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ON THE MICROLOCAL MODELLING OF TORSION OF RODS WITH ε-PERIODIC VARIABLE CROSS-SECTIONS

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Introduction

The aim of the paper is to propose a certain method of homogenization of constrained torsion problems for straight linear-elastic rods with periodic variable compact cross-sections. The method is based on the concepts of nonstandard analysis [4], taking into account a nonstandard homogenization approach outlined in [5, 6] as well as the notion of internal constraints, [1, 2]. The derived homogenized model of the rods under consideration can be a basis for an analysis of many special engineering problems.

1. Basic assumptions and internal constraints

We consider a straight linear-elastic rod with a variable compact bisymmetric cross-section. In the undeformed configuration the rod occupies a regular region Ω in the 3-space parametrized by the orthogonal Carthesian coordinates X_1, X_2, X_3. We assume that X_3 coincides with the rod axis and X_1, X_2 are paralell to the principal central inertia axes of an arbitrary cross-section $F(X_3)$, $X_3 \in [0, 1]$, cf. Fig. 1. We also assume that $F(X_3) = F(X_3+\varepsilon)$, $X_3 \in [0, 1-\varepsilon]$, i.e., that the rod has ε-periodic structure, with $\varepsilon \ll 1$.

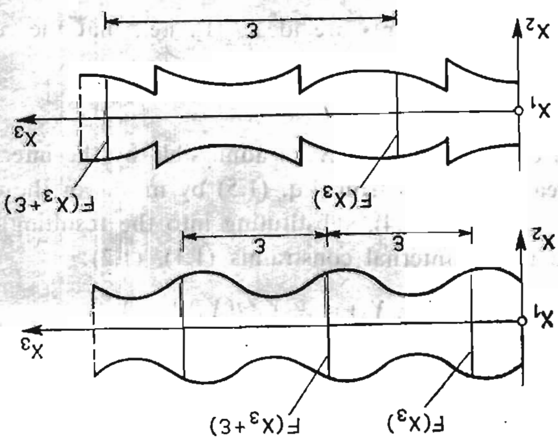

Fig. 1

We shall confine ourselves to the rod deformations $\chi_k = \chi_k(X, t)$, $X = (X_1, X_2, X_3) \in \in \Omega$, $t \in [0, t_f]$, t being the time coordinate, admissible by the internal constraints of the form*) $\chi^m_{,\alpha}\chi_{m,\beta} = \delta_{\alpha\beta}$, [2]. It means that projections of cross-sections of the deformed rod on the plane OX_1X_2 behave as rigid. Introducing the displacement vector field $\boldsymbol{u}(X, t) = = \chi(X, t) - X$, $X \in \Omega$, $t \in [0, t_f]$, after the linearization of constraints with respect to $\boldsymbol{u}(X, t)$, we arrive at the following explicit form of the internal constraints:

$$
\begin{aligned}
u_1 &= -\Theta(X_3, t)X_2 + \psi(X_3, t),\\
u_2 &= \Theta(X_3, t)X_1 + \varphi(X_3, t),\\
u_3 &= u_3(X_1, X_2, X_3, t),
\end{aligned} \tag{1.1}
$$

where $\Theta(\cdot)$, $\psi(\cdot)$, $\varphi(\cdot)$ are arbitrary differentiable functions.

Moreover, we introduce the extra constraints in the explicit form:

$$u_3(X_1, X_2, X_3, t) = \Phi(X_1, X_2)\zeta(X_3, t) + \eta(X_3, t), \tag{1.2}$$

where $\Phi(\cdot)$ is a certain a priori postulated function, depending on the shape of rod cross-sections, and $\zeta(\cdot)$, $\eta(\cdot)$ are arbitrary differentiable functions.
Functions $\Theta(\cdot)$, $\psi(\cdot)$, $\varphi(\cdot)$, $\zeta(\cdot)$, $\eta(\cdot)$, called the generalized coordinates [1], are assumed to be independent and defined on $[0, l] \times [0, t_f]$.

The motion of the constrained body is governed by the equation of motion, [1]:

$$T^{ij}_{,j} + \varrho b_i + \varrho r_i = \varrho \ddot{\chi}_i, \quad X \in \Omega, t \in [0, t_f], \tag{1.3}$$

where $\boldsymbol{T} = \boldsymbol{T}(X, t)$ is the Piola-Kirchhoff tensor of stress produced by the material reaction, $\varrho = \varrho(X)$ is the mass density in the reference configuration, $\boldsymbol{b} = \boldsymbol{b}(X, t)$ is the density of external loads and $\boldsymbol{r} = \boldsymbol{r}(X, t)$ denotes the density of unknown reaction body forces due to the internal constraints.

At the boundary $\partial\Omega$ of the rod the following conditions hold, [1]:

$$T^{ij}n_j = p_i + s_i, \text{ for almost every } X \in \partial\Omega,\ t \in [0, t_f], \tag{1.4}$$

where $\boldsymbol{n} = \boldsymbol{n}(X)$ is a unit autward normal to $\partial\Omega$, $\boldsymbol{p} = \boldsymbol{p}(X, t)$ are the known surface tractions and $\boldsymbol{s} = \boldsymbol{s}(X, t)$ stands for unknown surface reaction forces also due to the internal constraints.

We postulate that the constraints are ideal, [1], i.e., that the condition:

$$\int_{\Omega} \varrho \boldsymbol{r} \cdot \delta\chi d\Omega + \int_{\partial\Omega} \boldsymbol{s} \cdot \delta\chi d(\partial\Omega) = 0, \tag{1.5}$$

holds for any virtual displacements $\delta\chi(X, t)$ admissible by the internal constrains.

