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CONSTRAINTS IN SOILD MECHANICS. AN APPLICATION OF NONSTANDARD ANALYSIS

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1. Introduction

The concept of constraints in solid mechanics is usually utilized to formulate special cases of constitutive relations identifying with certain restriction imposed on pairs (χ, T) of a motion χ and a stress T. To be a constitutive relation such restriction must have a special form, i.e. it must fulfil certain neccesary conditions stated in the general theory of constitutive relations. The following Noll axiom is exactly one of these conditions, [9, p. 160].

The principle of determinism for simple materials. The stress at the place occupied by the body-point χ at the time t is determined by the history χ^t of the motion of the body up to the time t, i.e.

$$T(\chi(X, t), t) = \mathscr{F}(\nabla\chi^t_i(X, \cdot); X).$$

Here $\mathscr{F}(\cdot)$ denotes a sufficient regular mapping of histories $\nabla\chi^t$ of a gradient $\nabla\chi$ of a motion χ, body -points X onto symmetric Cauchy stress tensors.

The above principle of determinism will be called here a classical principle of determinism. However, there exist real materials for which forementioned defined principle leads to the theory not consistent with experiment. In such situations more general or alternative formulations of the principle of determinism should be applied. For example, if admissible motions of a body are subjected to constraints of the form:

$$w(X, \nabla\chi^T\nabla\chi(X, t)) = 0, \tag{1.1}$$

where $w(\cdot)$ is sufficient regular function with values in R^n then the following statement holds, [9 p. 176].

Principle of determinism for simple materials subject to constraints. The stress at the place occupied by the body-point X at the time t is determined by the history χ^t of the motion χ up to the time t only to within an arbitrary tensor that does no work in any motion compatible with the constraints. That is:

$$T(\chi(X, t), t) = G(\nabla\chi^t(X, \cdot); X)+N,$$

where the mapping $G(\cdot)$ need be sufficiently regular and defined only for arguments $\nabla\chi^t$ such as to satisfy the constraints, N being a stress for which the stress-power vanishes in any motion satisfying the constraints, i.e. $\mathrm{tr}(ND) = 0$ for each symmetric tensor D such that:

$$\frac{\partial w}{\partial(\chi^k_{,\alpha}\chi_{k,\beta})} D^{lm}(\chi^{-1})^{\alpha}_{,l}(\chi^{-1})^{\beta}_{,m} = 0.$$

The principle of determinism for simple materials with constraints is a generalisations of the classical principle. If there are no restrictions of the form (1.1), i.e. $w(\cdot) \doteq$ const., then $N = 0$ and both principles coincide.

The principle of determinism can be formulated in mechanics also in more general form, describing more general classes of physical situations. For example in [14] it can be found the following formulation of constraints:

$$w(\chi, \nabla\chi, \ldots, \nabla^p\chi) = 0$$

and in [1] we deal with constraints:

$$w(J, \nabla J, \ldots, \nabla^p J) = 0,$$
$$J \equiv (\chi, \chi^{(1)}, \ldots, \chi^{(q)}),$$

where p, q are natural numbers and numbers in scopes under the letters denote suitable time-derivative.

In the paper we apply nonstandard analysis as a mathematical tool derive new constitutive relations of mechanics from the known constitutive relations. Fundamental concepts of this approach are based on [7, 11, 12]. The aim of the paper is to prove that by applying concepts and methods of nonstandard analysis the principle of determinism for constitutiverelations with constraints can be obtained from the classical principle of determinism. This proof will be realized by a certain specification of nonstandard constitutive relations which are consistent with the classical principle of determinism. We are to show that this approach has the following attributes:

(i) it eliminates from the axioms of mechanics the principle of determinism with constraints,

(ii) it has a clear physical interpretation being based only on the classical principle of determinism,

(iii) it leads to a description of physical situations which cannot be described neither by the classical principle of determinism nor by the principle of determinism for constitutive relations with constraints.

In the paper the concept of a constitutive relation is understood in more general sense than that in most of the papers on this subject. Namely after a certain specification the constitutive relations will be treated as constitutive relations for the internal forces describing material properties of bodies — or as constitutive relations for the external forces describing interactions between a body and its exterior, cf. [11, 15].

