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ON CONTINUOUS DESCRIPTIONS OF THE DAMAGE

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1. Introductory remarks

A large group of structural materials are these whose deformation involves the nucleation and growth of internal damage. Both the deformation and damage growth are closely connected, which means that this is a usual response of this kind of materials to external loading. Moreover, at low temperature almost all materials suffer internal damage and therefore a description of their mechanical behaviour in the presence of damage plays an important role. Due to this importance there exist many different propositions of descriptions of the damage nucleation and growth. All these theories can be classified from the methodological point of view in the following way. The first group is constituted by empiric theories. They are based on experiments whose aim is to explain the fundamental mechanisms of nucleation and growth of microvoids. A formal description of such mechanisms provides the theories which use great number of parameters for description of a relatively narrow class of materials subjected to rather strongly limited types of external loadings. These are, in majority, metallurgical and physical works starting with investigations where electron microscopy plays a central role. The second group covers a priori theories. The body is treated as an isolated mechanical — rarely thermodynamic — system and theory is a formal description of the moste characteristic phenomena of the damage process. These theories — called phenomenological — are constructed in various ways. The largest part of these theories are such that describe a single macrocrack and its propagation. The main aim of this description is to establish the criterion of unstable crack propagation as a yield condition. This kind of model seems to be the most adequate for the description of the last phase of the body behaviour resulting from the micro-cracks coalescence. Formally these are description of the discrete process of fracture. To describe the body behaviour in the state preceding macro-cracks propagation, a number of authors have proposed "continualization" of the system of macro-cracks description. The most effective and formally faultless method of such "smearing" is the homogenization method. The localization of macro-effects in a particle resulting from such formal procedure leads to not always acceptable events and therefore a priori continuous theories have been proposed. The continuous theories based on the assumption that the effect of the micro-cracks nucleation and growth is measurable and that the measure function can be assumed to be regular enough. First such concept was proposed by L. M. Kachanov. This idea has been developed by many authors and resulted in a number of various theories. All these theories differ from each other in various definitions of the damage measures and their interpretations. This paper considers common features of the theories mentioned above.

From the methodological point of view the continuous theories of damage can be treated as a formal realization of the idea of the material structure with internal variables but with such additional assumptions which allow one to simplify the description of the body behaviour. In general terms these assumption leads to decoupling of the system of equations which governs the process. Sometimes these assumptions are formulated explicitly, and sometimes they are left to the readers guess.

This paper is concerned with the problem of kinematical admissibility of the damage descriptions which have been derived on the basis of the mentioned assumptions. The problem is discussed in detail in relation to some theories which have been chosen as an example but the validity of the conclusions is more general. Apart from the above discussion, there is proposed a physically reasonsable definition of the damage measure and such description of the damage process which allows one to overcome the kinematical admissibility problem. Finally it has been shown that the damage measures proposed so far can be obtained as a particular case of the one proposed here.

2. Formal foundation of damage continuous description

All theories which are considered here use the same basic notion which is their common feature. This notion has been understood in different ways, however, and its scope of meaning has also been different. For example in papers [18] and [5] this notion was identified with the so called "undamaged material" but in papers [1] and [14] "the fictitious undamaged configuration" was introduced. Moreover in none of all the quoted papers this notion was defined and readers must had to content themselves with intuitive meaning. Despite different scope of meaning the notion has been used in a similar way and for the same purpose. In this paper the compromising notion of "the undamaged body" was adopted and owing to this both the fictitious undamaged configuration and the fictitious undamaged material can be considered as well. The undamaged, fictitious body will be meant further as such a body that can represent the deformation process of the real damaged body in the sens which will be defined precisely each time. The knowledge of the fictitious body deformation allows one to determine both the deformation and the state of damage of the real body, the latter one by means of the values of the appropriate function of damage measure. A good illustration of the fictitious, undamaged body concept can be found in paper [18] by Sidoroff. This paper proposes the simplest decoupled description of damage process. Constitutive equation of the real body has been denoted by author as a functional:

$$\boldsymbol{\epsilon}(t) = \mathscr{E}\{\boldsymbol{\sigma}, D\}, \tag{2.1}$$

where D — denotes the macroscopic measure of internal degradation, or in other words — damage variable.

The response of an ideal undamaged material is obtained as the values of this functional when there is no damage i.e. setting $D = 0$ and denoted:

$$\mathscr{E}\{\sigma, D = 0\} = \mathscr{E}_0\{\sigma\}. \tag{2.2}$$

The definition quoted above has been used in the following "basic postulate"

— the damaged response of the material is obtained by replacement of the stress tensor $\boldsymbol{\sigma}$ by an effective stress tensor $\tilde{\boldsymbol{\sigma}}$ in the undamaged response:

$$\mathscr{E}(t) = \mathscr{E}\{\sigma, D\} = \mathscr{E}_0\{\tilde{\sigma}\}, \tag{2.3$_1$}$$

$$\tilde{\sigma} = \sigma/(1-D), \quad D = D(t) \in \langle 0,1 \rangle. \tag{2.3$_2$}$$

Next, this postulate has been used by Sidoroff for the formulation of the damaged response of a brittle elastic material. The undamaged response is described by Hooke's law, so replacing stress tensor $\boldsymbol{\sigma}$ by effective stress tensor $\tilde{\boldsymbol{\sigma}}$ one can obtain:

$$\boldsymbol{\epsilon} = \frac{1}{\tilde{E}}[\tilde{\boldsymbol{\sigma}}(1+\mu)-\mu \operatorname{tr}\tilde{\boldsymbol{\sigma}}\mathbf{1}] = \mathscr{E}_0(\tilde{\sigma}). \tag{2.4$_1$}$$

By using definition (2.3$_2$) the functional (2.4$_1$) can be in straight-forward way extended to damage response of elastic material:

$$\boldsymbol{\epsilon} = \frac{1}{(1-D)\tilde{E}}[\boldsymbol{\sigma}(1+\mu)-\mu \operatorname{tr}\boldsymbol{\sigma}\mathbf{1}] = \mathscr{E}(\sigma, D). \tag{2.4$_2$}$$

So, the real damaged material is modelled by the elastic material but subjected to the effective stress action.

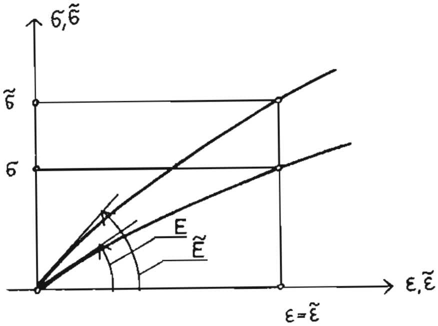

Fig. 1.

It is easy to see that the real damaged material can be modelled also by the elastic material subjected to the nominal stress action but with the appropriately modified properties by setting:

$$E = \tilde{E}(1-D),$$

$$\boldsymbol{\epsilon} = \frac{1}{E}[\boldsymbol{\sigma}(1+\mu)-\mu \operatorname{tr}\boldsymbol{\sigma}\,\mathbf{1}].$$

Denoting formally:

$$\tilde{\boldsymbol{\epsilon}} = \frac{1}{\tilde{E}}[\tilde{\boldsymbol{\sigma}}(1+\mu)-\mu \operatorname{tr}\tilde{\boldsymbol{\sigma}}\mathbf{1}],$$

and assuming $\boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}}$ one can obtain the relation between the strain tensors:

$$\boldsymbol{\epsilon} = \frac{\tilde{\boldsymbol{\epsilon}}}{1-D}. \tag{2.5}$$

It means — in accordance with intuition — that the same stress produces different strains in both regarded bodies; greater in the real damaged and lower in the fictitious undamaged one shown in formula (2.5).

