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APPROXIMATE ANALYTICAL SOLUTION FOR BERNOULLI-EULER BEAMS UNDER DIFFERENT BOUNDARY CONDITIONS WITH NON-LINEAR WINKLER TYPE FOUNDATION

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In this study, a powerful analytical method, known as Homotopy Analysis Method (HAM), is used to obtain an analytical solution to nonlinear ordinary deferential equations arising for Bernoulli-Euler beams with a non-linear foundation of the Winkler type. A comparison between the HAM solution and a solution obtained by a numerical method is made to show the accuracy of the method. It is shown that the present solution is valid for the whole domain of the solution and also for high nonlinear terms, where other methods such as the perturbation method fail to converge. The results clearly indicate that the convergence region can be controlled and adjusted by HAM. Finally, after validating the results, the effect of constant parameters on the deflection and slope for different boundary conditions is presented.

Key words: Bernoulli-Euler beam, Winkler type foundation, HAM

1. Introduction

Most of engineering problems are inherently nonlinear, especially those problems arising in fluid mechanics, heat transfer, large deformations, nonlinear dynamics and so on. Generally, nonlinear problems are difficult to solve, especially in an analytical manner. Some of these problems are solved using numerical techniques, and sometimes approximate analytical methods are used.

In numerical methods, stability and convergence should be considered to avoid divergence or inappropriate results. On the other hand, some analytical methods are incorprated, which apply perturbation artificial small parameter and δ-expansion method. Those methods can be applied for some specific problems or under special conditions. For example, the perturbation method is one of the well-known methods to solve nonlinear problems which are based on the existence of small/large parameters. This method in a lot of cases has deficiency and fails to solve nonlinear problems, a specially in those with no small/large parameters (Hale, 1969; Liao, 1992, 1997, 2003; Nayef, 1985; Sajid and Hayat, 2007).

Liao (1992) proposed one of semi-exact methods which does not need small/large parameters named as homotopy analysis method (HAM). This method has been extensively applied and still used to solve many types of nonlinear problems (chowdhury *et al.*, 2007; Kimiaeifar, 2010; Kimiaeifar *et al.*, 2009a,b; Nadeem *et al.*, 2010; "Oziş and Yıldırım, 2007a,b; Sohouli *et al.*, 2010; Wang *et al.*, 2008). In this paper, an analytical approach to obtain the solution describing deflection of beams with nonlinear Winkler type foundations is presented using HAM. It is supposed that the beam is subjected to arbitrarily distributed transverse load. There is a hyperbolic type relation for stress between the surface of the beam and the foundation which yields nonlinear terms in the equation for Bernoulli-Euler beams. Soldatos and Selvadurai (1983, 1985) proposed a solution to obtain flexure of this kind of beams by means of the perturbation method. They combined the Lyengar and Anantharamul method for characteristic eigenfunctions of a freely vibrating beam (Bishop and Johnson, 1960; Lyengar and Anantharamul, 1963; Soldatos and Selvadurai, 1985) with method of Galerkin (Bishop and Johnson, 1960; Kantorovich and Krylov, 1964; Lyengar and Anantharamul, 1963; Soldatos and Selvadurai, 1985). In this paper, a convenient and brief way to solve this problem by means of HAM is proposed.It is shown that the convergence region can be controlled by this method also for high nonlinear terms. At the end, HAM results with numerical ones are compared, and the convergence study is made.

2. Governing equation of the problem

Consider a beam which is subjected to an external transverse stress distribution $p(x)$ and width of b and with no foundation stress. The Bernoulli-Euler equation can be written as follows

$$EI\frac{d^4w}{dx^4} = bp(x) \tag{2.1}$$

where E is Young's modulus and I is moment of inertia. In the stated problem, it is assumed that the beam is subjected to an external transverse stress distribution $p(x)$ and also that there is a smooth and bilateral contact between the beam and foundation. The foundation is of the Winkler type with an assumption that the displacement occurs only under the load area, and outside this region displacements are zero (Selvadurai, 1979). There is also a nonlinear hyperbolic stress expressed by a relation between the deflection of the beam and foundation, which can be given as follows

$$q(x) = \frac{kw(x)}{1 + \mu w(x)} \tag{2.2}$$

The stress parameter q at any point of the foundation surface x relates the deflection w to the corresponding point stress. The parameter k indicates the modulus of the linear sub-grade reaction with dimensions of stress per unit length, and μ is a non-linear parameter with dimensions of $(\text{length})^{-1}$ indicating the nonlinear response of the elastic foundation. It is clear when μ is zero, Eq. (2.2) reduces to linear form (Soldatos and Selvadurai, 1985).