2. Physical foundations

Let be given the following objects:

(i) the set Γ of states γ of the mechanical system under consideration, i.e. assume that Γ is an open set in a certain topological space,

(ii) the set $\mathscr{P}$ of admissible evolutions $R \ni t \to \gamma(t) \in \Gamma$ of states of the mechanical system under consideration, i.e. the set of right-hand side differentiable functions of real variable; assume that this differentiation is well defined,

(iii) the dual pairing $(W, \langle \cdot, \cdot \rangle, W')$ of linear topological spaces in which W is the space of time rates $\dot{\gamma}$, W' is the space of reactions ϱ of the system and $\langle \dot{\gamma}, \varrho \rangle$ is the power of the reaction ϱ for the rate $\dot{\gamma}$, $\dot{\gamma} \in W$, $\varrho \in W'$,

(iv) the set H_t of histories $\gamma^{(t)}: R_t \to \Gamma$ of the evolution $\gamma(\cdot) \in \mathscr{P}$ of the system states up to the time t, defined for each $t \in R$, i.e.

$$\gamma^{(t)}(s) = \gamma(t-s)$$

for $t \in R$, $s \in R$, $s \geqslant 0$.

As a basis for our considerations the following requirement will be postulated.

Principle of determinism. For each time-instant $t \in R$ a reaction $\varrho(t)$ of the system is uniquely determined by the history $\gamma^{(t)} \in H_t$ of the evolution $\gamma(\cdot) \in \mathscr{P}$ up to the time t and by the rate $\dot{\gamma}(t)$ of change of a system state in the time t, i.e,

$$\varrho(t) = \varphi\big(t, \dot{\gamma}(t), \dot{\gamma}^{(t)}\big). \tag{2.1}$$

Introducing above and applying below concepts such as the state of the system, a reaction of the system, the rate of changing of a state of the system, etc., can have a different physical interpretation, which can be found in [11].

In a description of a mechanical system the concept of constraints is used in situations where it is impossible to receive so many informations to be sufficient to describe it by a constitutive relation satisfying the classical principle of determinism. Accepting here as a fundamental requirement the classical principle of determinism has then a superior authority with respect to other ones. The approach using in the paper is in agree with above premises because the concept of constraints is here a natural consequence of the classical principle of determinism.

3. Tools from nonstandard analysis

Let $\mathscr{X}$ be a nonempty set. From all sequences of points of $\mathscr{X}$ we shall distinguish the set C, $C \subset \mathscr{X}^N$, elements of which will be called the converging sequences. For each converging sequence $(x_n)_{n \in N}$ we assing exactly one point $\lim x_n \in \mathscr{X}$ which will be called a limit of $(x_n)_{n \in N}$. We will also say that each sequence $(x_n)_{n \in N} \in C$ converges to the limit $\lim x_n$. We assume that the operation $\lim: C \to \mathscr{X}$ fulfils the following conditions:

(i) each subsequence of a sequence converging to x, $x \in \mathscr{X}$, is a sequence converging to x,

(ii) the constant sequence with values equal to x, $x \in \mathscr{X}$, converges to x,

(iii) each sequence not converging to x, $x \in \mathscr{X}$, contains a subsequence which in turn does not contain any subsequence converging to x.

Then the pair $(\mathscr{X}, \lim)$ will be referred to as L'-space, cf. [6 p. 339]. Let C_A, $A \subset \mathscr{X}$, stands for the set of all converging sequences with values in A and let $P(\mathscr{X})$ be the power set of $\mathscr{X}$. Define two sequences $\mathrm{cl}_n\colon P(\mathscr{X}) \to P(\mathscr{X})$, $\mathrm{int}_n\colon P(\mathscr{X}) \to P(\mathscr{X})$, of operations, setting:

$$\mathrm{cl}_1(A) \equiv \{\lim x_n : (x_n)_{n\in N} \in C_A\},$$

$$\mathrm{cl}_{n+1}(A) \equiv \mathrm{cl}_1\mathrm{cl}_n(A),\ n \in N,$$

and:

$$\mathrm{int}_1(A) \equiv \mathscr{X}\setminus \mathrm{cl}_1(\mathscr{X}\setminus A),$$

$$\mathrm{int}_{n+1}(A) \equiv \mathrm{int}_1\mathrm{int}_n(A),\ n \in N,$$

for every $A \in P(\mathscr{X})$. It is easy to verify that each pair $(\mathscr{X}, \mathrm{cl}_n)$ $n \in N$, is a step-space, cf. [3], i.e. for each $n \in N$ the operation cl_n fulfils all conditions defining a closure operation in a topological space (possible except the requirement that cl_n^2 must be equal to cl_n). It is easy to introduce a topological structure in each L'-space by defining the closed sets as the sets D containing limits converging sequences of points belonging to the set D, cf. [2 p. 90]. This topology will be denoted by τ. If the operation lim fulfils the additional condition

(iv) if $\lim x_n = x$ and $\lim x_k^n = x_n$, $n \in N$, then there exist sequences $(n_i)_{i\in N}$, $(k_i)_{i\in N}$ of natural numbers for which $\lim x_{k_i}^{n_i} = x$,

then $\mathrm{cl}_n = \mathrm{cl}_m$ for each pair $(n, m) \in N^2$ and $\mathrm{cl} = \mathrm{cl}_n$, $n \in N$, is then a closure operation in topological space $(\mathscr{X}, \tau$, cf. [2, p. 90].