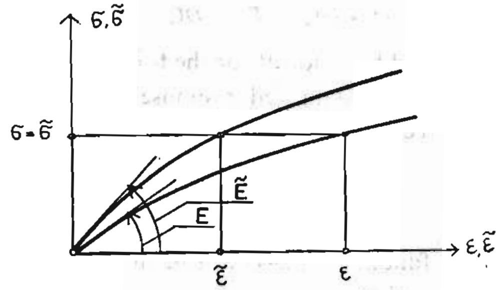

Fig. 2.

Relation (2.5) can be treated as the transformation rule which allows one to determine the state of strains on the basis of the strains of the fictitious body. The sense of formula (2.3_2) is illustrated in figure 1. It is easy to see that the state of real body in the presence of the damage can be described as a result of the fictitious relaxation. Similarly, the sense of transformation formula (2.5) for strains is illustrated in figure 2. It shows that the damage growth process can be treated as "the fictitious creep" of the fictitious body. The model of the fictitious body considered here in accordance with paper [18] represents a concept simplified so much that the most characteristic features of damage growth process are lost (damage measure does not depend on strains, the Poisson's ratio μ remains constant and the anisotropy of the process is ignored). A slightly better possibility of damaged body modelling was proposed in paper [5]. This concept is based on the assumption that penny-shaped microvoids develop in the planes normal to the direction of principal tension. Such a physically acceptable hipothesis distinguishes some directions and gives the possibility to describe the anisotropy of the damage growth process.

Let A denote the area of the cross-section of the specimen submitted to uniaxial tension. If the material of the specimen suffers damage, then some part of this area is occupied by voids. Denoting this part of the cross-section area that carries the stress with $\overset{*}{A}$, one can write the relation:

$$\overset{*}{A} = (1-\omega)A, \tag{2.6}$$

where ω — denotes the damage measure which is defined as ratio:

$$\frac{A-\overset{*}{A}}{A} = \frac{\Delta A}{A} \overset{\text{df}}{=} \omega; \qquad \omega \in \langle 0,1 \rangle. \tag{2.7}$$

Further, the author of the quoted paper assumes that strains of both real and fictitious bodies are equal to each other when measured in the direction of the tension. Assuming that it is direction x_3 this equivalence reads:

$$\varepsilon_{33} = \overset{*}{\varepsilon}_{33}. \tag{2.8}$$

It suggests that the damage growth process will be treated as a fictitious relaxation. Taking into account obvious relation:

$$P = \sigma A = \overset{*}{\sigma}\overset{*}{A} = \overset{*}{\sigma}(1-\omega)A,$$

one can see that the stresses are related to one other:

$$\overset{*}{\sigma} = \sigma/(1-\omega). \tag{2.9}$$

So, the elasticity moduli are related, too:

$$\overset{*}{E} = E(1-\omega). \tag{2.10}$$

Relation (2.9) looks as a relaxation rule but the equivalence $P = \sigma A = \overset{*}{\sigma}\overset{*}{A}$ leads to the confusion because the external force remains unchanged. It should be emphesitzed, however, that assumption $\varepsilon_3 = \overset{*}{\varepsilon}_3$ does not exclude the damage growth process which can be described as the fictitious creep but in the direction perpendicular to the x_3 axis. This effect is due to "plane deterioration" of the cross-section area in accordance with the assumed shape of the microvoids. This effect is shown in figure 3. The reasonability of the "trans-

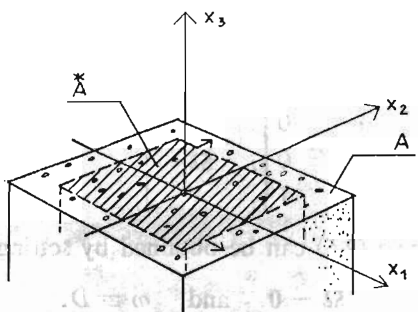

Fig. 3.

verse creep" mechanism is confirmed by the Poisson's ratio change observed in experiments. The rule of Poisson's ratio changing can be easily derived in the following way. Taking into account that both areas A and $\overset{*}{A}$ resulted from the deformation of the same area chosen in the reference configuration $A_0 = a^2$, the following relation holds:

$$\begin{aligned} A &= a^2(1-\varepsilon_2)^2 \cong a^2(1-2\varepsilon_2), \\ \overset{*}{A} &= a^2(1-\overset{*}{\varepsilon}_2)^2 \cong a^2(1-2\overset{*}{\varepsilon}_2), \end{aligned} \tag{2.11}$$

where: $\varepsilon_2 = \varepsilon_{22} = \varepsilon_{11}, \quad \overset{*}{\varepsilon}_2 = \overset{*}{\varepsilon}_{22} = \overset{*}{\varepsilon}_{11},$

then the area increment can be found in the following form:

$$\Delta A = 2a^2(\overset{*}{\varepsilon}_2 - \varepsilon_2). \tag{2.12$_1$}$$

On the other hand:

$$\Delta A = A\omega = a^2(1-2\varepsilon_2)\omega, \tag{2.12$_2$}$$

and by comparison:

$$\varepsilon_2 = \overset{*}{\varepsilon}_2/(1-\omega) - \omega/2(1-\omega). \tag{2.13}$$

This is the rule of the "transverse creep" which was mentioned before. Next, taking into account the relations:

$$\varepsilon_{11} = \varepsilon_{22} = -\mu\varepsilon_3, \quad \overset{*}{\varepsilon}_{11} = \overset{*}{\varepsilon}_{22} = -\overset{*}{\mu}\overset{*}{\varepsilon}_3,$$

one can easily find the relation:

$$\mu = \frac{1}{1-\omega}\left[\overset{*}{\mu} - \frac{\omega}{2(1-\omega)\overset{*}{\varepsilon}_3}\right], \tag{2.14}$$

which expresses the Poisson's ratio changing. So, in accordance with Chaboche concept the fictitious and real bodies differ from each other not only in the elasticity modulus but in the Poisson's ratio as well. It is the qualitatively new element of the considered concept in comparison with the previously quoted one.

Formulae (2.8) and (2.13) represent — as mentioned before — fictitious creep due to damage growth. It can be generalized as the strain transformation rule (this formal relation will be derived further in a different way):

$$\boldsymbol{\epsilon} = \mathbf{Q}\overset{*}{\boldsymbol{\epsilon}}\mathbf{Q}^{\mathsf{T}} + \boldsymbol{\Omega}, \tag{2.15}$$

where:

$$\mathbf{Q} = \begin{bmatrix} (1-\omega)^{-1/2} & 0 & 0 \\ 0 & (1-\omega)^{-1/2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\Omega} = \begin{bmatrix} -\omega/2(1-\omega) & 0 & 0 \\ 0 & -\omega/2(1-\omega) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Similiarly, transformation rule (2.5) can be obtained by setting:

$$\boldsymbol{\Omega} = \mathbf{0} \quad \text{and} \quad \omega \equiv D.$$

Now, it is easily seen that while Sidoroff's model leads to the isotropic fictitious creep, Chaboche's proposition allows one to describe the anisotropy of the damage growth. Intuitively it is quite obvious. In the first concept the microvoids were assumed to be spherical, whilst in the second there were assumed penny-shaped laing in the distinguished planes. Other generalizations will be discussed further.