By rearranging the Bernoulli-Euler equation with Eqs. (2.1) and (2.2), the governing equation of the beam leads to

$$EI\frac{d^4w(x)}{dx^4} + \frac{kbw(x)}{1 + \mu w(x)} = bp(x) \tag{2.3}$$

By changing the variables and defining a new term, equation (2.3) can be rewritten in the form below (Soldatos and Selvadurai, 1985)

$$\frac{d^4w(\xi)}{d\xi^4} + \frac{4w(\xi)}{1 + \mu w(\xi)} = \frac{4}{k}p(\xi) \qquad \xi = \lambda x \tag{2.4}$$

where

$$\lambda^{-1} = \Big(\frac{kb}{4EI}\Big)^{-\frac{1}{4}}$$

with dimensions of length, therefore $\xi = \lambda x$ is a non-dimensional spatial coordinate.

3. Fundamental of HAM method

To start the basic idea of HAM, the following differential equation is considered

$$\mathcal{N}[u(\xi)] = 0 \tag{3.1}$$

where $\mathcal{N}$ is a nonlinear operator, ξ indicates an independent variable, $u(\xi)$ is the answer to the nonlinear equation that is an unknown function. The so-called 0-th order deformation equation which includes the main idea of the presenting HAM method is constructed as follows (Liao, 1992, 2003)

$$(1-q)\mathcal{L}[\Phi(\xi;q)-u_0(\xi)] = q\hbar H(\xi)\mathcal{N}[\Phi(\xi;q)] \tag{3.2}$$

where $q \in [0,1]$ is the embedding parameter, $\hbar$ is a non-zero auxiliary parameter which controls the convergence region, $H(\xi)$ is also a non-zero arbitrary function, $\mathcal{L}$ is an auxiliary linear operator, $u_0(\xi)$ is an initial guess of $u(\xi)$, $\Phi)q;\xi)$ is the answer to the homotopy equation according to the variation of q parameter. From equation (3.2), it is obvious that

$$q=0 \ \rightarrow \ \Phi(\xi;0)=u_0(\xi) \qquad\qquad q=1 \ \rightarrow \ \Phi(\xi;1)=u(\xi) \tag{3.3}$$

It can be therefore concluded that, when q increases from 0 to 1, the solution $\Phi(\xi;q)$ varies from the initial guess $u_0(\xi)$ to the exact solution $u(\xi)$. Considering $\Phi(\xi;q)$ as a function of q and expanding Taylor series with respect to q, results in

$$\Phi(\xi;q) = u_0(\xi) + \sum_{m=1}^{\infty} u_m(\xi)q^m \tag{3.4}$$

where for an arbitrary value of $m \geqslant 1$

$$u_m(\xi) = \frac{1}{m!}\frac{\partial^m \Phi(\xi;q)}{\partial q^m}\bigg|_{q=0} \tag{3.5}$$

by choosing an appropriate auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function, the stated series in Eq. (3.4) converges to the exact solution to the problem at $q=1$, so it can be written

$$u(\xi) = u_0(\xi) + \sum_{m=1}^{\infty} u_m(\xi) \tag{3.6}$$

Equation (3.6) must be one of the original nonlinear answers. By setting $H(\xi)=1$ Eq. (3.2) is reduced to

$$(1-q)\mathcal{L}[\Phi(\xi;q)-u_0(\xi)] = q\hbar\mathcal{N}[\Phi(\xi;q)] \tag{3.7}$$

By differentiating Eq. (3.2) m times with respect to the embedding parameter q and setting $q=0$ and finally dividing them by $m!$, the so-called m-th order deformation equation for $m \geqslant 1$ is obtained

$$\mathcal{L}[u_m(\xi) - \chi_m u_{m-1}(\xi)] = \hbar H(\xi) R_m(\boldsymbol{u}_{m-1}) \tag{3.8}$$

where

$$
\begin{aligned}
&R_m(\boldsymbol{x}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}[\varPhi(\xi;q)]}{\partial q^{m-1}}\bigg|_{q=0} \\
&\boldsymbol{x}_{m-1} = \{u_0(\xi), u_1(\xi), \ldots, u_{m-1}(\xi)\} \\
&\chi_M = \begin{cases} 0\, quad & m \leqslant 1 \\ 1 & m > 1 \end{cases}
\end{aligned}
\tag{3.9}
$$

4. Application of HAM to solution of the problem

As it was mentioned above, equation (2.3) is nonlinear and cannot be solved by regular methods, so HAM technique is applied to present an analytical solution. By rewriting equation (2.4), it is obtained

$$
(1+\mu w)\frac{d^4 w}{d\xi^4} + 4w = (1+\mu w)\frac{4}{k}p(\xi) \tag{4.1}
$$

Based on Eq. (3.1), it is obtained

$$
\mathcal{N}[w(\xi)] = (1+\mu w)\frac{d^4 w}{d\xi^4} + 4w - (1+\mu w)\frac{4}{k}p(\xi) \tag{4.2}
$$