Eliminating the reaction forces from Eq. (1.5) by means of the equations of motion (1.3) and boundary conditions (1.4), substituting into the resulting relations the virtual displacements related to the internal constraints (1.1), (1.2):

$$
\begin{aligned}
\delta\chi_1(X_1, X_2, X_3) &= -X_2\tilde{\Theta}(X_3) + \tilde{\psi}(X_3),\\
\delta\chi_2(X_1, X_2, X_3) &= X_1\tilde{\Theta}(X_3) + \tilde{\varphi}(X_3),\\
\delta\chi_3(X_1, X_2, X_3) &= \Phi(X_1, X_2)\tilde{\zeta}(X_2) + \tilde{\eta}(X_3),
\end{aligned} \tag{1.6}
$$

*) The Latin indices take the values 1, 2, 3; the Greek ones take the values 1, 2. Summation convention holds for all kinds of indices.

where $\tilde{\psi}(\cdot)$, $\tilde{\varphi}(\cdot)$, $\tilde{\Theta}(\cdot)$, $\tilde{\zeta}(\cdot)$, $\tilde{\eta}(\cdot)$, are arbitrary independent sufficiently regular functions, we arrive at the following system of variational equations:

$$\int_{\Omega} T^{13}\tilde{\psi}d\Omega + \int_{\Omega}[-\varrho b_1 + \varrho(-X_2\ddot{\Theta}+\ddot{\psi})]\tilde{\psi}d\Omega - \int_{\partial\Omega} p_1\tilde{\psi}d(\partial\Omega) = 0,$$

$$\int_{\Omega} T^{23}\tilde{\varphi}d\Omega + \int_{\Omega}[-\varrho b_2 + \varrho(X_1\ddot{\Theta}+\ddot{\varphi})]\tilde{\varphi}d\Omega - \int_{\partial\Omega} p_2\tilde{\varphi}d(\partial\Omega) = 0,$$

$$\int_{\Omega}(T^{23}X_1 - T^{13}X_2)\tilde{\Theta}_{,3}d\Omega + \int_{\Omega}[(\varrho b_1 X_2 - \varrho b_2 X_1) - \varrho(-X_2\ddot{\Theta}+\ddot{\psi})X_2 +$$

$$+\varrho(X_1\ddot{\Theta}+\ddot{\varphi})X_1]\tilde{\Theta}d\Omega - \int_{\partial\Omega}(p_2X_1 - p_1X_2)\tilde{\Theta}d(\partial\Omega) = 0, \tag{1.7}$$

$$\int_{\Omega}[(T^{31}\Phi_{,1} + T^{32}\Phi_{,2})\tilde{\zeta} + T^{33}\Phi\tilde{\zeta}_{,3}]d\Omega + \int_{\Omega}(-\varrho b_3\Phi + \varrho\Phi^2\ddot{\zeta} +$$

$$+\varrho\Phi\ddot{\eta})\tilde{\zeta}d\Omega - \int_{\partial\Omega} p_3\Phi\tilde{\zeta}d(\partial\Omega) = 0,$$

$$\int_{\Omega} T^{33}\tilde{\eta}_{,3}d\Omega + \int_{\Omega}(-\varrho b_3 + \varrho\Phi\ddot{\zeta} + \varrho\ddot{\eta})\tilde{\eta}d\Omega - \int_{\partial\Omega} p_3\tilde{\eta}d(\partial\Omega) = 0,$$

which has to hold for any $\tilde{\Theta}(\cdot)$, $\tilde{\psi}(\cdot)$, $\tilde{\varphi}(\cdot)$, $\tilde{\zeta}(\cdot)$, $\tilde{\eta}(\cdot)$.

For homogeneous isotropic materials the well known stress-strain relations yield:

$$\begin{aligned} T^{13} &= \mu(-X_2\Theta_{,3} + \psi_{,3} + \zeta\Phi_{;1}), \\ T^{23} &= \mu(X_1\Theta_{,3} + \varphi_{3,} + \zeta\Phi_{,2}), \\ T^{33} &= (\lambda+2\mu)(\Phi\zeta_{,3} + \eta_{,3}), \end{aligned} \tag{1.8}$$

where μ and λ are Lamé modulae.

Denoting:

$$\begin{aligned} S_1 &= S_1(X_3) \equiv \int_{F(X_3)} X_2 dF, \\ S_2 &= S_2(X_3) \equiv \int_{F(X_3)} X_1 dF, \\ I_0 &= I_0(X_3) \equiv \int_{F(X_3)} (X_1^2 + X_2^2)dF, \\ I_s &= I_s(X_3) \equiv \int_{F(X_3)} (X_1\Phi_{,2} - X_2\Phi_{,1})dF, \\ I &= I(X_3) \equiv \int_{F(X_3)} \Phi^2 dF, \\ I_k &= I_k(X_3) \equiv \int_{F(X_3)} [(\Phi_{,1})^2 + (\Phi_{,2})^2]dF, \\ K_1 &= K_1(X_3) \equiv \int_{F(X_3)} \Phi_{,2}dF, \end{aligned} \tag{1.9}$$

$$K_2 = K_2(X_3) \equiv \int_{F(X_3)} \Phi_{,1} dF,$$

$$S_\Phi = S_\Phi(X_3) \equiv \int_{F(X_3)} \Phi dF,$$

and

$$\begin{aligned}
\mathring{p}_k &\equiv \int_{\partial F(X_3)} \sqrt{g(\gamma, X_3)}\, p_k d(\partial F) && \text{for } X_3 \in (0, 1),\\
P_k &\equiv \int_F p_k dF && \text{for } X_3 = 0 \text{ and } X_3 = 1,\\
m_s &\equiv \int_{\partial F(X_3)} \sqrt{g(\gamma, X_3)}\, (p_2 X_1 - p_1 X_2) d(\partial F) && \text{for } X_3 \in (0, 1),\\
M_s &\equiv \int_F (p_2 X_1 - p_1 X_2) dF && \text{for } X_3 = 0 \text{ and } X_3 = 1,\\
m_\Phi &\equiv \int_{\partial F(X_3)} \sqrt{g(\gamma, X_3)}\, p_3 \Phi d(\partial F) && \text{for } X_3 \in (0, 1),\\
M_\Phi &\equiv \int_F p_3 \Phi dF && \text{for } X_3 = 0 \text{ and } X_3 = 1,
\end{aligned} \tag{1.10}$$