The damage measure ω — represents a new state variable and for the completeness of the considered concepts the constitutive equation should be added. Assuming this equation in a general form:

$$\omega = \omega(\overset{*}{\boldsymbol{\epsilon}}, t), \tag{2.16}$$

one can find the deformation of the real body and the nominal stress on the knowledge of the states of strain and stress of the fictitious body. Taking into account transformation rule for stress (2.9) one can easily come to the conclusion that effective stresses have to satisfy the following equation:

$$\operatorname{div}\overset{*}{\boldsymbol{\sigma}} - \overset{*}{\boldsymbol{\sigma}}(\operatorname{grad}\omega)/(1-\omega) + \mathbf{b}/(1-\omega) = 0, \tag{2.17}$$

which represents the condition of the statical admissibility of the state of stress in the real body (equivalent to $\operatorname{div}\boldsymbol{\sigma}+\mathbf{b} = 0$). In other words, for statical admissibility of the nominal stress, the fictitious body should be submitted to the fictitious body forces action which are defined as below:

$$\overset{*}{\mathbf{b}} \overset{\mathrm{df}}{=} [\mathbf{b} - \overset{*}{\boldsymbol{\sigma}}(\operatorname{grad}\omega)]/(1-\omega). \tag{2.18}$$

In a particular case when ω is a constant scalar function $\mathbf{b} = \overset{*}{\mathbf{b}}$, but in general it does not to be true. Similarly, the transformation rule for strain implies that for kinematical admissibility of the real body strain (i.e. $\operatorname{Inc}\boldsymbol{\epsilon} = 0$) the following condition should be fulfiled:

$$(\operatorname{Inc}\mathbf{Q})\overset{*}{\boldsymbol{\epsilon}}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}(\operatorname{Inc}\overset{*}{\boldsymbol{\epsilon}})\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\overset{*}{\boldsymbol{\epsilon}}\operatorname{Inc}(\mathbf{Q}^{\mathrm{T}}) + \operatorname{Inc}\boldsymbol{\Omega} = 0, \tag{2.19}$$

where: $\operatorname{Inc} \overset{\mathrm{df}}{=} \epsilon^{imk}\epsilon^{jnl}\nabla_m\nabla_n$

In fact this a restriction put on the constitutive equation of the damage measure evolution. It becomes obvious of the kinematical admissibility of the fictitious body strain is required. In this case the considered condition takes the following form:

$$(\operatorname{Inc}\mathbf{Q})\overset{*}{\boldsymbol{\epsilon}}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\overset{*}{\boldsymbol{\epsilon}}(\operatorname{Inc}\mathbf{Q}^{\mathrm{T}}) + \operatorname{Inc}\boldsymbol{\Omega} = 0, \tag{2.20}$$

which can be treated as the condition of the kinematical admissibility of the damage evolution equation. In particular cases condition (2.20) can be satisfied but in general it represents a non-trivial and very strong restriction of the class of the admissible constitutive equations of the damage.

3. On some features of damage continuous descriptions

One of the main directions of Kachanov's concept evolution is such a modification of damage measure which allows one to describe the anisotropy of the damage growth process. The theories proposed first by Murakami and Ohno [14] and next improved by Murakami and Imazumi [15] represent the result of such investigations. It has been obtained as a generalization of the previously quoted idea according to the Chaboche's paper. Some considerations dealing with the tetrahedron damage led the authors to the hipothesis that the damage mechanism is like the one mentioned in Chaboche's paper but generally in three principal directions simultanously. In other words, in accordance with Murakami's hypothesis, the microvoids should have the form of elipsoids. This basic hypothesis allows the authors to define the tensorial damage measure in such a way that to every vectorial measure of the area in the real damaged body one can associate an appropriate area in the ficititious undamaged body:

$$\overset{*}{\mathbf{S}} = (\mathbf{1}-\boldsymbol{\Omega})\mathbf{S}, \quad \overset{*}{\mathbf{S}} = \overset{*}{S}\overset{*}{\boldsymbol{\nu}}, \quad \mathbf{S} = S\boldsymbol{\nu}. \tag{3.1}$$

Here the original notation in paper [14] was used. The asterisk denotes the quantities related to the fictitious configuration. Symmetric tensor $\mathbf{\Omega}$ has been used to define a new tensor:

$$\mathbf{\Phi} \overset{\mathrm{df}}{=} (\mathbf{1}-\mathbf{\Omega})^{-1}, \tag{3.2}$$

which is called the damage effects tensor. Next, on the basis of the Cauchy stress tensor defined for the real body the effective stress tensor has been defined over the undamaged configuration $\overset{*}{B}_t$:

$$\boldsymbol{\sigma}\mathbf{S} = \overset{*}{\boldsymbol{\sigma}}\overset{*}{\mathbf{S}}, \quad \overset{*}{\boldsymbol{\sigma}} \overset{\mathrm{df}}{=} \mathbf{\Phi}\boldsymbol{\sigma}. \tag{3.3}$$

Because of lack of symmetry of tensor $\boldsymbol{\sigma}$, the following stress measure has been introduced:

$$\boldsymbol{\tau} = 1/2(\overset{*}{\boldsymbol{\sigma}}+\overset{*}{\boldsymbol{\sigma}}{}^{\mathrm{T}}). \tag{3.4}$$

The above definitions have been completed with the following constitutive equations:

$$\begin{aligned} \mathbf{D} &= \mathbf{G}(\boldsymbol{\sigma}, \mathbf{\Omega}, K, \Theta), \\ \mathring{\mathbf{\Omega}} &= \mathbf{H}(\boldsymbol{\sigma}, \mathbf{\Omega}, K, \Theta), \end{aligned} \tag{3.5}$$

where:

$\mathbf{D}$ — strain rate tensor,
K — hardening parameter,
Θ — temperature.

So, the real body has been treated as the material structure with internal variables. Further, however, the authors propose another form of the constitutive equation of damage tensor:

$$\mathring{\mathbf{\Omega}} = \overset{*}{\mathbf{H}}(\boldsymbol{\tau}, \Phi, K, \Theta), \tag{3.6}$$

which is considered to be equivalent to form (3.5). Taking into account the definitions of the tensors fields $\boldsymbol{\tau}$ and $\mathbf{\Phi}$ and the domains of these fields, formula (3.6) should be replaced by the following:

$$\Phi = \overset{*}{\mathbf{H}}(\boldsymbol{\tau}, \Phi, K, \Theta). \tag{3.7}$$

This last form is more correct and now the equation of damage measure is defined over the fictitious body. Further, the authors propose to rewrite the first of equations (3.5) as related to the fictitious body:

$$\overset{*}{D} = \overset{*}{\mathbf{G}}(\boldsymbol{\tau}, \mathbf{\Phi}, K, \Theta). \tag{3.8}$$