The nonlinear part $\mathcal{N}[w(\xi)]$ must be choosen as the same as the equation of the problem to avoid failure in the solution procedure, so it is described as

$$
\mathcal{N}[\varPhi(\xi;q)] = [1+\mu\varPhi(\xi;q)]\frac{d^4 \mu\varPhi(\xi;q)}{d\xi^4} + 4\mu\varPhi(\xi;q) - [1+\mu\varPhi(\xi;q)]\frac{4}{k}p(\xi) \tag{4.3}
$$

Either because of small values of μ or, in the region of validity of small deflection theory, $|w| \ll 1$, the linear operator is chosen as below

$$
\mathcal{L}[u] = \frac{d^4 u}{d\xi^4} + 4u \tag{4.4}
$$

According to (3.5) and by differentiating $\mathcal{N}[\varPhi(\xi;q)]$ m-times with respect to q, the result is

$$
\begin{aligned}
R_m(\boldsymbol{x}_{m-1}) = {}& \frac{d^4 u_{m-1}}{d\xi^4} + \mu\sum_{i=0}^{m-1} u_i \frac{d^4 u_{m-i-1}}{d\xi^4} + 4u_{m-1} \\
& - 4\mu\frac{p(\xi)}{k}u_{m-1} - (1-\chi_m)\frac{4}{k}p(\xi)
\end{aligned}
\tag{4.5}
$$

In general and all cases, it can always be written that

$$u_m = \chi_m u_{m-1} + \mathcal{L}^{-1}[\hbar R_m(\boldsymbol{u}_{m-1})] \tag{4.6}$$

where

$$\begin{aligned}\mathcal{L}^{-1}[R_m] = \sum_{i=1}^{4} u_{m_i} \int \frac{R_m(\xi)W_i(\xi)}{W(\xi)}\, d\xi + c_1 \mathrm{e}^{\xi}\cos\xi + c_2\mathrm{e}^{\xi}\sin\xi \\ + c_3\mathrm{e}^{-\xi}\cos\xi + c_4\mathrm{e}^{-\xi}\sin\xi\end{aligned} \tag{4.7}$$

where $W(\xi)$ is the Wronskian of $\{u_{m_1}, u_{m_2}, u_{m_3}, u_{m_4}\}$ and $W_i(\xi)$ is the determinant obtained from the Wronskian by substituting the column $(0,0,0,1)$ into the i-th column, the coefficients $\{c_1, c_2, c_3, c_4\}$ are gained by applying initial conditions and u_{m_i}, u_{m_2}, u_{m_3}, u_{m_4} for all values of m which are $\mathrm{e}^{-\xi}\cos\xi$, $\mathrm{e}^{-\xi}\sin\xi$, $\mathrm{e}^{\xi}\sin\xi$, $\mathrm{e}^{\xi}\cos\xi$, respectively.

By expanding Eq. (4.7) and simplifying, one obtains

$$\begin{aligned}\mathcal{L}^{-1}[R_m] = -\frac{1}{4}\int_0^{\xi} [\sinh(\xi-\tau)\cos(\xi-\tau) - \sin(\xi-\tau)\cosh(\xi-\tau)]R_m(\xi)\, d\tau \\ + c_1\mathrm{e}^{\xi}\cos\xi + c_2\mathrm{e}^{\xi}\sin\xi + c_3\mathrm{e}^{-\xi}\cos\xi + c_4\mathrm{e}^{-\xi}\sin\xi\end{aligned} \tag{4.8}$$

As u_0 plays an important role in the convergence of the solution, u_0 can be obtained from the following equation

$$\frac{d^4u_0}{d\xi^4} + 4u_0 = \frac{4}{k}p(\xi) \tag{4.9}$$

Although it is not always essential to obtain u_0 from Eq. (4.9), when all boundary conditions are zero or initial displacements are considered, u_0 can be a constant function. Therefore, the base function will be an exponential and trigonometric one.