where γ is the parameter of the curve $\partial F(X_3)$ and $g(\gamma, X_3)$ is the discriminant of the first quadric form of the lateral surface of the rod, and substituting the RHS of Eq. (1.8) into (1.7) under the extra assumption $b_i(X_1, X_2, X_3, t) = \text{const}$, after simple calculations we arrive at the system of the five variational equations for the unknown generalized coordinates $\Theta(X_3, t)$, $\psi(X_3, t)$, $\varphi(X_3, t)$, $\zeta(X_3, t)$, $\eta(X_3, t)$, $X_3 \in [0, 1]$, $t \in [0, t_f]$:

$$\begin{aligned}
&\mu \int_0^1 (I_0 \Theta_{,3} + S_2 \varphi_{,3} - S_1 \psi_{,3} + I_s \zeta) \tilde{\Theta}_{,3} dX_3 + \varrho b_1 \int_0^1 S_1 \tilde{\Theta} dX_3 +\\
&\quad - \varrho b_2 \int_0^1 S_2 \tilde{\Theta} dX_3 + \varrho \int_0^1 I_0 \ddot{\Theta} \tilde{\Theta} dX_3 - \varrho \int_0^1 S_1 \ddot{\psi} \tilde{\Theta} dX_3 + \varrho \int_0^1 S_2 \ddot{\varphi} \tilde{\Theta} dX_3 +\\
&\quad - \int_0^1 m_s \tilde{\Theta} dX_3 - M_s(0, t) \tilde{\Theta}(0) - M_s(1, t) \tilde{\Theta}(1) = 0,\\
&\mu \int_0^1 (-S_1 \Theta_{,3} + F \psi_{,3} + K_2 \zeta) \tilde{\psi}_{,3} dX_3 - \varrho b_1 \int_0^1 F \tilde{\psi} dX_3 + \varrho \int_0^1 (-S_1 \ddot{\Theta} + F \ddot{\psi}) \tilde{\psi} dX_3 +\\
&\quad - \int_0^1 \mathring{p}_1 \tilde{\psi} dX_3 - P_1(0, t) \tilde{\psi}(0) - P_1(1, t) \tilde{\psi}(1) = 0,\\
&\mu \int_0^1 (S_2 \Theta_{,3} + F \varphi_{,3} + K_1 \zeta) \tilde{\varphi}_{,3} dX_3 - \varrho b_2 \int_0^1 F \tilde{\varphi} dX_3 + \varrho \int_0^1 (S_2 \ddot{\Theta} + F \ddot{\varphi})\\
&\quad \tilde{\varphi} dX_3 - \int_0^1 \mathring{p}_2 \tilde{\varphi} dX_3 - P_2(0, t) \tilde{\varphi}(0) - P_2(1, t) \tilde{\varphi}(1) = 0,
\end{aligned} \tag{1.11}$$

$$\int_0^1 [\mu(I_s\Theta_{,3}+K_2\psi_{,3}+K_1\varphi_{,3}+I_k\zeta)\tilde{\xi}+$$

$$+(\lambda+2\mu)(I\zeta_{,3}+S_\Phi\eta_{,3})\tilde{\zeta}_{,3}]dX_3-\varrho b_3\int_0^1 S_\Phi\tilde{\zeta}dX_3+\varrho\int_0^1 I\ddot{\zeta}\tilde{\zeta}dX_3+$$

$$+\varrho\int_0^1 S_\Phi\ddot{\eta}\tilde{\zeta}dX_3-\int_0^1 m_\Phi\tilde{\zeta}dX_3-M_\Phi(0,t)\tilde{\zeta}(0)-M_\Phi(1,t)\tilde{\zeta}(1)=0,$$

$$(\lambda+2\mu)\int_0^1 (S_\Phi\zeta_{,3}+F\eta_{,3})\tilde{\eta}_{,3}dX_3-\varrho b_3\int_0^1 F\tilde{\eta}dX_3+\varrho\int_0^1 S_\Phi\ddot{\zeta}\tilde{\eta}dX_3+$$

$$+\varrho\int_0^1 F\ddot{\eta}\tilde{\eta}dX_3-\int_0^1 \mathring{p}_3\tilde{\eta}dX_3-P_3(0,t)\tilde{\eta}(0)-P_3(1,t)\tilde{\eta}(1)=0,$$

which have to hold for arbitrary $\tilde{\Theta}(\cdot), \tilde{\psi}(\cdot), \tilde{\varphi}(\cdot), \tilde{\zeta}(\cdot), \tilde{\eta}(\cdot)$.

The variational system (1.11) leads, after fulfilling the by parts integrations and applying the divergence theorem as well as the du Bois lemma, to the system differential equations for the generalized coordinates $\Theta(\cdot,t)$, $\psi(\cdot,t)$, $\varphi(\cdot,t)$ $\zeta(\cdot,t)$, $\eta(\cdot,t)$, $t\in[0,t_f]$. However, for the rods under consideration (with the periodically variable cross-sections), the resulting system of differential equations has the variable ε-periodic coefficients. For small values of ε, as related to the rod length l, the obtaining of solutions to such systems, even using numerical calculations, is rather complicated. That is why we are going to approximate this system of differential equations by a certain system of differential equations with the constant coefficients. The procedure applied below will be based on the nonstandard homogenization ideas developed in [3, 5, 6] and is referred to as the microlocal modelling approach.

2. Microlocal modelling

We shall use the method of microlocal modelling based on the concepts of the nonstandard analysis [4], the general formulation of which was outlined in [5, 6].

In the general case micro-effects can be due to the existence of a certain small length parameter ε which characterizes the microstructure of the body. In the case under consideration the parameter ε describes the periodic oscillations of the rod cross-sections. We tacitly assume that ε is a small parameter, $\varepsilon \ll 1$, and hence we deal with certain "micro"-effects due to the variability of the cross-section.