So, in view of (3.3) and (3.4) one can find the following formula equivalent to (3.8) (which cannot be found in [14]):

$$2D = \mathbf{G}\{\Phi \overset{*}{G}{}^{-1}(\overset{*}{\mathbf{D}}, \mathbf{\Phi}, K, \Theta)+[\overset{*}{G}{}^{-1}(\overset{*}{\mathbf{D}}, \mathbf{\Phi}, K, \Theta)]^{\mathrm{T}}\mathbf{\Phi}^{\mathrm{T}}\}, \tag{3.9}$$

which represents the transformation rule for the deformation rate of both considered bodies, real and fictitious one. It means that the description of the damage process has been decoupled. The similarity between the concept considered here and the one quoted previously according to the Chaboche's paper can be shown on the basis of the kinematical analysis. For this purpose it is enough to choose an appropriate mapping of the fictitious body on the real one. In case of Sidoroff's concept, the damage effect is equivalent to the homogenous creep" and the following form of mapping is suggested:

$$\mathbf{x} = \overset{*}{\mathbf{x}}(1-\omega)^{-1/2}. \tag{3.10}$$

The deformation gradient takes the form:

$$\mathbf{\Phi} = \frac{\partial \mathbf{x}}{\partial \overset{*}{\mathbf{x}}} = \mathbf{1}(1-\omega)^{-1/2}. \tag{3.11}$$

Now in virtue of the well known formula for vectorial element of area transformation [22]:

$$da = J(\mathbf{F}^{-1})^{\mathrm{T}} dA, \qquad J = \det \mathbf{F}, \tag{3.12}$$

where: $dA = dA(\mathrm{N})$ — area of the element chosen in the reference configuration,
$da = da(n)$ — area of the element but in the current configuration,
$\mathbf{F}$ — deformation gradient,

and changing the notation appropriately:

$$\mathbf{F} \doteq \mathbf{\Phi},$$
$$J \doteq J_{\Phi} = \det\mathbf{\Phi},$$
$$da \doteq A,$$
$$dA \doteq \overset{*}{A},$$

one can obtain $A = \overset{*}{A}\dfrac{1}{1-\omega}$.

In case of the concept according to Chaboche's papaer, the mapping of the fictitious body on the real one can be taken in the form:

$$\begin{aligned} x_1 &= \overset{*}{x}_1(1-\omega)^{-1/2}, \\ x_2 &= \overset{*}{x}_2(1-\omega)^{-1/2}, \\ x_3 &= \overset{*}{x}_3. \end{aligned} \tag{3.13}$$

Now one can calculate the deformation gradient and in the same way as previously find the formulae for area transformation:

$$A_1 = A_2 = \frac{\overset{*}{A}_1}{(1-\omega)^{-1/2}} = \frac{\overset{*}{A}_2}{(1-\omega)^{-1/2}},$$

which agrees with formula (2.5).

$$A_3 = \frac{\overset{*}{A}_3}{(1-\omega)}, \tag{3.14}$$

Finally for Murakami and Ohno idea the mapping of the fictitious body onto the real one should be taken in the form:

$$\begin{aligned} x_1 &= (1-\Omega_1)^{-1/2}(1-\Omega_2)^{1/2}(1-\Omega_3)^{1/2}\overset{*}{x}_1, \\ x_2 &= (1-\Omega_2)^{-1/2}(1-\Omega_1)^{1/2}(1-\Omega_3)^{1/2}\overset{*}{x}_2, \\ x_3 &= (1-\Omega_3)^{-1/2}(1-\Omega_2)^{1/2}(1-\Omega_1)^{1/2}\overset{*}{x}_3, \end{aligned} \tag{3.15}$$

$$\boldsymbol{\Phi} = \begin{bmatrix} (1-\Omega_1)^{-1/2}[(1-\Omega_2)(1-\Omega_3)]^{1/2}, & 0, & 0 \\ 0, & (1-\Omega_2)^{-1/2}[(1-\Omega_1)(1-\Omega_3)]^{1/2}, & 0 \\ 0, & 0, & (1-\Omega_3)^{-1/2}[(1-\Omega_1)(1-\Omega_2)]^{1/2} \end{bmatrix}. \tag{3.16}$$

Since on an additional assumption $\det\boldsymbol{\Phi} = 1$, which is conjectured in paper [14], the formulae for area transformation read:

$$A_i = \overset{*}{A}_i(1-\Omega_i)^{-1}. \tag{3.17}$$

Of course the above formula is identical with (3.1) after substitution $\overset{*}{S} = \overset{*}{A}$ and $S = A$. The mentioned results show that phenomenological foundation of the discussed continuous theories of the damage process is in fact of kinematical character. So, for further analysis of the various propositions of the damage process description the kinematical framework will be the most convenient. Therefore the next section will be devoted to the damage growth process.

4. Kinematic of the damage growth process

The existing continuous theories of damage are kinematically linearized. Generally the damaged and undamaged configurations are distinguished one from each other but not from undamaged and deformed configuration of the body. Probably the result of such identification on configurations overlooks the fact that the effective stress tensor is a Piola-Kirchhoff pseudostress tensor defined over the deformed but undamaged configuration. The latter remark is very important because it dispels all doubts concerning thermodynamic and static admissibility of the effective stress tensor. Moreover, the problem kinematical admissibility of the damage evolution law has been omitted due to such a priori linearization. For all the above reasons attention is further paid to the precise description of the damage growth process. It requires the following postulates to be accepted:

Postulate 1

Over the body particles one can define function regular enough which plays the role of the damage measure.

Postulate 2

Such motion that damage measure function keeps constant value represents the motion of the fictitious body.

These postulates and an additional assumption that the body which suffers the damage can be treated as an isolated, mechanical system allow one to investigate the motion of the real body as related to the fictitious one, when the latter one is deformed but undamaged. Both considered bodies will be regarded with respect to this reference configuration. So, the fictitious body can be treated as the real body but with internal constraints. These constraints are selected in such way that in the motion which satisfies them the deformation due to damage growth dissapears. The motion of both bodies is shown in figure 4.

Let the real body motion in relation to the reference configuration be denoted:

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t), \qquad B_R \to B_t, \tag{4.1$_1$}$$

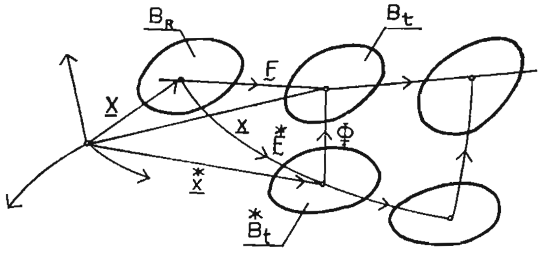

Fig. 4.

and the motion of the fictitious body in relation to the same configuration, respectively:

$$\overset{*}{\mathbf{x}} = \overset{*}{\mathbf{x}}(\mathbf{X}, t), \quad B_R \rightarrow \overset{*}{B}_t. \tag{4.1$_2$}$$

Then the real body motion but in the relation to the fictitious body can be denoted as the mapping:

$$\mathbf{x} = \mathbf{x}(\overset{*}{\mathbf{x}}, t), \quad \overset{*}{B}_t \rightarrow B_t. \tag{4.1$_3$}$$

In view of (4.1$_2$) the mapping $B_R \rightarrow \overset{*}{B}_i \rightarrow B_t$ can be written in the form:

$$\mathbf{x} = \mathbf{x}[\overset{*}{\mathbf{x}}(\mathbf{X}, t), t]. \tag{4.1$_4$}$$