Similarly to such topological concepts as: the monad, the standard part operation, the F-limit operation, we are going to define, for any $n \in N$ and for any L'-space, new concepts of n-monad, n-standard part operation and F-limit operation. To this aid let the pair $(\mathscr{X}, \lim)$ be a L'-space and let $\mathscr{X}$, lim be objects in a certain full structure $\mathfrak{M}$. Let $^*\mathfrak{M}$ be an enlargement of $\mathfrak{M}$. We have $^*\mathscr{X} \in {}^*\mathfrak{M}$ and $\lim \in {}^*\mathfrak{M}$ (here and below we write lim instead of *lim). The pair $(^*\mathscr{X}, \lim)$ is considered here as a QL'-space. For $x \in \mathscr{X}$ and $n \in N$ define:

$$\mathrm{Mon}_n(x) \equiv \cap\{^*A : A \in P(\mathscr{X}),\ x \in A = \mathrm{int}_n A\}. \tag{3.1}$$

Denoting by $\mu_\tau(x)$ the monad of x in the topological space $(\mathscr{X}, \tau)$ it is easy to verify that the following inclusions:

$$\mathrm{Mon}_1(x) \supset \mathrm{Mon}_2(x) \supset \mathrm{Mon}_3(x) \supset \ldots,$$

as well as the equality:

$$\bigcap_{n\in N} \mathrm{Mon}_n(x) = \mu_\tau(x)$$

hold. The L'-space $(\mathscr{X}, \lim)$ will be called n-Hausdorff, $n \in N$, if $x = y$ is implied by $\mathrm{Mon}_n(x) = \mathrm{Mon}_n(y)$. It is easy to see that if L'-space $(\mathscr{X}, \lim)$ is n-Hausdorff, for a certain $n \in N$, then the topological space $(\mathscr{X}, \tau)$ is a Hausdorff space.

Now let L'-space be n-Hausdorff for a certain $n \in N$. Then in every n-monad $\mathrm{Mon}_n(x)$, $x \in \mathscr{X}$, there is exactly one standard point. For each pair $(x, y) \in \mathscr{X} \times {}^*\mathscr{X}$ we shall write $\mathrm{st}_n y = x$ if $y \in \mathrm{Mon}_n(x)$. The aforemention operation $\mathrm{st}_n\colon {}^*\mathscr{X} \to \mathscr{X}$ will be considered as the n-standard part operation. The domain of st_n is equal to $\bigcup\{\mathrm{Mon}_n(x) : x \in \mathscr{X}\}$. A sequence $(x_n)_{n\in {}^*N}$ of points of $^*\mathscr{X}$ will be called F-converging if there exist a point

$x, x \in \mathscr{X}$, and a hypernatural number $\lambda_0 \in {}^*N \setminus N$, such that the relation $x_\nu \in \mathrm{Mon}_n(x)$ holds for every $\nu \in {}^*N \setminus N$, $\nu < \lambda_0$. Points from $\mathrm{Mon}_n(x)$ will be considered as F_n-limits of $(x_n)_{n \in {}^*N}$.

The concepts of n-Hausdorff L'-space, n-monad, n-standard part operation, F_n-limit, operation will be used below only in the case of $n = 1$. In the sequel instead of a 1-monad a 1-Hausdorff space, etc., we shall use the terms: a monad, a Hausdorff L'-space, etc., respectively.

Now let T stands for a fixed topological regular space and 2^T be the set of all closed subsets of T. Let define a convergence in 2^T setting $(A_n)_{n \in N} \in C$ iff for some $A \in 2^T$ the following statements holds:

(i) $\limsup A_n = A$, i.e. each neibourhood of any point from A has a nonempty intersections with almost every set A_n, $n \in N$,

(ii) $\liminf A_n = A$, i.e. each neibourhood of any point from A has a non-empty intersection with infinite number of sets A_n, $n \in N$.