The method of microlocal modelling, applied below, is based on two assumptions, [3]. Firstly, we have to introduce so called homogenization hypothezis, which states here that the problem $P_{(\varepsilon)}$ described by the system (1.11) for a sufficiently small ε, $\varepsilon \ll 1$, can be approximated by the pertinent problem $P_{(\varepsilon/n)}$ in which ε is replaced by ε/n, for $n=2,3,4,\ldots$. Hence, using the known theorems of the nonstandard analysis, [4], the problem $P_{(\varepsilon)}$ can be approximated by the problem $P_{(\varepsilon/\check{\omega})}$, where $\check{\omega}$ is a certain infinite natural number. Obviously, the problem $P_{(\varepsilon/\check{\omega})}$ can be properly formulated only within the nonstandard analysis structure, where the infinite as well as infinitely small positive numbers are well defined. As it is known, [4], to every known mathematical entity Ψ corresponds in the nonstandard analysis so called standard entity, denoted by $^*\Psi$. Now we can formulate the second basic assumption of the microlocal modelling, which is referred to as the m i c r o l o c a l a p p r o x i m a t i o n a s s u m p t i o n. To formulate this assumption let $\Theta^{\check{\omega}}(X_3, t)$, $\psi^{\check{\omega}}(X_3, t)$, $\varphi^{\check{\omega}}(X_3, t)$, $\zeta^{\check{\omega}}(X_3, t)$, $\eta^{\check{\omega}}(X_3, t)$, $X_3 \in {}^*[0, 1]$, $t \in {}^*[0, t_f]$, be the unknown solution of the (nonstandard) problem $P_{(\varepsilon/\check{\omega})}$. The microlocal approximation postulates that we look for the approximate solution to $P_{(\varepsilon/\check{\omega})}$ in the class of functions given by:

$$
\begin{aligned}
\Theta^{\check{\omega}}(X_3, t) &= {}^*\Theta_0(X_3, t) + {}^*\Theta_a(X_3, t)\overset{a}{h}_{\check{\omega}}(X_3),\\
\psi^{\check{\omega}}(X_3, t) &= {}^*\psi_0(X_3, t) + {}^*\psi_a(X_3, t)\overset{a}{h}_{\check{\omega}}(X_3),\\
\varphi^{\check{\omega}}(X_3, t) &= {}^*\varphi_0(X_3, t) + {}^*\varphi_a(X_3, t)\overset{a}{h}_{\check{\omega}}(X_3),\\
\zeta^{\check{\omega}}(X_3, t) &= {}^*\zeta_0(X_3, t) + {}^*\zeta_a(X_3, t)\overset{a}{h}_{\check{\omega}}(X_3),\\
\eta^{\check{\omega}}(X_3, t) &= {}^*\eta_0(X_3, t) + {}^*\eta_a(X_3, t)\overset{a}{h}_{\check{\omega}}(X_3),
\end{aligned}
\tag{2.1}
$$

where $a = 1, 2, \ldots, n$, (summation convention holds), $\overset{a}{h}_{\check{\omega}}(X_3) \equiv \frac{1}{\check{\omega}}\, {}^*h^a(\check{\omega}X_3)$, $h^a(\cdot)$ are postulated a priori ε-periodic regular functions, such that $\int_0^\varepsilon h^a_{,3}(X_3)dX_3 = 0$, and $\Theta_0(\cdot, t)$, $\Theta_a(\cdot, t)$, $\psi_0(\cdot, t)$, $\psi_a(\cdot, t)$, $\varphi_0(\cdot, t)$, $\varphi_a(\cdot, t)$, $\zeta_0(\cdot, t)$, $\zeta_a(\cdot, t)$, $\eta_0(\cdot, t)$, $\eta_a(\cdot, t)$ are sufficiently regular unknown functions.

The unknowns $\psi_0(X_3, t)$, $\varphi_0(X_3, t)$, $\eta_0(X_3, t)$, will be called macro- displacements in direction of X_1, X_2, X_3 — axes, respectively, $\Theta_0(X_3, t)$ are called macro-rotations, $\Phi(X_1, X_2)\zeta_0(X_3, t)$ now represents the non rigid out of plane deformations of the cross-sections. Functions $\Theta_0(\cdot)$, $\psi_0(\cdot)$, $\varphi_0(\cdot)$, $\zeta_0(\cdot)$, $\eta_0(\cdot)$ will be called generalized macro-deformations. Functions $\Theta_a(\cdot)$, $\psi_a(\cdot)$, $\varphi_a(\cdot)$, $\zeta_a(\cdot)$, $\eta_a(\cdot)$ describe the effects due to the micro-periodic structure of the rod are called the microlocal parameters.

Substituting the RHS of Eqs. (2.1) into conditions (1.11) and assuming that X_3 — axis passes through the mass centers of all cross-sections (hence $S_\alpha(X_3) = 0$) we obtain the following variational equations system for $\Theta_0(\cdot)$, $\Theta_a(\cdot)$, $\psi_0(\cdot)$, $\psi_a(\cdot)$, $\varphi_0(\cdot)$, $\varphi_a(\cdot)$, $\zeta_0(\cdot)$, $\zeta_a(\cdot)$, $\eta_0(\cdot)$, $\eta_a(\cdot)$,

$$
\begin{aligned}
&\mu \int_0^1 [I_0({}^*\Theta_{0,3} + \underline{{}^*\Theta_{a,3}\overset{a}{h}_{\check{\omega}}} + {}^*\Theta_a \overset{a}{h}_{\check{\omega},3}) + I_s({}^*\zeta_0 + \underline{{}^*\zeta_a \overset{a}{h}_{\check{\omega}}})\\
&({}^*\tilde{\Theta}_{0,3} + \underline{{}^*\tilde{\Theta}_{b,3}\overset{b}{h}_{\check{\omega}}} + {}^*\tilde{\Theta}_b \overset{b}{h}_{\check{\omega},3})dX_3 + \varrho \int_0^1 I_0({}^*\ddot{\Theta}_0 + \underline{{}^*\ddot{\Theta}_a \overset{a}{h}_{\check{\omega}}})\\
&({}^*\tilde{\Theta}_0 + \underline{{}^*\tilde{\Theta}_b \overset{b}{h}_{\check{\omega}}})dX_3 - \int_0^1 m_s({}^*\tilde{\Theta}_0 + \underline{{}^*\tilde{\Theta}_b \overset{b}{h}_{\check{\omega}}})dX_3 +\\
&\quad - M_s(0, t)[{}^*\tilde{\Theta}_0(0) + \underline{{}^*\tilde{\Theta}_b(0, t)\overset{b}{h}_{\check{\omega}}(0)}] - M_s(1, t)[{}^*\tilde{\Theta}_0(1) +\\
&\quad + \underline{{}^*\tilde{\Theta}_b(1)\overset{b}{h}_{\check{\omega}}(1)}] = 0,
\end{aligned}
$$