In the case when the beam is subjected to a line pressure load, P, with dimension of force per unit lengthat a point α, the external distribution stress can expressed by the Dirac delta function (Soldatos and Selvadurai, 1985)

$$p(\xi) = \overline{P}\delta(\xi-\alpha) \qquad\qquad \overline{P} = \lambda P \tag{4.10}$$

As mentioned

$$\lambda^{-1} = \Big(\frac{kb}{4EI}\Big)^{-\frac{1}{4}}$$

then u_0 is obtained

$$
\begin{aligned}
u_0 = \frac{2\overline{P}}{k}\widetilde{H}(\xi-\alpha)\Big\{&\mathrm{e}^{\xi-\alpha}[\sin(\xi-\alpha)-\cos(\xi-\alpha)] \\
&+\mathrm{e}^{\alpha-\xi}[\sin(\xi-\alpha)+\cos(\xi-\alpha)]\Big\}+c_1\mathrm{e}^{\xi}\cos\xi+c_2\mathrm{e}^{\xi}\sin\xi \\
&+c_3\mathrm{e}^{-\xi}\cos\xi+c_4\mathrm{e}^{-\xi}\sin\xi
\end{aligned}
\tag{4.11}
$$

By converting the Heaviside function $\widetilde{H}(\xi-\alpha)$ to a piecewise function, it can be easily differentiated and integrated from u_0, which makes the calculation and computation much easier. Therefore, the Heaviside function is defined as follows

$$
\widetilde{H}(\xi-\alpha)=\begin{cases} 1 & \xi \geqslant \alpha \\ 0 & \xi < \alpha \end{cases}
\tag{4.12}
$$

So by some recessive calculation and computing u_m from the previous terms and equations, it can be written

$$
W(\xi)=u_0(\xi)+\sum_{m=1}^{\infty}u_m(\xi)
\tag{4.13}
$$

The accuracy of the solution by considering some examples for different boundary conditions is investigated in the coming parts of the paper.

5. Examples

5.1. Finite beam subjected to an initial displacement

Consider a beam resting on a Winkler foundation of the hyperbolic type and subjected to the following set of inhomogeneous boundary conditions for Eq. (2.4)

$$
w(0)=\delta \qquad w'(0)=w''(\beta)=w'''(\beta)=0
\tag{5.1}
$$

where β is nondimensional form of the beam length. It is assumed that the beam has two edges, the edge in which $\xi=0$ is subjected to an initial displacement δ and rotation is prevented, and at $\xi=\beta$ the edge is free.

In this case, the force is zero and the deformation depends on the initial displacement, as $p(\xi)=0$ Eq. (2.4) changes to the following equation

$$
\frac{d^4w}{d\xi^4}+\frac{4w}{1+\mu w}=0 \qquad \xi=\lambda x
\tag{5.2}
$$

Inhomogeneous boundary conditions for u_0 should be chosen, and for $m \geqslant 1$ the boundary conditions are homogeneous

$$
\begin{aligned}
&u_0(0) = \delta \qquad && u_0'(0) = u_0''(\beta) = u_0'''(\beta) = 0 && \\
&u_m(0) = 0 \qquad && u_m'(0) = u_m''(\beta) = u_m'''(\beta) = 0 \qquad && m \geqslant 1
\end{aligned}
\tag{5.3}
$$

To solve this kind of non-linear equation and obtaining an analytical solution, a system of coefficient equations should be solved. For comparing this method with the numerical one, the problem for definite values of μ and δ is solved. The results are presented in Tables 1 and 2. $w(\xi)$ shows the beam deflection with respect to the initial displacement δ, it should be pointed out that the increasing of the order of HAM solution series increases the accuracy of the results. The numerical methods are based on non-linear finite difference method (Ames, 1977; Hildebrand, 1968), and there is a good agreement between the analytical and numerical solutions.

Table 1. Comparison between the HAM and numerical solution for the beam deflection $w(\xi)$, $\beta = 5$, $\delta = 0.1$ and $\mu = 1$

ξ	$W(\xi)$			
	2-nd order	4-th order	6-th order	numeric
0.0	0.100000000	0.100000000	0.100000000	0.100000000
1.0	0.051618111	0.051619511	0.051619516	0.051619513
2.0	0.007165896	0.007166002	0.007166089	0.007166025
3.0	−0.0042674816	−0.004267464	−0.004267464	−0.004267485
4.0	−0.0026255686	−0.002625633	−0.002625633	−0.002625653
5.0	0.000701062	0.000701029	0.000701029	0.000701059

Table 2. Comparison between the HAM and numerical solution for the beam deflection $w(\xi)$, $\beta = 6$, $\delta = 0.6$ and $\mu = 0.5$

ξ	$W(\xi)$			
	2-nd order	4-th order	6-th order	numeric
0.0	0.60000000	0.60000000	0.60000000	0.60000000
1.0	0.31819265	0.31839107	0.31839764	0.31839794
2.0	0.04958304	0.04965315	0.04965534	0.04965556
3.0	0.02334683	−0.02334443	−0.02334441	−0.02334441
4.0	−0.01632195	−0.01633095	−0.01633125	−0.01633139
5.0	−0.00279692	−0.00280073	−0.00280085	−0.00280085
6.0	0.005679063	0.00568124	0.00568132	0.00568145