The set 2^T with the convergence of sequences of sets defining above, determines a certain L'-space, [6 p. 188], which will be denoted here by $(2^T, \lim)$. An important result, [10], is that this L'-space is Hausdorff (i.e. 1-Hausdorff) and:

$$\mathrm{Mon}(A) = \{B \in {}^*(2^T) : {}^\circ B = A\}, \quad A \in 2^T, \tag{3.2}$$

where ${}^\circ B$ stands for the standard part of the set B. It means that the standard part operation in L'-space $(2^T, \lim)$ is equal to the standard part operation of (closed) subsets of T. Moreover, $F\text{-}\lim A_n = \mathrm{Mon}(A)$ provided that:

$$(\exists \lambda_0 \in {}^*N \setminus N)(\forall n \in {}^*N \setminus N)[[n < \lambda_0] \Rightarrow [A = {}^\circ A_n]],$$

for each F-converging sequence $(A_n)_{n \in {}^*N}$ of closed subsets $A_n \in {}^*(2^T)$.

4. From microconstitutive relations to macroconstitutive relations.

Now we are going to formulate the method which enable us to obtain new constitutive relations from the known constitutive relations. The known constitutive relations are here relations satisfying the following form of the classical principle of determinism (2.1):

$$\varrho(t) = \tilde{\varphi}_t(\gamma(t), \dot{\gamma}(t), \gamma^{(t)}), \tag{D}$$

where function $\tilde{\varphi}_t : \Gamma \times W \times H_t \to W'$, for every $t \in R$, is defined by $\tilde{\varphi}_t(\gamma(t), \dot{\gamma}(t), \gamma^{(t)}) \equiv \equiv \varphi(t, \dot{\gamma}(t), \gamma^{(t)})$. The formula (D) is a starting point of our considerations. In the sequel arguments t and $\gamma^{(t)}_*$, $\gamma^{(t)} \in H_t$, $t \in R$, will be treated as parameters; for the sake of simplicity they will be omitted. So (D) has a form:

$$\varrho = \tilde{\varphi}(\gamma, \dot{\gamma}); \ \tilde{\varphi} : \Gamma \times W \to W'. \tag{4.1}$$

Let us assume that the set $U(\gamma) \equiv \mathrm{dom}\, \tilde{\varphi}(\gamma, \cdot)$, for every $\gamma \in \Gamma$, is open in W. In a particular case Eq. (4.1) reduces to $\varrho = \tilde{\varphi}(\gamma)$.

Let Φ be a set of functions $\tilde{\varphi} : \Gamma \to W \to W'$ which are assumed to describe physical situations defined by (D). Hence we conclude that the set Φ depends on parameters t and $\gamma^{(t)}(\cdot)$. In agreement with physical premises, Φ is an infinite set. Every function

$\tilde{\varphi} \in \Phi$ will be called a constitutive relation. It is not assumed here that every constitutive elation being an element of Φ has a physical sense.

Let $\mathfrak{M} \equiv (A_\tau)_{\tau \in \mathcal{T}}$ be a full structure in which sets R, Γ, W, W' are separated objects of the type $A_{(0)}$. Passing to an enlargement $^*\mathfrak{M}$ of $\mathfrak{M}$, elements of $^*\Phi$ will be called microconstitutive relations:

$$\varrho = \tilde{\varphi}(\gamma, \dot{\gamma}); \quad \tilde{\varphi}: {}^*\Gamma \times {}^*W \to {}^*W', \; \tilde{\varphi} \in {}^*\Phi. \tag{4.2}$$

For every microconstitutive relation $\tilde{\varphi}$, $\tilde{\varphi} \in {}^*\Phi$, and for every $\gamma, \gamma \in \Gamma$, we have dom $\tilde{\varphi}(\varphi, \cdot) = {}^*U(\gamma)$. Every function $\tilde{\varphi}$, $\tilde{\varphi} \in {}^*\Phi$, is an internal relation but not necessary standard.

Let us assume that the set Γ is a topological Hausdorff space satisfying the first axiom of countability. For every state γ, $\gamma \in \Gamma$, we denote by $(\sigma_n(\gamma))_{n \in N}$ the neibourhood-basis of γ in Γ. In the space $2^{W'}$ of all closed subsets of W' we shall introduce a L'-space structure setting $T := W'$ in L'-space $(2^T, \lim)$. It is possible to introduce such structure by means of considerations of Sec. 3, provided that W' is regular. Let us define sequences $(\mathscr{R}^{\tilde{\varphi}}_n(\gamma, w))_{n \in N}$ setting

$$\mathscr{R}^{\tilde{\varphi}}_n(\gamma, w) \equiv \{\varrho = \tilde{\varphi}(\bar{\gamma}, \bar{w}) : (\bar{\gamma}, \bar{w}) \in \sigma_n(\gamma) \times B(w, r_0/n)\}, \tag{4.3}$$

where $r_0 \in R_+$, $\tilde{\gamma} \in {}^*\Phi$ and $B(w, r_0/n)$ is an open ball in W with a center w and a radius r_0/n, $n \in {}^*N$. Let Ψ_0 be a subset of $^*\Phi$ satisfying comditions:

(i) for every $\tilde{\varphi} \in \Psi_0$, $\gamma \in {}^*\Gamma$, $w \in {}^*W$, sequences $(\overline{\mathscr{R}}^{\tilde{\varphi}}_n(\gamma, w))_{n \in {}^*N}$ of closures of sets defined by Eq. (4.3) have F-limits (in the sense precised in Sec. 3),

(ii) there exist 1° a standard state γ, $\gamma \in \Gamma$, 2° a standard velocity $\dot{\gamma}$, $\dot{\gamma} \in W$, 3° a nonstandard number $\lambda_0 \equiv \lambda_0(\tilde{\gamma})$, $\lambda_0 \in {}^*N \setminus N$, such that:

$$^\circ[\overline{\mathscr{R}}^{\tilde{\varphi}}_\nu(\bar{\gamma}, \bar{w})] \neq \emptyset, \qquad (\tilde{\varphi}, \bar{\gamma}, \bar{w}) \in \Psi_0 \times \mu(\gamma) \times \mu(w), \tag{4.4}$$

where the closures $\overline{\mathscr{R}}^{\tilde{\varphi}}_\nu(\bar{\gamma}, \bar{w})$ of sets $\mathscr{R}^{\tilde{\varphi}}_\nu(\bar{\gamma}, \bar{w})$, for every $\nu \in {}^*N \setminus N$, $\nu < \lambda_0$, are F-limits of the sequences $(\overline{\mathscr{R}}^{\tilde{\varphi}}_n(\gamma, w))_{n \in {}^*N}$ and where the sets $\overline{\mathscr{R}}^{\tilde{\varphi}}_\nu(\bar{\gamma}, \bar{w})$ do not depend on $(\bar{\gamma}, \bar{w}) \in \mu(\gamma) \times \mu(w)$.

In Ψ_0 we introduce an equivalence reiation $\sim$, setting $\tilde{\varphi}_1 \sim \tilde{\varphi}_2$ provided that:

$$(\forall \nu \in {}^*N \setminus N)[[\nu < \min(\lambda_0(\tilde{\varphi}_1), \lambda_0(\tilde{\varphi}_2))] \Rightarrow [{}^\circ R^{\tilde{\varphi}_1}_\nu(\gamma, w) = {}^\circ R^{\tilde{\varphi}_2}_\nu(\gamma, w)]],$$

holds for $(\tilde{\varphi}_1, \tilde{\varphi}_2) \in \Psi_0^2$. The equivalence class determined by the microconstitutive relation $\tilde{\gamma} \in \Psi_0$ and the pertinent quotient set will be denoted by $\pi(\tilde{\varphi})$ and Π respectively. Setting:

$$\mathscr{R}^{\pi(\varphi)}(\gamma, w) \equiv {}^\circ[\mathscr{R}^{\tilde{\varphi}}_\nu(\gamma, w)], \tag{4.5}$$

for $(\tilde{\varphi}, \gamma) \in \Psi_0 \times \Gamma$, the relation:

$$\varrho \in \mathscr{R}^\pi(\gamma, w), \quad \mathscr{R}^{\pi(\cdot)}: \Gamma \times W \to 2^{W'}, \tag{4.6}$$

will be called a macroconstitutive relation generated by a microconstitutive relation $\tilde{\varphi}$, $\tilde{\varphi} \in \Psi_0$, provided that $\pi = \pi(\tilde{\varphi})$. It is a macroidealisation of physical situation described by microconstitutive relation given by (4.2). It is important that $\mathscr{R}^\pi(\gamma, w)$ is a closed set in W' but not necessary bounded. Microconstitutive relations $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ will be considered as nondiscernible if they generate the same macroconstitutive relation. Equality of classes $\pi(\tilde{\varphi}_1)$ and $\pi(\varphi_2)$ is equivalent to nondiscernibless of microconstitutive relations φ_1 and φ_2. Introducing parameters t and $\gamma^{(t)}$ it can be formulate the following proposition.