$$\mu \int_0^1 [F(^*\psi_{0,3} + \underline{^*\psi_{a,3} h_\omega^a} + {^*\psi_a} h_{\omega,3}^a) + K_2(^*\zeta_0 + \underline{^*\zeta_a h_\omega^a})] \tag{2.2}$$

$$(^*\tilde{\psi}_{0,3} + \underline{^*\tilde{\psi}_{b,3} h_\omega^b} + {^*\tilde{\psi}_b} h_{\omega,3}^b) dX_3 - \varrho b_1 \int_0^1 F(^*\tilde{\psi}_0 + \underline{^*\tilde{\psi}_b h_\omega^b}) dX_3 +$$

$$+ \varrho \int_0^1 F(^*\ddot{\psi}_0 + \underline{^*\ddot{\psi}_a h_\omega^a})(^*\tilde{\psi}_0 + \underline{^*\tilde{\psi}_b h_\omega^b}) dX_3 - \int_0^1 \mathring{p}_1 (^*\tilde{\psi}_0 + \underline{^*\tilde{\psi}_b h_\omega^b}) dX_3 +$$

$$- P_1(0, t)[^*\tilde{\psi}_0(0) + \underline{^*\tilde{\psi}_b(0) h_\omega^b(0)}] - P_1(1, t)[^*\tilde{\psi}_0(1) +$$

$$+ \underline{^*\tilde{\psi}_b(1) h_\omega^b(1)}] = 0,$$

$$\mu \int_0^1 [F(^*\varphi_{0,3} + \underline{^*\varphi_{a,3} h_\omega^a} + {^*\varphi_a} h_{\omega,3}^a) + K_1(^*\zeta_0 + \underline{^*\zeta_a h_\omega^a})]$$

$$(^*\varphi_{0,3} + \underline{^*\tilde{\varphi}_{b,3} h_\omega^b} + {^*\tilde{\varphi}_b} h_{\omega,3}^b) dX_3 - \varrho b_2 \int_0^1 F(^*\tilde{\varphi}_0 + \underline{^*\tilde{\varphi}_b h_\omega^b}) dX_3 +$$

$$+ \varrho \int_0^1 F(^*\ddot{\varphi}_0 + \underline{^*\ddot{\varphi}_a h_\omega^a})(^*\tilde{\varphi}_0 + \underline{\tilde{\varphi}_b h_\omega^b}) dX_3 +$$

$$- \int_0^1 \mathring{p}_2 (^*\varphi_0 + \underline{\tilde{\varphi}_b h_\omega^b}) dX_3 - P_2(0.\ t)[^*\tilde{\varphi}_0(0) + \underline{^*\tilde{\varphi}_b(0) h_\omega^b(0)}] +$$

$$- P_2(1, t)[^*\tilde{\varphi}_0(1) + \underline{^*\tilde{\varphi}_b(1) h_\omega^b(1)}] = 0,$$

$$\int_0^1 \{\mu [I_s(^*\Theta_{0,3} + \underline{^*\Theta_{a,3} h_\omega^a} + {^*\Theta_a} h_{\omega,3}^a) + I_k(^*\zeta_0 + \underline{^*\zeta_a h_\omega^a})]$$

$$(^*\tilde{\zeta}_0 + \underline{^*\tilde{\zeta}_b h_\omega^b}) + (\lambda + 2\mu)[I(^*\zeta_{0,3} + \underline{^*\zeta_{a,3} h_\omega^a} + \underline{^*\zeta_a h_{\omega,3}^a}) +$$

$$+ S_\Phi(^*\eta_{0,3} + \underline{^*\eta_{a,3} h_\omega^a} + {^*\eta_a} h_{\omega,3}^a)](^*\tilde{\zeta}_{0,3} + \underline{^*\tilde{\zeta}_{b,3} h_\omega^b} + {^*\tilde{\zeta}_b} h_{\omega,3}^b)\} dX_3 +$$

$$- \varrho b_3 \int_0^1 S_\Phi(^*\tilde{\zeta}_0 + \underline{^*\tilde{\zeta}_b h_\omega^b}) dX_3 + \varrho \int_0^1 I(^*\ddot{\zeta}_0 + \underline{^*\ddot{\zeta}_a h_\omega^a})(^*\tilde{\zeta}_0 + \underline{^*\tilde{\zeta}_b h_\omega^b}) dX_3 +$$

$$+ \varrho \int_0^1 S_\Phi(^*\ddot{\eta}_0 + \underline{^*\ddot{\eta}_a h_\omega})(^*\tilde{\zeta}_0 + \underline{^*\zeta_b h_\omega^b}) dX_3 - \int_0^1 m_\Phi(^*\tilde{\zeta}_0 + \underline{^*\tilde{\zeta}_b h_\omega^b}) dX_3 +$$

$$- M_\Phi(0, t)[^*\tilde{\zeta}_0(0) + \underline{^*\zeta_b(0) h_\omega^b(0)}] - M_\Phi(1, t)[^*\tilde{\zeta}_0(1) + {^*\tilde{\zeta}_b}(1)$$

$$\underline{h_\omega^b(1)}] = 0,$$

$$(\lambda + 2\mu) \int_0^1 \lfloor S_\Phi(^*\zeta_{0,3} + \underline{^*\zeta_{a,3} h_\omega^a} + {^*\zeta_a} h_{\omega,3}^a) + F(^*\eta_{0,3} + \underline{^*\eta_{a,3} h_\omega^a} +$$

$$+ {^*\eta_a} h_{\omega,3}^a)(^*\tilde{\eta}_{0,3} + \underline{^*\tilde{\eta}_{b,3} h_\omega^b} + {^*\tilde{\eta}_b} h_{\omega,3}^b) dX_3 - \varrho b_3 \int_0^1 F(^*\tilde{\eta}_0 + \underline{^*\tilde{\eta}_b h_\omega^b}) dX_3 +$$