The first and second terms of the solution series by assuming $u_0(\xi) = \delta$ are as follows

$$\begin{aligned} w(\xi) = u_0(\xi) + \sum_{i=1}^{\infty} u_i(\xi) &= \delta - \frac{2}{5}\hbar\delta\cos^4\xi + \frac{19}{20}\hbar\delta + \frac{2}{5}\hbar\delta\cos^2\xi \\ &- \frac{1}{20}\hbar\delta\sin\xi\sin 3\xi + \frac{1}{20}\hbar\delta\cos\xi\cos 3\xi + c_1 e^{\xi}\cos\xi \\ &+ c_2 e^{\xi}\sin\xi + c_3 e^{-\xi}\cos\xi + c_4 e^{-\xi}\sin\xi + \ldots \end{aligned} \tag{5.4}$$

The coefficients c_1, c_2, c_3, c_4 are obtained by using the boundary conditions stated in Eq. $(5.3)_2$. As it was mentioned before, $\hbar$ is the homotopy auxiliary parameter, and μ is a nonlinear parameter, and δ is initial boundary condition.

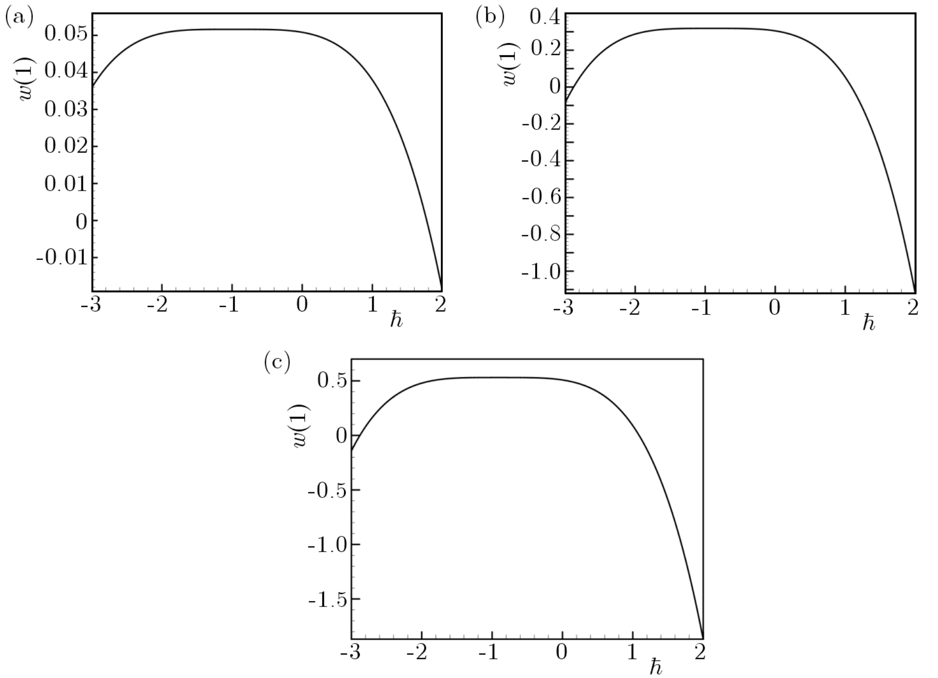

Fig. 1. $\hbar$-curves with the 6-th order approximation showing the convergence of HAM results; w is the deflection of the beam at a desired point versus $\hbar$, which indicates the region where the results converge; (a) $\beta = 5$, $\delta = 0.1$, $\mu = 1$, (b) $\beta = 6$, $\delta = 0.6$, $\mu = 0.5$, (c) $\beta = 10$, $\delta = 1$, $\mu = 0.3$

First of all, a convergence study should be done to assure the solution. In order to define a region such that the solution series is independent on $\hbar$, a multiple of curves are plotted in Fig. 1. The region where the distribution of the beam deflection parameter w versus $\hbar$ is a horizontal line is known as the convergence region. For the finite beam subjected to an initial displacement, $\hbar$-curves are shown in Fig. 1. To compare the solution with the numerical one, some constant parameters for μ, β, δ are considered and the results indicate good agreement between the HAM and numerical solution as shown in Table 3 and Tables 1 and 2.