Proposition. For every microconstitutive relation $\tilde{\varphi}$, $\tilde{\varphi} \in \Psi_0$, there exists in $\mathfrak{M}$ a macroconstitutive relation $\mathscr{R}_t^{\pi(\varphi)}(\gamma^{(t)}, \cdot)$ generated by $\tilde{\varphi}$, i.e. there exists in $\mathfrak{M}$ a set of reactions, closed in W', uniquely determined by $\gamma(t), \dot{\gamma}(t)$ and $\gamma^{(t)}$. A relation $\mathscr{R}_t^{\pi(\tilde{\varphi})}(\gamma^{(t)}, \cdot)$ not depend on a choice of a microconstitutive relation from the class $\pi(\tilde{\varphi})$, i.e. it is the same for each pair of microconstitutive relations. So (4.2) implies:

$$\varrho(t) \in \mathscr{R}_t^{\pi(\tilde{\varphi})}(\gamma^{(t)}, \gamma(t), \dot{\gamma}(t)). \tag{GD}$$

Above proposition will be considered as the general principle of determinism and the family of multifunctions:

$$\Gamma \ni \gamma \to \Delta_t^{\gamma^{(t)}}(\gamma) \equiv \{w \in W\colon (\gamma, w) \in \mathrm{dom}\mathscr{R}_t^{\pi(\tilde{\varphi})}(\gamma^{(t)}, \cdot)\}, \tag{4.7}$$

where:

$$\mathrm{dom}\mathscr{R}_t^{\pi(\tilde{\varphi})}(\gamma^{(t)}, \cdot) \equiv \{(\gamma, w) \in \Gamma \times W\colon \mathscr{R}_t^{\pi(\tilde{\varphi})}(\gamma^{(t)}, \gamma, w) \neq \Phi\},$$

will be formed constraints. We will describe below physical situations for which constraints (4.7) do not depend on the history $\gamma^{(t)}$, $\gamma^{(t)} \in H_t$. So, we shall also define $\Delta_t(\gamma) \equiv \Delta_t^{\gamma^{(t)}}(\gamma)$. From now on and from Eq. (4.4) we conclude that evolutions $\gamma(\cdot) \in \mathscr{P}$, satisfying for every $t \in R$ the condition $\dot{\gamma}(t) \in \Delta_t(\gamma(t))$, exist. So, for each $t \in R$ and $\gamma \in \Gamma$ the set $\Delta_t(\gamma)$ is the set of all rates $\dot{\gamma}$ of state γ at the time t. However, macroconstitutive relations as well as constitutive relations from the set Φ not necessary have physical sense.

5. From the general principle of determinism to the principle of determinism for constitutive relations with constraints

The formalism presented in Sec. 4 leads from microconstitutive relations satisfying the classical principle of determinism (D) to the macroconstitutive relations satisfying the general principle of determinism (GD). The idea of such passage is in splitting the set Ψ_0 of microconstitutive relations into disjointed classes. To every class is assigned the value of the operation $\tilde{\varphi} \to \mathscr{R}_t^{\pi(\tilde{\varphi})}(\cdot)$ on an arbitrary element φ of this class. This mapping is one to one and the operation mentioned above is additive if at least one from the components is standard, i.e.:

$$\mathscr{R}^{\pi(\tilde{\varphi}+{}^*\psi)}(\cdot) = \mathscr{R}^{\pi(\tilde{\varphi})}(\cdot) + \mathscr{R}^{\pi({}^*\psi)}(\cdot).$$

The operation $\Phi \to {}^*\Phi$ together with the choice of the set Φ of constitutive relations leads to the set ${}^*\Phi$. The choice of the operation $\Phi \to {}^*\Phi$ seems to be natural, because Φ and ${}^*\Phi$ represent the same physical object in different structures $\mathfrak{M}$ and ${}^*\mathfrak{M}$ respectively. The restriction of the considerations to the set Ψ_0, which is the domain of the operation $\tilde{\varphi} \to \mathscr{R}_t^{\pi(\tilde{\varphi})}(\cdot)$, has a character of a regularisation assumtion and is made only for securing mathematical correctness of the proposed approach. Now the question arise: What constitutive relations already known in mechanics can be obtained on that way from a certain microconstitutive relation $\tilde{\varphi}$, $\tilde{\varphi} \in \Psi_0$?

Answer yes to this question can be easily obtained for relations satisfying the classical principle of determinism (D) by setting $\tilde{\varphi} := {}^*\psi$ where $\psi: \Gamma \times W \to W'$ is a function for which every element of the family $\{\mathrm{dom}\,\psi(\gamma, \cdot): \gamma \in \Gamma\}$ is open. It is not so easy to obtain a result related to the question for more wide class of constitutive relations. In Sec. 6 we shall obtain results for certain special cases of constitutive relations, namely we shall found solutions to the following problem:

Problem. Let t be a fixed time instant, $t \in R$, and let be given:

(i) constraints $\Gamma \ni \gamma \to \Delta_s(\gamma) \subset W$, $s \in R$,

(ii) the family of functions $\psi_s: \Gamma \times W \times H_s \to W$, $s \in R$, sufficiently regular and that for every $s \in R$ and for every pair $(\gamma, \gamma^{(s)}) \in \Gamma \times H_s$ inclusion:

$$\Delta_s(\gamma) \subset \mathrm{dom}\,\psi_s(\gamma, \cdot, \gamma^{(s)})$$

holds. We are to find a microconstitutive relation which generate the macroconstitutive relation:

$$\varrho(t) \in \psi_t(\gamma(t), \gamma(t), \gamma^{(t)}) + N_{\Delta_t(\gamma(t))}(\dot{\gamma}(t)). \tag{5.1}$$

In Eq. (5.1) $N_{\Delta_t(\gamma(t))}(\gamma(t))$ is a cone, normal to the set $\Delta_t(\gamma(t))$ in a point $\dot{\gamma}(t) \in \Delta_t(\gamma(t))$, defined as follows. Let $\Delta \subset W$ and $w \in W$. First we define a cone tangent to Δ at a point w, setting, [8]:

$$T_\Delta(w) \equiv \liminf_{\substack{\Delta \ni \bar{w} \to w \\ t \downarrow 0}} t^{-1}(\Delta - \bar{w}),$$

where lim inf is taken in the Hausdorff sense [4, p. 147].

A cone normal to Δ is the set defined by:

$$N_\Delta(w) \equiv \{\varrho \in W': \langle \dot{\gamma}, \varrho \rangle \geqslant 0, \bar{w} \in T_\Delta(w)\}.$$

Note that if Δ is a closed set in a separable Banach space (and hence in all special cases examined in Sec. 6) then, [8]:

$$N_\Delta(w) = \{\varrho \in W': \varrho/\|\varrho\| \in \partial d_\Delta(w)\} \cup \{0\},$$

where ∂d_Δ is the subgradient of the function $d_\Delta: W \to R_+$ defined by:

$$d_\Delta(w) \equiv \inf\{\|w - \bar{w}\|: \bar{w} \in \Delta\}.$$

A solution to the aforemention problem can be given by an arbitrary microconstitutive relation which generates a macroconstitutive relation satisfying the following principle of determinism.

The principle of determinism for constitutive relations with constraints. The reaction $\varrho(t)$ of the system at the time t is determined by a history $\gamma^{(t)} \in H_t$, up to the time t by a state $\gamma(t)$ and by a rate $\dot{\gamma}(t)$ with an accuracy to an additive term ϱ, $\varrho \in W'$, having nonnegative power:

$$\langle \dot{\gamma}, \varrho \rangle \geqslant 0, \tag{5.2}$$

on every rate $\dot{\gamma}$, $\dot{\gamma} \in W$, admissible by constraints, i.e. on every rate belonging to the set $\Delta_t(\gamma(t))$.

In the forementioned principle of determinism the condition (5.2) can be changed by the alternative condition:

$$\langle \dot{\gamma}, \varrho \rangle = 0, \tag{5.2.1}$$

provided that for every $t \in R$ and every $\gamma(\cdot) \in \mathscr{P}$ the set $\Delta_t(\gamma(t))$ is a certain linear space.

6. Special cases

6.1. Firstly let us assume that: 1° Γ is an open set in a certain linear space $\tilde{W}$ for which $\dim \tilde{W} = \dim W < +\infty$, 2° constraints are holonomic, i.e. for every $t \in R$ equality:

$$\Delta_t(\gamma) = T_{[\Delta_t]}(\gamma), \gamma \in [\Delta_t], \tag{6.1}$$

where:

$$[\Delta_t] \equiv \{\gamma \in \tilde{W}: \Delta_t(\gamma) \neq \Phi\}, \tag{6.2}$$

holds. Moreover let for every $\dot{\gamma} \in \Delta_t(\gamma)$ equality:

$$N_{\Delta_t(\gamma)}(\dot{\gamma}) = N_{[\Delta_t]}(\gamma) \tag{6.3}$$

holds. Then it can be proved that, [10], there exists a microconstitutive relation $\tilde{\varphi} \in \Psi_0$ which generates the following macroconstitutive relation:

$$\varrho(t) \in \psi_t(\gamma(t), \dot{\gamma}(t), \gamma^t) + N_{[\Delta_t]}(\gamma).$$

This result is equivalent to the principle of determinism stated below.

Principle of determinism for constitutive relations with holonomic constraints in spaces of finite dimension. The reaction $\varrho(t)$ of the system at the time t is determined by a history $\gamma^{(t)}$ of the system up to time t, by a state $\gamma(t)$ and by a rate $\dot{\gamma}(t)$ with an accuracy to an additive term having nonnegative power:

$$\langle \dot{\gamma}, \varrho \rangle \geqslant 0 \tag{6.4}$$

on every rate $\dot{\gamma}$, $\dot{\gamma} \in W$, admissible by constraints, i.e. on every rate belonging to the set $T_{[\Delta_t]}(\gamma(t))$.