$$+\varrho\int_0^1 S_\Phi(*\ddot{\zeta}_0+\underline{*\ddot{\zeta}_a h_\omega^a})(*\tilde{\eta}_0+\underline{*\tilde{\eta}_b h_\omega^b})dX_3+$$

$$+\varrho\int_0^1 F(*\ddot{\eta}_0+\underline{*\ddot{\eta}_a h_\omega^a})(*\tilde{\eta}_0+\underline{*\tilde{\eta}_b h_\omega^b})dX_3-\int_0^1 \mathring{p}_3(*\tilde{\eta}_0+\underline{*\tilde{\eta}_b h_\omega^b})dX_3+$$

$$-P_3(0,t)[*\tilde{\eta}_0(0)+\underline{*\tilde{\eta}_b(0)h_\omega^b(0)}]-P_3(1,t)[*\tilde{\eta}_0(1)+*\tilde{\eta}_b(1)h_\omega^b(1)]=0,$$

which has to hold for any sufficiently regular $\tilde{\Theta}_0(\cdot)$, $\tilde{\Theta}_b(\cdot)$, $\tilde{\psi}_0(\cdot)$, $\tilde{\psi}_b(\cdot)$, $\tilde{\varphi}_0(\cdot)$, $\tilde{\varphi}_b(\cdot)$, $\tilde{\zeta}_0(\cdot)$, $\tilde{\zeta}_b(\cdot)$, $\tilde{\eta}_0(\cdot)$, $\tilde{\eta}_b(\cdot)$.

The underlined terms in (2.2) as infinitely small will be neglected. Taking into account the independence of functions: $\tilde{\Theta}_0(\cdot)$, $\tilde{\Theta}_b(\cdot)$, $\tilde{\psi}_0(\cdot)$, $\tilde{\psi}_b(\cdot)$, $\tilde{\varphi}_0(\cdot)$, $\tilde{\varphi}_b(\cdot)$, $\tilde{\zeta}_0(\cdot)$, $\tilde{\zeta}_b(\cdot)$, $\tilde{\eta}_0(\cdot)$, $\tilde{\eta}_b(\cdot)$ we obtain the system of $5(n+1)$ equations of the form:

$$\mu\int_0^1(F*\psi_{0,3}+Fh_{\omega,3}^a*\psi_a+K_2*\zeta_0)*\tilde{\psi}_{0,3}dX_3-\varrho b_1\int_0^1 F*\tilde{\psi}_0dX_3+$$

$$+\varrho\int_0^1 F*\ddot{\psi}_0*\tilde{\psi}_0dX_3-\int_0^1\mathring{p}_1*\tilde{\psi}_0dX_3-P_1(0,t)*\tilde{\psi}_0(0)+$$

$$-P_1(1,t)*\psi_0(1)\simeq 0,$$

$$\mu\int_0^1(Fh_{\omega,3}^b*\psi_{0,3}+Fh_{\omega,3}^a h_{\omega,3}^b*\psi_a+K_2h_{\omega,3}^b*\zeta_0)*\tilde{\psi}_bdX_3\simeq 0, \tag{2.3}$$

$$\mu\int_0^1(F*\varphi_{0,3}+Fh_{\omega,3}^{a}*\varphi_a+K_1*\zeta_0)*\tilde{\varphi}_{0,3}dX_3-\varrho b_2\int_0^1 F*\tilde{\varphi}_0dX_3+$$

$$+\varrho\int_0^1 F*\ddot{\varphi}_0*\tilde{\varphi}_0dX_3-\int_0^1\mathring{p}_2*\tilde{\varphi}_0dX_3-P_2(0,t)*\tilde{\varphi}_0(0)+$$

$$-P_2(1,t)*\varphi_0(1)\simeq 0,$$

$$\mu\int_0^1(Fh_{\omega,3}^b*\varphi_{0,3}+Fh_{\omega,3}^a h_{\omega,3}^b*\varphi_a+K_1h_{\omega,3}^b*\zeta_0)*\tilde{\varphi}_bdX_3\simeq 0,$$

$$\mu\int_0^1(I_0*\Theta_{0,3}+I_0h_{\omega,3}^a*\Theta_a+I_s*\zeta_0)*\tilde{\Theta}_{0,3}dX_3+\varrho\int_0^1 I_0*\ddot{\Theta}_0*\tilde{\Theta}_0dX_3+$$

$$-\int_0^1 m_s*\tilde{\Theta}_0dX_3-M_s(0,t)*\tilde{\Theta}_0(0)-M_s(1,t)*\tilde{\Theta}_0(1)\simeq 0,$$

$$\mu\int_0^1(I_0h_{\omega,3}^b*\Theta_{0,3}+I_0h_{\omega,3}^a h_{\omega,3}^b*\Theta_a+I_sh_{\omega,3}^b*\zeta_0)*\tilde{\Theta}_bdX_3\simeq 0,$$

$$\int_0^1[\mu(I_s*\Theta_{0,3}+I_sh_{\omega,3}^a*\Theta_a+I_k*\zeta_0)*\tilde{\zeta}_0+(\lambda+2\mu)$$

$$(I^*\zeta_{0,3}+Ih^a_{\omega,3}{}^*\zeta_a+S_\Phi{}^*\eta_{0,3}+S_\Phi h^a_{\omega,3}{}^*\eta_a)^*\tilde{\zeta}_{0,3}\,dX_3+$$

$$-\varrho b_3\int_0^1 S_\Phi{}^*\tilde{\zeta}_0\,dX_3+\varrho\int_0^1 I^*\ddot{\zeta}_0{}^*\tilde{\zeta}_0\,dX_3+\varrho\int_0^1 S^*_\Phi\ddot{\eta}_0{}^*\tilde{\zeta}_0\,dX_3+$$

$$-\int_0^1 m_\Phi{}^*\tilde{\zeta}_0\,dX_3-M_\Phi(0,t)^*\tilde{\zeta}_0(0)-M_\Phi(1,t)^*\tilde{\zeta}_0(1)\simeq 0,$$

$$(\lambda+2\mu)\int_0^1(Ih^b_{\omega,3}\zeta_{0,3}+Ih^a_{\omega,3}h^b_{\omega,3}{}^*\zeta_a+S_\Phi h^b_{\omega,3}{}^*\eta_{0,3}+S_\Phi h^a_{\omega,3}h^b_{\omega,3}{}^*\eta_a)$$