Table 3. Comparison between the HAM and numerical solution for the beam deflection $w(\xi)$, $\beta = 10$, $\delta = 1$ and $\mu = 0.3$

ξ	$W(\xi)$			
	2-nd order	4-th order	6-th order	numeric
0.0	1.00000000	1.00000000	1.00000000	1.00000000
2.0	0.08269054	0.08280740	0.08281105	0.08281123
4.0	−0.02636454	−0.02637893	−0.02637942	−0.02637945
6.0	0.00148376	0.00148316	0.00148315	0.00148315
8.0	0.00032402	0.00032436	0.00032437	0.00032438
10.0	−0.000162186	−0.00016231	−0.00016231	−0.00016231

5.2. Finite beam subjected to a line load

In another case, the beam is subjected to a line load resting on a Winkler foundation of the hyperbolic type. The beam is subjected to external loading, which is aline load P with dimension of force per unit length. Because of discontinuity of stress distribution, the line load in form of the Dirac delta function is used (Sohouli *et al.*, 2010)

$$p(\xi) = \overline{P}\delta(\xi - \alpha) \qquad \overline{P} = \lambda P \tag{5.5}$$

where α is the point in which a single line load is applied. The boundary conditions of the beam edges are arbitrary, and it can be assumed that the beam is free-free or clamped-free or simply supported. A clamped-clamped beam with a single line load appied to the point α is chosen. The boundary conditions are expressed as follows

$$w(0) = w'(0) = w(\beta) = w'(\beta) = 0 \tag{5.6}$$

By assuming $u_0(\xi) = 0$ in the first and second term of the series, the beam deflection $w(\xi)$ will be as follows

$$
\begin{aligned}
w(\xi) &= u_0(\xi) + \sum_{i=1}^{\infty} u_i(\xi) = \frac{\overline{P}}{2k}\hbar[\widetilde{H}(\xi-\alpha)+1] \\
&\cdot \Big\{ \mathrm{e}^{\xi-\alpha}[\sin(\xi-\alpha)-\cos(\xi-\alpha)] + \mathrm{e}^{\alpha-\xi}[\sin(\xi-\alpha)+\cos(\xi-\alpha)] \Big\} \\
&+ c_1\mathrm{e}^{\xi}\cos\xi + c_2\mathrm{e}^{\xi}\sin\xi + c_3\mathrm{e}^{-\xi}\cos\xi + c_4\mathrm{e}^{-\xi}\sin\xi + \ldots
\end{aligned}
\tag{5.7}
$$

To compare the present solution with numerical one, some scalar parameters such as μ, β, $\overline{P}/k$ are arbitrarily defined and some values have been attributed to them. The results are presented in Tables 4 and 5. $w(\xi)$ is the beam deflection under a single line load P which is expressed in a non-dimensional form $\overline{P}/k$. It is obvious that by increasing the order of HAM iterations, the accuracy increases. A numerical method based on a non-linear finite difference technique is employed to compare the results. It should be mentioned that the number of mesh nodes in this part is increased because of the jump of the Dirac delta function (Courant and Hilbert, 1962; Li and Wong, 2008).

Table 4. Comparison between the HAM and numerical solution with accuracy of 10E-6 for the beam deflection $w(\xi)$, $\beta = 5$, $\overline{P}/k = 1$, $\mu = 1$ and $\alpha = 2.5$

ξ	$W(\xi)$			
	2-nd order	4-th order	6-th order	numeric
0.0	0.000000	0.000000	0.000000	0.000000
1.0	0.162611	0.163896	0.163766	0.163767
2.0	0.557089	0.559202	0.558912	0.558917
2.5	0.665190	0.667336	0.667035	0.667030
3.0	0.557089	0.559202	0.558912	0.558917
4.0	0.162611	0.163896	0.163766	0.163767
5.0	0.000000	0.000000	0.000000	0.000000

The Dirac delta function influences numerical computation, therefore another expression for the Dirac delta function is considered as follows

$$
p(\xi) = \overline{P}\delta(\xi-\alpha) = \frac{\overline{P}}{2}\lim_{\varepsilon\to 0}\Big(\frac{1}{\varepsilon}\mathrm{e}^{-\frac{|\xi-\alpha|}{\varepsilon}} \tag{5.8}
$$

In some cases, the delta function is described as an expansion of the sine Fourier series

$$
p(\xi) = \overline{P}\delta(\xi-\alpha) = \frac{2\overline{P}}{\beta}\sum_{n=0}^{\infty}\sin\frac{n\pi\alpha}{\beta}\sin\frac{n\pi\xi}{\beta} \tag{5.9}
$$

where β is length of the beam. This problem is solved by the finite difference method. Results of expanding of the Fourier series with 300 orders are shown

Table 5. Comparison between the HAM and numerical solution with accuracy of 10E-6 for the beam deflection $w(\xi)$, $\beta = 10$, $\overline{P}/k = 2$, $\mu = 0.5$ and $\alpha = 5$

ξ	$W(\xi)$			
	2-nd order	4-th order	6-th order	numeric
0.0	0.0000000	0.0000000	0.00000000	0.0000000
2.0	−0.0449447	−0.0438416	−0.0437668	−0.0437651
4.0	0.7519053	0.7630867	0.7632160	0.7632141
5.0	0.1332533	1.3776853	1.3777917	1.3777917
6.0	0.7632141	0.7630867	0.7632160	0.7632141
8.0	−0.0449447	−0.0438416	−0.0437668	−0.0437651
10.0	0.0000000	0.0000000	0.0000000	0.0000000

in Tables 4 and 5. The convergence study of the solution is presented in Fig. 2. The $\hbar$-curves indicate the interval of solution convergence, which is $[-2, -1]$. From Tables 4 and 5 it can be concluded that the error is less than 10E-6, and the analytical solution converges with a high order of accuracy.

