following data: $E^{(m)} = 4.31$ GPa, $E^{(f)} = 398$ GPa, $\nu = 0.35$ ($M = 85.24$) are presented. The optimal angles θ_1 and α correspond to the minimum values of the strain energy.

Table 2

k	$l=0$		$l=0.3$		$l=0.5$	
	θ_1	α	θ_1	α	θ_1	α
−1	0	90	83.690	90	79.849	90
−0.9	90	90	83.375	90	79.374	90
−0.8	90	90	83.025	90	78.855	90
−0.7	0	90	82.638	90	78.286	90
−0.6	90	90	82.206	90	77.660	90
−0.5	90	90	81.722	90	76.968	90
−0.4	90	90	81.176	90	76.202	90
−0.3	90	90	80.557	90	75.784	88.400
−0.2	90	90	79.849	90	73.997	89.707
−0.1	90	0	79.033	90	73.320	90
0	90	0	78.025	0	70.458	0
0.1	73.980	32.040	76.880	0	0.026	69.563
0.2	66.844	46.312	0.328	77.070	0	69.647
0.3	61.960	56.080	4.498	73.880	1.844	68.288
0.4	58.180	63.640	8.294	71.796	7.293	64.279
0.5	55.100	69.800	9.779	71.795	11.237	62.391
0.6	52.510	74.980	9.733	72.774	12.509	62.783
0.7	50.270	79.460	9.121	73.930	12.565	63.960
0.8	48.300	83.400	8.334	75.001	12.205	65.282
0.9	46.600	86.800	7.499	75.918	11.690	66.564
1	$\forall\theta_1$	90	6.667	76.667	11.127	67.746

4. Conclusions

From a study of the results, we may conclude that if the composite element is subjected to the normal loads only: $p = q$, $l = 0$ ($k = 1$, $l = 0$) the strain energy attains to minimum for the two orthogonal systems of fibres, making with the axis a certain angle Θ_1. If $p = q$, $\tau \neq 0$ ($k = 1$, $l \neq 0$) then the following relation of the optimal angles $2\Theta_1 + \alpha$ is equal to 90°. The values of the angles Θ_1 and α are dependent on the ratio l and on the material elastic constants values and on the volumetric content of the fibres, respectively.

If the composite element is subjected to the normal loads only $p = kq$, $\tau = 0$ ($l = 0$) then for $-1 < k \leq 0$ the optimal angles Θ_1 and α attain to the values 0° or 90° according to the solution (3.7). For $0 < k < 1$ the relation for the optimal angles $2\Theta_1 + \alpha$ is equal to 180°, the values of the angles are dependent on the ratio of the normal loads k and on the material elastic constants values and on volumetric content of the fibres, respectively.

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Optymalne ukierunkowanie włókien w elemencie kompozytowym

Streszczenie

W pracy rozpatrzono element kompozytowy, w którym matryca uzbrojona jest dwiema rodzinami równoległych włókien. Optymalne kierunki ułożenia włókien wyznaczono z kryterium minimum energii odkształcenia. Zamieszczono rozwiązania numeryczne otrzymane przy różnych obciążeniach normalnych i stycznych.

Manuscript received November 21, 1990; accepted for print April 20, 1992