$$\tilde{\zeta}_b\,dX_3\simeq 0,$$

$$(\lambda+2\mu)\int_0^1(S_\Phi{}^*\zeta_{0,3}+S_\Phi h^a_{\omega,3}{}^*\zeta_a+F^*\eta_{0,3}+Fh^a_{\omega,3}{}^*\eta_a)^*\tilde{\eta}_{0,3}\,dX_3+$$

$$-\varrho b_3\int_0^1 F^*\tilde{\eta}_0\,dX_3+\varrho\int_0^1 S_\Phi{}^*\ddot{\zeta}_0{}^*\tilde{\eta}_0\,dX_3+\varrho\int_0^1 F^*\ddot{\eta}_0{}^*\tilde{\eta}_0\,dX_3+$$

$$-\int_0^1 \mathring{p}_3{}^*\tilde{\eta}_0\,dX_3-P_3(0,t)^*\tilde{\eta}_0(0)-P_3(1,t)^*\tilde{\eta}_0(1)\simeq 0,$$

$$(\lambda+2\mu)\int_0^1(S_\Phi h^b_{\omega,3}{}^*\zeta_{0,3}+S_\Phi h^a_{\omega,3}h^b_{\omega,3}{}^*\zeta_a+Fh^b_{\omega,3}{}^*\eta_{0,3}+Fh^a_{\omega,3}h^b_{\omega,3}{}^*\eta_a)$$

$$^*\tilde{\eta}_b\,dX_3\simeq 0.$$

Define $\langle f\rangle=\frac{1}{\varepsilon}\int_0^\varepsilon f(X_3)dX_3$ for any integrable ε-periodic function $f(\cdot)$. Now, using the known theorem of the nonstandard integral calculus, [5], which states that:

$$st\int_0^1 f(X_3)^*g(X_3)\,dX_3=\langle f\rangle\int_0^1 g(X_3)\,dX_3, \tag{2.4}$$

we obtain the following equation system:

$$\begin{aligned}
&\mu(\langle I_0\rangle\Theta_{0,33}+\langle I_0h^a_{,3}\rangle\Theta_{a,3}+\langle I_s\rangle\zeta_{0,3})=\varrho\langle I_0\rangle\ddot{\Theta}_0-m_s,\\
&\langle I_0h^b_{,3}\rangle\Theta_{0,3}+\langle I_0h^a_{,3}h^b_{,3}\rangle\Theta_a+\langle I_sh^b_{,3}\rangle\zeta_0=0,\\
&\mu(\langle F\rangle\psi_{0,33}+\langle Fh^a_{,3}\rangle\psi_{a,3}+\langle K_2\rangle\zeta_{0,3})=-\varrho b_1\langle F\rangle+\varrho\langle F\rangle\ddot{\psi}_0-\mathring{p}_1,\\
&\langle Fh^b_{,3}\rangle\psi_{0,3}+\langle Fh^a_{,3}h^b_{,3}\rangle\psi_a+\langle K_2h^b_{,3}\rangle\zeta_0=0,\\
&\mu(\langle F\rangle\varphi_{0,33}+\langle Fh^a_{,3}\rangle\varphi_{a,3}+\langle K_1\rangle\zeta_{0,3})=-\varrho b_2\langle F\rangle+\varrho\langle F\rangle\ddot{\varphi}_0-\ddot{p}_2,\\
&\langle Fh^b_{,3}\rangle\varphi_{0,3}+\langle Fh^a_{,3}h^b_{,3}\rangle\varphi_a+\langle K_1h^b_{,3}\rangle\zeta_0=0,\\
&-\mu(\langle I_s\rangle\Theta_{0,3}+\langle I_sh^a_{,3}\rangle\Theta_a+\langle I_k\rangle\zeta_0)+(\lambda+2\mu)\\
&(\langle I\rangle\zeta_{0,33}+\langle Ih^a_{,3}\rangle\zeta_{a,3}+\langle S_\Phi\rangle\eta_{0,33}+\langle S_\Phi h^a_{,3}\rangle\eta_{a,3})=
\end{aligned} \tag{2.5}$$

$$= -\varrho b_3\langle S_\Phi\rangle + \varrho\langle I\rangle\ddot{\zeta} + \varrho\langle S_\Phi\rangle\ddot{\eta}_0 - m_\Phi,$$

$$\langle Ih^b_{,3}\rangle\zeta_{0,3} + \langle Ih^a_{,3}h^b_{,3}\rangle\zeta_a + \langle S_\Phi h^b_{,3}\rangle\eta_{0,3} + \langle S_\Phi h^a_{,3}h^b_{,3}\rangle\eta_a = 0,$$

$$(\lambda+2\mu)(\langle S_\Phi\rangle\zeta_{0,33} + \langle S_\Phi h^a_{,3}\rangle\zeta_{a,3} + \langle F\rangle\eta_{0,33} + \langle Fh^a_{,3}\rangle\eta_{a,3}) =$$

$$= -\varrho b_3\langle F\rangle + \varrho\langle S_\Phi\rangle\ddot{\zeta}_0 + \varrho\langle F\rangle\ddot{\eta}_0 - \mathring{p}_3,$$

$$\langle S_\Phi h^b_{,3}\rangle\zeta_{0,3} + \langle S_\Phi h^a_{,3}h^b_{,3}\rangle\zeta_a + \langle Fh^b_{,3}\rangle\eta_{0,3} + \langle Fh^a_{,3}h^b_{,3}\rangle\eta_a = 0,$$

for $X_3 \in (0, 1)$, $t \in [0, t_f]$,
and boundary conditions:

$$\begin{aligned}
&\mu(\langle I_0\rangle\Theta_{0,3} + \langle I_0 h^a_{,3}\rangle\Theta_a + \langle I_s\rangle\zeta_0) = M_s n_3,\\
&\mu(\langle F\rangle\psi_{0,3} + \langle Fh^a_{,3}\rangle\psi_a + \langle K_2\rangle\zeta_0) = P_1 n_3,\\
&\mu(\langle F\rangle\varphi_{0,3} + \langle Fh^a_{,3}\rangle\varphi_a + \langle K_1\rangle\zeta_0) = P_2 n_3,\\
&(\lambda+2\mu)(\langle I\rangle\zeta_{0,3} + \langle Ih^a_{,3}\rangle\zeta_a + \langle S_\Phi\rangle\eta_{0,3} + \langle S_\Phi h^a_{,3}\rangle\eta_a) = M_\Phi n_3,\\
&(\lambda+2\mu)(\langle S_\Phi\rangle\zeta_{0,3} + \langle S_\Phi h^a_{,3}\rangle\zeta_a + \langle F\rangle\eta_{0,3} + \langle Fh^a_{,3}\rangle\eta_a) = P_3 n_3,
\end{aligned} \tag{2.6}$$

for $X_3 = 0$, $X_3 = 1$, $t \in [0, t_f]$.