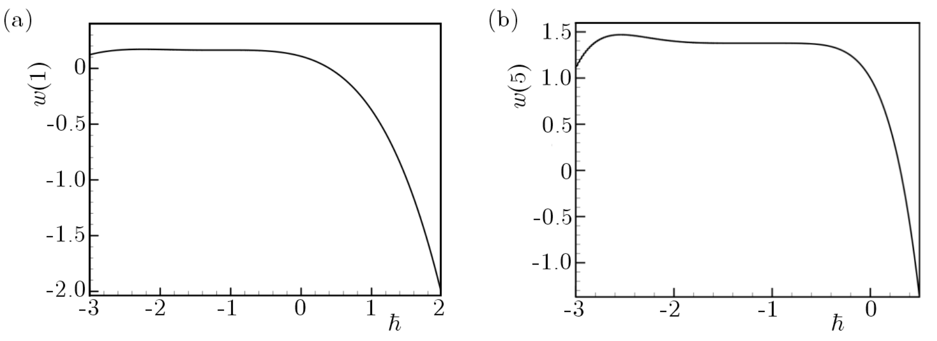

Fig. 2. $\hbar$-curves with the 6-th order approximation showing the convergence of HAM results; w is the deflection of the beam at a desired point versus $\hbar$, which indicates the region where the results converge; (a) $\beta = 5$, $\overline{P}/k = 1$, $\mu = 1$, $\alpha = 2.5$, (b) $\beta = 10$, $\overline{P}/k = 2$, $\mu = 1$, $\alpha = 5$

5.3. Problems with high nonlinearity

By considering Eq. (2.3), it can be pointed out that the coefficient μ has an effective influence on nonlinearity of the problem. The nonlinearity becomes more intensive when the value of $w(\xi)$ increases more than one. For example, when $|\mu w| \ll 1$ the nonlinearity of the problem is weak and the answer converges after a few iterations, hence in the most of cases, the perturbation method can not converge. Therefore, a solution should be presented in such a way so that it could control the convergence. In the HAM method, by controlling the $\hbar$ parameter, we can ensure the convergence of the problem. It was indicated above that the parameter μ has a very effective influence on the problem nonlinearity, and increasing the μ causes the nonlinearity even more intense. In this case, the order of the HAM method should be increased, and also the $\hbar$ parameter should be properly controlled as shown in Fig. 3. It means that in problems with intense nonlinearity the order of iterations in the solution series should be increased to have more accurate results. Therefore, a proper $\hbar$ for the desired order of HAM solution should be considered.

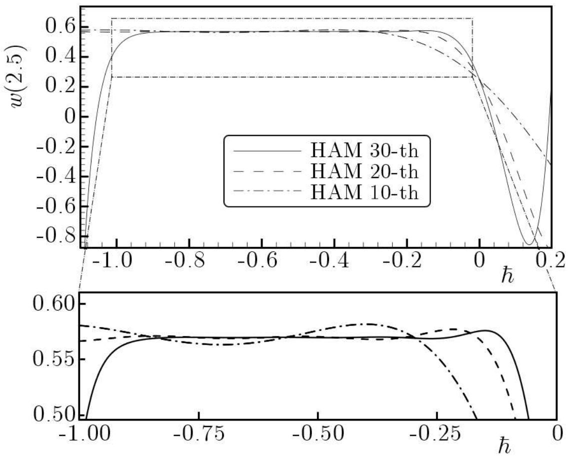

Fig. 3. $\hbar$-curve indicating the convergence of HAM results with respect to the order of HAM method and various values of $\hbar$. w shows the deflection of the beam at $\xi = 2.5$

To investigate this case, a beam on a Winkler foundation of the hyperbolic type with a single line load is considered. It is assumed that $\mu = 6$ and also for having rational displacement and preventing too large deformation, $\overline{P}/k = 0.5$ is chosen. The load is applied in the middle of the beam, and the length parameter β is equal to 5. The comparison between the numerical solution and HAM is presented in Table 6. Again, it can be easily seen that the solution converges and the results are in good agreement with the numerical ones.