Now, the deformation of the real body with respect to the reference configuration is defined by the deformation gradient:

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \{x^i_{,\alpha}\}. \tag{4.2}$$

Similarly, the deformation of the fictitious body in relation to the same reference configuration defines the deformation gradient:

$$\overset{*}{\mathbf{F}} = \frac{\partial \overset{*}{\mathbf{x}}}{\partial \mathbf{X}} = \{\overset{*}{x}{}^{\Delta}_{,\alpha}\}. \tag{4.3}$$

Finally, the deformation of the real body but in relation to the fictitious body is determined bu the following deformation gradient:

$$\mathbf{\Phi} = \frac{\partial \mathbf{x}}{\partial \overset{*}{\mathbf{x}}} = \{x^i_{,\Delta}\}. \tag{4.4}$$

This is the deformation due to micro-cracks field growth and nucleation. Using (4.1$_4$) it is easy to find the following relation between the deformation gradients just defined:

$$\begin{gathered} \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}}{\partial \overset{*}{\mathbf{x}}} \frac{\partial \overset{*}{\mathbf{x}}}{\partial \mathbf{X}} = \{x^i_{,\Delta} \overset{*}{x}{}^{\Delta}_{,\alpha}\}, \\ \mathbf{F} = \mathbf{\Phi} \overset{*}{\mathbf{F}}. \end{gathered} \tag{4.5}$$

Now in view of usual definitions other deformation measure can be found. For the real body:

$$\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F} = \{x^i_{,\alpha}x^j_{,\beta}g_{ij}\} \text{ — Green's deformation tensor,}$$
$$\mathbf{c} = (\mathbf{F}^{-1})^{\mathsf{T}}\mathbf{F}^{-1} = \{X^{\alpha}_{,i}X^{\beta}_{,j}G_{\alpha\beta}\} \text{ — Cauchy's deformation tensor,}$$
$$2\mathbf{E} = \mathbf{C}-\mathbf{G};\ 2E_{\alpha\beta} = C_{\alpha\beta}-G_{\alpha\beta} \text{ — lagrangian strain tensor,}$$
$$2\mathbf{e} = \mathbf{g}-\mathbf{c};\ 2e_{ij} = g_{ij}-c_{ij} \text{ — eulerian strain tensor.}$$

These are the deformation measures defined in relation to the reference configuration common for both bodies. In case of the real body one can define the deformation measure in relation to the fictitious body which is represented by configuration $\overset{*}{B}_t$:

$$\mathbf{C}_{\varphi} = \mathbf{\Phi}^{\mathsf{T}}\mathbf{\Phi} = \{x^i_{,\Delta}x^j_{,\Gamma}g_{ij}\}$$ — deformation tensor defined over the B_t which can be called "lagrangian-like",

$$\mathbf{c}_{\varphi} = (\mathbf{\Phi}^{-1})^{\mathsf{T}}\mathbf{\Phi}^{-1} = \{\overset{*}{x}^{\Gamma}_{,i}\overset{*}{x}^{\Delta}_{,j}g_{\Gamma\Delta}\}$$ — deformation tensor but defined over the B_t configuration which can be called eulerian.

The strain of the real body in relation to the fictitious body can be defined in the two different ways:

$$d\mathbf{x}\,d\mathbf{x}^{\mathsf{T}} - d\overset{*}{\mathbf{x}}\,d\overset{*}{\mathbf{x}}^{\mathsf{T}} = (x^i_{,\Gamma}x^j_{,\Delta}g_{ij}-g_{\Gamma\Delta})\,d\overset{*}{x}^{\Gamma}d\overset{*}{x}^{\Delta} = (\overset{*}{C}_{\Gamma\Delta}-g_{\Gamma\Delta})\,d\overset{*}{x}^{\Gamma}d\overset{*}{x}^{\Delta},$$

or briefly:

$$2\mathbf{e}_{\varphi} = \mathbf{C}_{\varphi}-\mathbf{g}. \tag{4.6}$$

This is of course a tensor field defined over the fictitious configuration $\overset{*}{B}_t$ and can be called "lagrangian-like". On the other hand, the same strain can be calculated in the following way:

$$d\mathbf{x}\,d\mathbf{x}^{\mathsf{T}} - d\overset{*}{\mathbf{x}}\,d\overset{*}{\mathbf{x}}^{\mathsf{T}} = (x^i_{,\alpha}x^j_{,\beta}g_{ij}-\overset{*}{x}^{\Gamma}_{,\alpha}\overset{*}{x}^{\Delta}_{,\beta}g_{\Gamma\Delta})\,dX^{\alpha}dX^{\beta} = (C_{\alpha\beta}-\overset{*}{C}_{\alpha\beta})\,dX^{\alpha}dX^{\beta},$$

or:

$$\mathbf{E}_{\varphi} = \mathbf{E}-\overset{*}{\mathbf{E}}. \tag{4.7}$$

Now this is the tensor field defined over the reference configuration, thus it represents exactly lagrangian strain tensor. In view of formula (4.5) one can find the relation between the deformations of both considered bodies. The most useful one take the forms:

$$\mathbf{c} = (\mathbf{F}^{-1})^{\mathsf{T}}\mathbf{F}^{-1} = (\mathbf{\Phi}^{-1})^{\mathsf{T}}(\overset{*}{\mathbf{F}}^{-1})^{\mathsf{T}}\overset{*}{\mathbf{F}}^{-1}\mathbf{\Phi}^{-1} = (\mathbf{\Phi}^{-1})^{\mathsf{T}}\overset{*}{\mathbf{c}}\mathbf{\Phi}, \tag{4.8$_1$}$$
$$\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F} = \overset{*}{\mathbf{F}}^{\mathsf{T}}\mathbf{\Phi}^{\mathsf{T}}\mathbf{\Phi}\overset{*}{\mathbf{F}} = \overset{*}{\mathbf{F}}^{\mathsf{T}}\mathbf{c}_{\varphi}\overset{*}{\mathbf{F}}. \tag{4.8$_2$}$$

Since — as mentioned before — the damage growth effect is represented as the deformation of the real body in relation to the fictitious one, then the surface deterioration due to this damage can be calculated in form:

$$d\mathbf{a} = J_{\varphi}\mathbf{\Phi}^{-1}d\overset{*}{\mathbf{a}};\quad J_{\varphi} = \det\mathbf{\Phi}.$$

Taking into account that:

$$\Delta d\mathbf{a} = d\mathbf{a}-d\overset{*}{\mathbf{a}},$$

one can finally find:

$$\Delta da = (J_\varphi \boldsymbol{\Phi}^{-1} - g)\overset{*}{da}. \tag{4.9}$$

Now the damage measure can be defined as the tensor:

$$\boldsymbol{\Psi} \overset{\mathrm{df}}{=} J_\varphi (\boldsymbol{\Phi}^{-1})^{\mathrm{T}} - g. \tag{4.10}$$

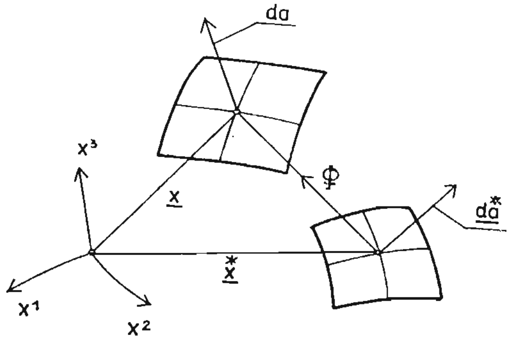

Fig. 5.