Eqs. (2.5) represent the system of $5(n+1)$ linear partial differential equations with the constant coefficients. Hence for the quasi-static case the exact analytical solution to this system (with the corresponding boundary conditions (2.6)) may be explicitly obtained; if we neglect inertia forces the Eqs. (2.5) will constitute the linear differential equations system of the first order for $5n$ microlocal parameters $\Theta_a(\cdot, t)$, $\psi_a(\cdot, t)$, $\varphi_a(\cdot, t)$, $\zeta_a(\cdot, t)$, $\eta_a(\cdot, t)$ and of the second order for 5 generalized macro-deformations $\Theta_0(\cdot, t)$, $\psi_0(\cdot, t)$, $\varphi_0(\cdot, t)$, $\zeta_0(\cdot, t)$, $\eta_0(\cdot, t)$.

Now assume that a certain solution to the problem given by (2.5), (2.6) has been obtained. Then the following evaluations hold, [5]:

$$\begin{aligned}
&\Theta(X_3, t) \sim \Theta_0(X_3, t), \quad \Theta_{,3}(X_3, t) \sim \Theta_{0,3}(X_3, t) + \Theta_a(X_3, t)h^a_{,3}(X_3),\\
&\psi(X_3, t) \sim \psi_0(X_3, t), \quad \psi_{,3}(X_3, t) \sim \psi_{0,3}(X_3, t) + \psi_a(X_3, t)h^a_{,3}(X_3),\\
&\varphi(X_3, t) \sim \varphi_0(X_3, t), \quad \varphi_{,3}(X_3, t) \sim \varphi_{0,3}(X_3, t) + \varphi_a(X_3, t)h^a_{,3}(X_3),\\
&\zeta(X_3, t) \sim \zeta_0(X_3, t), \quad \zeta_{,3}(X_3, t) \sim \zeta_{0,3}(X_3, t) + \zeta_a(X_3, t)h^a_{,3}(X_3),\\
&\eta(X_3, t) \sim \eta_0(X_3, t), \quad \eta_{,3}(X_3, t) \sim \eta_{0,3}(X_3, t) + \eta_a(X_3, t)h^a_{,3}(X_3).
\end{aligned} \tag{2.7}$$

Hence we see that the microlocal parameters $\Theta_a(X_3, t)$, $\psi_a(X_3, t)$, $\varphi_a(X_3, t)$, $\zeta_a(X_3, t)$, $\eta_a(X_3, t)$ have the neglectible influence on the displacement field (1.1), (1.2), but they play an essential role if we calculate the stresses (1.8), (the terms of the form $\xi_a h^a_{,3}$ are not small as compared with $\xi_{0,3}$). On the basis of $\Theta(X_3, t)$, $\psi(X_3, t)$, $\varphi(X_3, t)$, $\zeta(X_3, t)$, $\eta(X_3, t)$ we can calculate the reaction forces produced by the internal constraints (1.1), (1.2) using Eqs. (1.3) and Eqs. (1.4). Taking into account the criterion of the physical correctness of constraints, [1], we can also determine the applicability range of the obtained solution.

3. Final remarks

In this paper we have shown that applying the microlocal modelling approach (based on the concepts of nonstandard analysis) to the problems of torsion of rods with periodically variable cross-section we arrive to the system of differential equations with the constant coefficients (2.5) and to the pertinent natural boundary conditions (2.6). It has to be emphasized that the microlocal parameters can be eliminated from the foregoing system of equations and hence we obtain what will be called the system of effective equations for the torsional problems of rods with ε-periodic variable cross-section. The general results obtained in this paper will be analysed and illustrated in the forthcoming papers on this subject.

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Р е з ю м е

МИКРОЛОКАЛЬНОЕ МОДЕЛИРОВАНИЕ ПРОБЛЕМЫ КРУЧЕНИЯ СТЕРЖНЯ С ε-ПЕРИОДИЧЕСКОМ СЕЧЕНИЕМ

Цель настоящего сообщения — представление некоторого метода гомогенизации проблемы стесненного кручения прямого, линейно — упругого стержня, которого сечение меняется ε-периодически. Используя методы нестандартного анализа [4], употребляется нестандартную гомогенизацию (микролокальное моделирование), применение которой к механике ввел Ч. Возьняк [5,6], для решения некаторой технической теории кручения стержней, полученной в рамках аналитической механики тел с внутренными связями [1,2]. Полученная модель является базой анализа многих инжинерных проблем.

S t r e s z c z e n i e

O MIKROLOKALNYM MODELOWANIU ZAGADNIENIA SKRĘCANIA PRĘTA O OKRESOWO ZMIENNYM PRZEKROJU

Celem pracy jest przedstawienie pewnej metody homogenizacji problemu nieswobodnego skręcania prostego, liniowo sprężystego pręta o okresowo zmieniającym się zwartym przekroju. Korzystając z metod analizy niestandardowej [4], stosuje się homogenizację niestandardową (modelowanie mikrolokalne), której zastosowanie w mechanice zapoczątkował Cz. Woźniak [5, 6], celem rozwiązania pewnej technicznej teorii skręcania prętów, otrzymanej w ramach mechaniki analitycznej kontinuum materialnego [1, 2]. Otrzymany model może stanowić podstawę analizy wielu zagadnień inżynierskich.

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