It should be noted that Eq (2.2) indicates the nonlinear response of the foundation due to the beam deflection stress response, which behaves like a resistant factor preventing the beam from forced deflection. Increasing the value of μ will decrease the magnitude of the foundation stress; hence, the resistance

Table 6. Comparison between the HAM and numerical solution with accuracy of 10E-6 for the beam deflection $w(\xi)$, $\beta = 5$, $\overline{P}/k = 0.5$, $\mu = 6$ and $\alpha = 2.5$

ξ	$W(\xi)$			
	2-nd order	4-th order	6-th order	numeric
0.0	0.0000000	0.0000000	0.0000000	0.0000000
1.0	0.1654173	0.1667064	0.1670509	0.1670520
2.0	0.4897607	0.4931670	0.4939046	0.4938936
2.5	0.5652989	0.5690297	0.5698224	0.5698091
3.0	0.4897607	0.4931670	0.4939046	0.4938936
4.0	0.1654173	0.1667064	0.1670509	0.1670520
5.0	0.0000000	0.0000000	0.0000000	0.0000000

factor will decrease and the beam will be more deformable. It occurs that increasing μ will increase deformation variation two times greater with respect to the situation with zero nonlinearity ($\mu = 0$). Figure 4 shows the deflection of the clamped-clamped beam under the single line load by considering different values of the nonlinear term of μ.

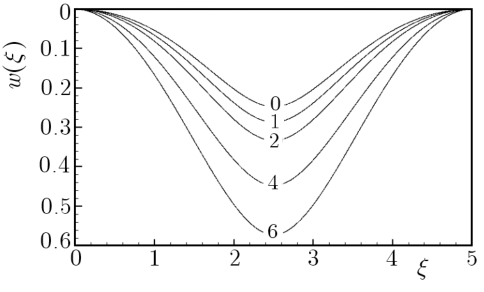

Fig. 4. Deflection of the clamped-clamped beam obtained by the HAM method with respect to various values of μ; $\beta = 5$, $\overline{P}/k = 0.5$, $\alpha = 2.5$

6. Conclusion

In this paper, an analytical solution was presented for the Bernoulli-Euler beam with a non-linear Winkler type foundation. It was found with a high domain of convergence and by a few iterations in the solution procedure using the HAM method. The problem was solved for different boundary conditions and constant parameters and also for high nonlinearity conditions. It was shown that in some cases, the accuracy is 10E-7 by only the 6-th order of approximation. The convergence of the results was investigated by plotting different $\hbar$-curves for different boundary conditions. In most cases, $\hbar = -1$ was the appropriate value for good stability in the desired $\hbar$ point.

However, in the existing solutions, two approximations were used to obtain the results. HAM solution was obtained with a high accuracy and a few iterations made.

There is also a benefit in the HAM solution comparing to other references (Sohouli *et al.*, 2010), as can be mentioned that those methods are based on the perturbation method combined with orthogonal functions expansion, in which the orthogonal functions are obtained from solving characteristic eigenfunctions of a freely vibrating beam. So there are two approximations in their solution, the first approximation is in the perturbation expansion and the second one is in the orthogonal functions. They affect the accuracy of the solution. Moreover, that method leads to two problems, one is the extraction of the eigenfunctions of a freely vibrating beam, and the second is solving the problem with an asymptotic expansion with the perturbation method. The HAM method gives a brief solution with benefit of control convergence by the parameter $\hbar$ that guarantees the convergence for variety of coefficients. This paper presents an analytical solution for any kind of initial conditions and distributed loads which can be easily applied to other Bernoulli-Euler beams with non-linear elastic foundations and other nonlinear parameters.

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Przybliżone rozwiązanie analityczne dla belek Bernoulliego-Eulera opartych na nieliniowym podłożu winklerowskim przy różnych warunkach zamocowania

Streszczenie

W pracy przedstawiono zastosowanie bardzo wszechstronnej metody homotopii (HAM) do uzyskania analitycznego rozwiązania nieliniowych równań różniczkowych opisujących drgania belek Bernoulliego-Eulera spoczywających na nieliniowym podłożu winklerowskim. W celu zaprezentowania dokładności metody, uzyskane wyniki porównano z rezultatami symulacji numerycznych. Wykazano, że tak otrzymane rozwiązanie jest ważne w całej dziedzinie, także przy uwzględnieniu członów nieliniowych wyższego rzędu. Inne metody, m.in. analiza perturbacyjna, przestają być w takich przypadkach zbieżne. Przeprowadzone badania wyraźnie dowodzą, że obszar zbieżności może być monitorowany i dostosowywany w ramach metody HAM. Na zakończenie rozważań, po weryfikacji obliczeń, przedyskutowano wpływ stałych parametrów układu na ugięcie i kąt ugięcia belek przy równych warunkach zamocowania.

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