This is a two-points tensor field, i.e. a tensor defined over a pair of the configurations: a deformed but undamaged configuration and corresponding damaged configuration. Of course in general it is non-symmetric tensor. This is the operator which associates every element $\overset{*}{da}$ with the vectorial measure of the area deterioration.

So, the knowledge of the fictitious body motion and deformation in relation to configuration B_R (common with the real body) allows one to determine the damage measure and next the deformation caused by this damage. The real body deformation can be found as the composition of both above deformations.

The damage measure is a new state function and requires an appropriate constutive equation (similarly as in the previous concepts the damage parameter — ω)

In view of the polar decomposition theorem damage tensor $\boldsymbol{\Psi}$ can be decomposed into an orthogonal tensor $\mathbf{R}_\Psi$ which describes the rigid rotation and a symmetric tensor $\boldsymbol{\Omega}$ which describes the damage or deterioration of the body structure:

$$\boldsymbol{\Psi} = \mathbf{R}_\Psi^{\mathrm{T}} \boldsymbol{\Omega}. \tag{4.11}$$

Taking into account that in the framework of the material continuum model which is used here the particle has no rotational degrees of freedom and that the rigid motion does not cause damage growth, one can assume $\mathbf{R}_\Psi = \overset{*}{\mathbf{R}}$. Thus the constitutive equation defines the symmetric part of tensor $\boldsymbol{\Psi}$ and in accordance with the determinism and objectivity aksioms takes the general form:

$$\overset{\circ}{\boldsymbol{\Omega}} = \mathcal{G}\{\overset{*}{\mathbf{U}}(\mathbf{X}, t), \ldots \boldsymbol{\Omega}, \overset{*}{\mathbf{k}}(\mathbf{X}), \overset{*}{\mathbf{x}}(\mathbf{X}), t\}, \tag{4.12}$$

where: $\overset{*}{\mathbf{U}}$, $\overset{*}{\mathbf{F}} = \overset{*}{\mathbf{R}}\overset{*}{\mathbf{U}}$,

$\overset{\circ}{\boldsymbol{\Omega}}$ — Zaremba-Jaumann derivative.

The domain of this equation is the undeformed and undamaged configuration B_R. It was emphasised in form of dependence of the initial anisotropy. But even an initialy isotropic and homogenous material becomes anisotropic due to damage. This last anisotropy can be called a secondary one. The proposed form of the constitutive equation describes both the initial and secondary anisotropy as well. One can easily see it if the constitutive equation is written in the form:

$$\mathring{\mathbf{\Omega}} = \mathcal{G}\{\overset{*}{U}_k, \overset{*}{\mathbf{N}}_k \dots \mathbf{\Omega}, \overset{*}{k}(\mathbf{X}), \overset{*}{\mathbf{x}}(\mathbf{X}), t\},$$

where $\overset{*}{\mathbf{N}}$ — principal vectors of the strain tensor,

$$\overset{*}{\mathbf{U}}\overset{*}{\mathbf{N}}_k = \overset{*}{U}_k\overset{*}{\mathbf{N}}_k.$$

The simplest illustration of such a constitutive equation can be obtained as the result of the assumption that damage effect is an uniform, volumetric deterioration:

$$\mathbf{\Omega} = \mathbf{\Psi} = \mathbf{1}\frac{\varkappa}{3}\,\mathrm{tr}\overset{*}{\mathbf{U}}.$$

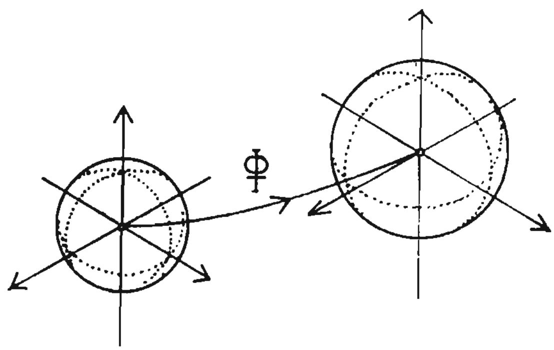

Fig. 6.

Taking into consideration that $\Delta d\mathbf{a} = \mathbf{\Psi} d\overset{*}{\mathbf{a}}$ or $\Delta da_i = \Psi_i^A d\overset{*}{a}_A$ one can obtain:

$$\Delta d\mathbf{a} = \frac{\varkappa}{3}\,\mathrm{tr}\overset{*}{\mathbf{U}} d\overset{*}{\mathbf{a}},$$

and the comparison with the transformation rule for the model according to Sidoroff's paper gives:

$$\frac{\omega}{1-\omega} = \frac{\varkappa}{3}\,\mathrm{tr}\overset{*}{\mathbf{U}} \quad \text{or} \quad \omega = \frac{\overset{*}{U}_k}{\frac{3}{\varkappa} + \overset{*}{U}_k},$$

and after linearization:

$$\omega = \frac{\overset{*}{\varepsilon}}{\frac{3}{\varkappa} + \overset{*}{\varepsilon}}, \tag{4.13}$$

where $\overset{*}{\varepsilon}$ — volumetric strain.

Formula (4.13) is the constitutive equation which has been implied by a particular phenomenological postulate. According to this postulate the damage growth leads to the volumetric deformation (independently of the fictitious body deformation). It would be physically reasonable to assume that only positive deformation causes damage growth. So, the constitutive law reads:

$$\mathbf{\Omega} = \begin{cases} \mathbf{1}\dfrac{\varkappa}{3}\,\mathrm{tr}\overset{*}{\mathbf{U}} & \forall\overset{*}{\mathbf{U}};\quad \mathrm{tr}\overset{*}{\mathbf{U}} > 0 \\ \mathbf{0} & \forall\overset{*}{\mathbf{U}}:\quad \mathrm{tr}\overset{*}{\mathbf{U}} < 0 \end{cases}.$$

It can be called an anisonomic equation of the evolution and written in form:

$$\mathbf{\Omega} = \mathbf{1}\frac{\varkappa}{3}H(\mathrm{tr}\overset{*}{\mathbf{U}})\,\mathrm{tr}\overset{*}{\mathbf{U}},$$

where $H(\mathrm{tr}\overset{*}{\mathbf{U}})$ — Heaviside's function.

The evolution law for the damage parameter in terms of linear theory takes the form:

$$\omega = \begin{cases} \dfrac{\overset{*}{\varepsilon}}{\dfrac{3}{\varkappa} + \overset{*}{\varepsilon}}; & \forall\overset{*}{\varepsilon} > 0 \\ 0 \quad ; & \forall\overset{*}{\varepsilon} < 0 \end{cases}.$$

Another phenomenological assumption here was that the micro-damage develops only on the planes which are perpendicular to the principal tension directions. In this case constitutive equation takes the form:

$$\mathbf{\Omega} = \mathbf{N}_\alpha \varkappa_\alpha \overset{*}{U}_\alpha H(\overset{*}{U}_\alpha).$$

where: $\overset{*}{\mathbf{N}}_\alpha$; $\overset{*}{\mathbf{U}}\overset{*}{\mathbf{N}}_\alpha = \overset{*}{U}_\alpha \overset{*}{\mathbf{N}}_\alpha$.

It is an anisotropic and anisonomic constitutive equation of the damage growth.

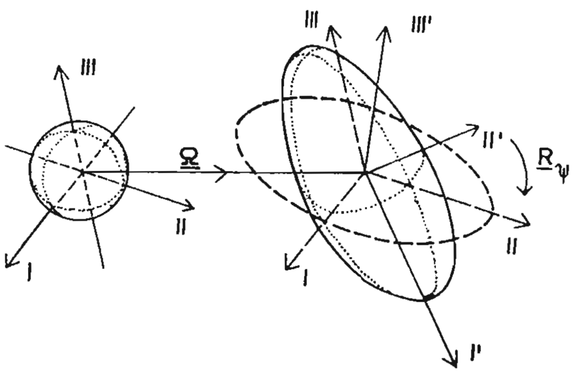

Fig. 7.

Other examples of the constitutive equation of the damage can be proposed in various forms but it was not the main aim of this paper. The simplest which were shown should illustrate the most essential properties ot the proposed description.

The kinematic description of the body behaviour in the presence of the damage is the same for all possible phenomenological assumptions. **The phenomenological postulates are expressed in terms of the constitutive equation of the damage evolution.**

For further illustration of the proposed description the concepts of the fictitious body quoted previously will be used again. In the first of them — in accordance with Sidoroff's paper — the mapping of the fictitious body on the damaged configuration has form (3.10). In view of this relation one can find the transformation rule for strain in form (2.5). Of course the strain measure $\boldsymbol{\epsilon}$ and $\overset{*}{\boldsymbol{\epsilon}}$ should be understood as linearized. Using the definition of the eulerian measures as their counterparts:

$$\boldsymbol{\epsilon} = 1/2(\mathbf{g}-\mathbf{c}^{-1}), \quad \mathbf{c}^{-1} = \mathbf{F}^{-1}(\mathbf{F}')^{\mathsf{T}},$$
$$\overset{*}{\boldsymbol{\epsilon}} = 1/2(\mathbf{g}-\overset{*}{\mathbf{c}}{}^{-1}), \quad \overset{*}{\mathbf{c}}{}^{-1} = \overset{*}{\mathbf{F}}{}^{-1}(\mathbf{F}^{-1})^{\mathsf{T}}, \tag{4.14}$$

and in view of (4.8_1)

$$\mathbf{c}^{-1} = \boldsymbol{\Phi}(\overset{*}{\mathbf{c}}{}^{-1})\boldsymbol{\Phi}^{\mathsf{T}},$$

it is easy to derive the relation:

$$\boldsymbol{\epsilon} = \boldsymbol{\Phi}\overset{*}{\boldsymbol{\epsilon}}\boldsymbol{\Phi}^{\mathsf{T}}+1/2(1-\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathsf{T}}).$$

Introducing now definition:

$$\boldsymbol{\epsilon}_{\varphi} \overset{\mathrm{df}}{=} (\mathbf{1}-\mathbf{C}_{\varphi}^{-1}), \quad \mathbf{C}_{\varphi}^{-1} = \boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathsf{T}},$$

one can finally write:

$$\boldsymbol{\epsilon} = \boldsymbol{\Phi}\overset{*}{\boldsymbol{\epsilon}}\boldsymbol{\Phi}^{\mathsf{T}}+\boldsymbol{\epsilon}_{\varphi}. \tag{4.16}$$

This last form is identical with formula (2.15) which was obtained previously in a different, purely formal, way.

Substituting formula (3.11) for the gradient $\boldsymbol{\Phi}$, it is easy to find:

$$\boldsymbol{\epsilon} = (1-\omega)^{-1}\overset{*}{\boldsymbol{\epsilon}}-\frac{\omega}{2(1-\omega)}\mathbf{1}. \tag{4.17}$$

For the sake of conformability with formula (2.5) one should assume $\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathsf{T}} = 1$ which leads to the equivalence $\boldsymbol{\epsilon}_{\varphi} = 0$. It means that damaged and undamaged configurations are undistinguishable which is typical of linearized kinematics.

In case of the concept according to Chaboche's paper the mapping of the fictitious body on the real one should be taken in form (3.13) and gradient of damage respectively:

$$\Phi_{11} = \Phi_{22} = (1-\omega)^{-1/2}, \quad \Phi_{33} = 1.$$

Since in view of (4.16) one can obtain:

$$\varepsilon_{11} = \overset{*}{\varepsilon}_{11}(1-\omega)^{-1}-\frac{\omega}{2(1-\omega)},$$
$$\varepsilon_{22} = \overset{*}{\varepsilon}_{22}(1-\omega)^{-1}-\frac{\omega}{2(1-\omega)},$$
$$\varepsilon_{33} = \overset{*}{\varepsilon}_{33},$$

and formulae are exactly as (2.15) obtained previously.

In the last considered concept, according to Murakami's paper, the usual euclidean strain measure can be used because the authors declare their theory as non-linear:

$$2\mathbf{e} = \mathbf{g}-\mathbf{c}, \quad 2\overset{*}{\mathbf{e}} = \mathbf{g}-\overset{*}{\mathbf{c}}.$$

After substitution to (4.8_1) one can find:

$$\mathbf{e} = (\mathbf{\Phi}^{-1})^{\mathrm{T}}\overset{*}{\mathbf{e}}\mathbf{\Phi}^{-1}+\hat{\mathbf{e}}_{\varphi},$$

where: $\hat{\mathbf{e}}_{\varphi} \overset{\mathrm{df}}{=} 1/2(\mathbf{1}-(\mathbf{\Phi}^{-1})^{\mathrm{T}}\mathbf{\Phi}^{-1})$.
In view of (3.16) one can derive:

$$e_i = (1-\Omega_i)^2\overset{*}{e}_i+\frac{\Omega_i}{2}(2-\Omega_i), \tag{4.18}$$

(under assumption $\det\mathbf{\Phi} = 1$ which means that the deformation due to damage are "small") The derived transformation formula allows one to describe the state of the body in the presence of damage as the result of "the fictitious creep".

Similarly, one can find transformation formula for the stress, respectively. It allows one to describe the damage process as "the fictitious relaxation". As mentioned before, the tensor of effective stress is Piola-Kirchhoff's pseudostress tensor but defined over the deformed and undamaged configuration. So, in accordance with definition:

$$\overset{*}{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(\mathbf{\Phi}^{-1})^{\mathrm{T}}\det\mathbf{\Phi},$$

for the first of considered concepts (according to Sidoroff) one can obtain by substituting (3.11):

$$\overset{*}{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(1-\omega)^{1/2}(1-\omega)^{-3/2} = \boldsymbol{\sigma}(1-\omega)^{-1}. \ ^{*)}$$

This relation is identical to (2.3). In case of the second concept (i.e. according to Chaboche) the following relations can be derived in the same way:

$$\overset{*}{\sigma}_3 = \sigma(1-\omega)^{-1},$$

$$\overset{*}{\sigma}_2 = \overset{*}{\sigma}_1 = 0,$$

which agree exactly with formula (2.9).
Finally for Murakami's concept the effective stress tensor takes the form:

$$\overset{*}{\boldsymbol{\sigma}} = \{\delta_{ij}\sigma_{ij}(1-\Omega_i)^{-1}\},$$

which is the same as in paper [14]. Of course there is no reason for such arbitrary symmetryzation as was made in the quoted paper. For each of the considered concepts the damage tensor $\mathbf{\Psi}$ can be defined in accordance with (4.8). In case of the first concept the damage tensor takes the form:

$$\mathbf{\Psi} = \mathbf{1}\frac{\omega}{1-\omega}. \tag{4.19}$$

*) here $\det\Phi = 1$ was assumed because in (18) the linear approximation of is considered.

In case of the second, the same damage measure reads:

$$\boldsymbol{\Psi} = \begin{bmatrix} \frac{\omega(2-\omega)}{(1-\omega)^{1/2}} & 0 & 0 \\ 0 & \frac{\omega(2-\omega)}{(1-\omega)^{1/2}} & 0 \\ 0 & 0 & \frac{\omega}{1-\omega} \end{bmatrix}. \tag{4.20}$$

Finally in case of Murakami's concept the damage tensor can be written:

$$\boldsymbol{\Psi} = \{\delta_{ij}\Omega_i\}, \tag{4.21}$$

and one can see that it is an identical tensor as the one defined in paper [14].

The above discussion shows that the quoted concepts of the fictitious body can be derived as the particular case of the proposed description of the damage growth process. In other words, these concepts represent simply interpretation of the proposed kinematical damage measure in the framework of the non-linear kinematics. Such an approach allows one to explain the role of simplifications which was adopted and the result of these assumptions. It is discussed more detail further. The proposed description of the damage process allows one to interpret correctly the effective stress tensor and its physical meaning as well. Such correcteness of the stress measure definition guarantees the fictitious body concept to be statically admissible. In other words, the motions of both considered bodies are due to the equivalent external forces systems.

The attention, however, should be paid to the kinematical admissibility problem. The deformation of the real body should of course be kinematicaly admissible and it means the deformation tensor should stand for the metric tensor of the euclidean space. It implies that the Riemann-Christoffel's tensor defined over the tensor **c** has to be equal zero:

$$\mathscr{R}(\mathbf{c}) = \mathbf{R}^{(c)} = \mathbf{0}, \tag{4.22}$$

where: $\mathscr{R} \overset{\mathrm{df}}{=} \partial_j \Gamma^l_{ik} - \partial_k \Gamma^l_{ij} + \Gamma^m_{ik}\Gamma^l_{mj} - \Gamma^m_{ij}\Gamma^l_{mk}$.

Hence follows the condition:

$$\mathscr{R}[(\boldsymbol{\Phi}^{-1})^{\mathrm{T}}]\overset{*}{\mathbf{c}}\boldsymbol{\Phi}^{-1} + (\boldsymbol{\Phi}^{-1})^{\mathrm{T}}\mathscr{R}[\overset{*}{\mathbf{c}}]\boldsymbol{\Phi}^{-1} + (\boldsymbol{\Phi})^{\mathrm{T}}\overset{*}{\mathbf{c}}\mathscr{R}[\Phi^{-1}] = \mathbf{0}. \tag{4.23}$$

Assuming the deformation of the fictitious body to be kinematically admissible, too, condition (4.23) takes the form of the restriction put on the damage constitutive equation:

$$\mathscr{R}[(\boldsymbol{\Phi}^{-1})^{\mathrm{T}}]\overset{*}{\mathbf{c}}\boldsymbol{\Phi}^{-1} + (\boldsymbol{\Phi}^{-1})^{\mathrm{T}}\overset{*}{\mathbf{c}}\mathscr{R}[\boldsymbol{\Phi}^{-1}] = \mathbf{0}. \tag{4.24}$$

The fictitious body motion has no physical sense, however, and it is reasonable to assume that deformation tensor $\overset{*}{\mathbf{c}}$ does not have to be a metric tensor of the euclidean space. The geometrical sense of such an assumption is illustrated in figure 8.

Such an assumption, however, makes the equations:

$$2\mathbf{e} = \nabla\overset{*}{\mathbf{u}}^{\mathrm{T}} + \nabla\overset{*}{\mathbf{u}} - \nabla\overset{*}{\mathbf{u}}^{\mathrm{T}}\nabla\overset{*}{\mathbf{u}},$$

appear unsolvable.

This problem can be overcome by an adequate selection of the reference configuration. Firstly it can be taken as the eigen-stressed configuration secondly it can be chosen as the

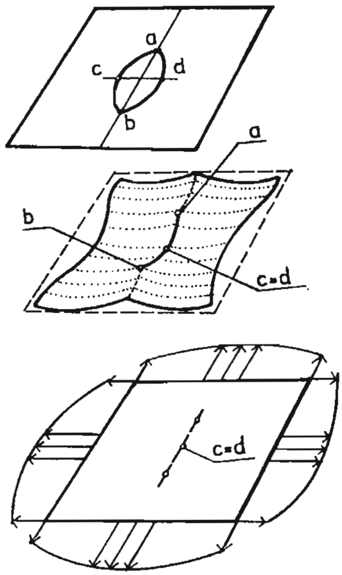

Fig. 8.

system of locally tangent reference configurations. In the latter case one can determine the motion of a particle neighbourhood small enough only [16].
Nevertheless, this possibility seems to be more convenient because the eigen-stressed configuration is not materially isomorphic and therefore constitutive equation should be modified appropriately.
Using the eigen-stressed configuration or a system of locally tangent configurations makes the damage constitutive equation free of any restrictions. The advantage of decoupling of the system of equations, however, can be essentially reduced because the searching of the eigen-stress tensor can appear a very difficault task.

5. Conclusions

The continual theories which utilize the fictitious body concept are based on the following assumption:
— to the body particles one can associate the regular function which represents the damage measure
— the damage growth considered in the framework of an isolated mechanical system can be treated as a strain transformation which is equivalent to the body structure deterioration.
These assumptions express the phenomenological sense of the proposed continuous theories. The concept of the fictitious body allows one to decouple the system of equations which governs the deformation process. The decoupling, however, can appear semingly because the condition of the kinematical admissibility of the real body deformation requires

using eigen-stressed or local tangent reference configuration for the fictitious body. It can be very inconvenient. The proposed description of the body behaviour in the presence of damage allows one to explain such problems as lack of symmetry of effective stress tensor and carry out linear aproximation in a fully consistent way.
The proposed kinematic damage measure has the obvious physical interpretation and allows one to describe the anisotropy of the damage process in a natural way.
As has been shown — at least some of the existing theories can be obtained as a particular case of the proposed general description.

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Р е з ю м е

КОНТИНУАЛЬНЫЕ ОПИСАНИЯ ПОВРЕЖДЕНИЙ

В настоящей работе излагается применение контуального описания в процессе развития повреждений. Кроме того рассматриваются общие качества теорий основанных на концепции „фиктивного тела".

S t r e s z c z e n i e

KONTYNUALNE OPISY ZNISZCZENIA

W pracy omówiono propozycje uogólnionej, kinematycznej miary uszkodzenia. Przedyskutowano inne znane propozycje kontynualnych opisów wykazując, że mogą być otrzymane jako przypadek szczególny proponowanego uogólnienia.

Wykazano, że równania konstytutywne odpowiedzi strukturalnej i odpowiedzi dynamicznej nie mogą być — poza szczególnie prostymi przypadkami — rozprzężone.

Praca wplynęla do Redakcji dnia 28 grudnia 1987 roku.