As before in the forementioned principle of determinism the condition (6.4) can be changed by the alternative condition

$$\langle \dot{\gamma}, \varrho \rangle = 0 \tag{6.5}$$

provided that for every $t \in R$ and every $\gamma(\cdot) \in \mathscr{P}$ the set $T_{[\Delta_t]}(\gamma(t))$ is a certain linear space.

6.2. Now assume that: 1° Γ is a certain Riemanian manifold and 2° the set $\Delta_t(\gamma(t))$, for every $(t, \gamma(\cdot)) \in R \times \mathscr{P}$ is a conformal image of a non-empty closed convex set in R^n or a diffeomorphici mage of a closed set in R^n with C^1-boundary. Then it can be proved that, [10], there exists a microconstitutive relation $\tilde{\varphi} \in \Psi_0$ which generates the following macroconstitutive relation:

$$\varrho(t) \in \psi_t(\gamma(t), \dot{\gamma}(t)) + N_{\Delta_t(\gamma_t)}(\dot{\gamma}(t)).$$

This result is equivalent to the principle of determinism for constitutive relations with constraints in its general form stated in Sec. 5 provided that Γ is a Riemanian manifold.

6.3. At last let us assume that (D) has a form:

$$\varrho(t) = \sigma'(\gamma(t)),$$

where $\sigma: \Gamma \to R$ is a certain function Gateaux differentiable in every point of the set Γ which is assumed to be an open subset of a certain separable Hilbert space $\tilde{W}$. Then the spaces $\tilde{W}$ and W are isomorphic and will be identified below. Moreover let us assume that constraints are holonomic, i.e. that equalities (6.1), (6.2) and (6.3) holds. Then it can be proved that, [10], there exists a microconstitutive relation $\tilde{\varphi} \in \Psi_0$ which generates the following macroconstitutive relation:

$$\varrho(t) \in \varepsilon'(\gamma(t)) + N_{[\Delta_t]}(\gamma(t)),$$

where $\varepsilon: \Gamma \to R$ denote the known Gateaux differentiable function, $[\Delta_t]$ is assumed to be a non-empty convex closed set. This result is equivalent to the following principle of determinism:

The principle of determinism for potential constitutive relations with holonomic constraints in Hilbert spaces. The reaction $\varrho(t)$ of the system at the time t is determined by a state $\gamma(t)$ of the system with an accuracy to an additive term ϱ having nonnegative power $\langle \dot{\gamma}, \varrho \rangle \geqslant 0$ on every rate $\dot{\gamma}$, $\dot{\gamma} \in W$, admissible by constraints, i.e. on every rate belonging to the set $T_{[\Delta_t]}(\gamma(t))$.

As before in the forementioned principle of determinism inequality $\langle \dot{\gamma}, \varrho \rangle \geqslant 0$ can be changed by the alternative condition (6.5) provided that for every $t \in R$ the set $T_{[\Delta_t]}(\gamma(t))$ is a certain linear space.

7. Final remarks

In the paper the following results are obtained:

(i) An approach of formulating new constitutive relations of mechanics starting from the known relations. The known relations satisfy the classical principle of determinism.

(ii) It is proved that, in the proposed approach, constitutive relations with constraints are special cases of constitutive relations without constraints.

(iii) The principle of determinism for constitutive relations with constraints is deduced from the classical principle of determinism, where no constraints are taken into account.

(iv) A generalisation of some topological concepts of nonstandard analysis to analogical concepts in L'-spaces is discussed.

(v) It is proved that the standard operation in L'-space of closed subsets of a regular topological space T coincides with the standard part operation of closed sets in a topological space T.

Results (ii) and (iii) can be generalized without difficulties for more wide class of constitutive relations than that described in the paper. This generalisation is related to the relations in which the reaction of the system depends on fields in RHS of (D) as well as on elements of a certain fibre bundle, [5], and to the relations in which (D) is replaced by

$$\varrho(t) = \tilde{\varphi}_t(\tilde{\gamma}(t), \dot{\tilde{\gamma}}(t), \tilde{\gamma}^{(t)}),$$

where $\tilde{\gamma}(\cdot) \equiv (\gamma(\cdot), \Theta(\cdot))$ is a pair of an evolution $\gamma(\cdot) \in \mathscr{P}$ and a temperature-field $\Theta(\cdot)$. In this case, applying the method proposed in the paper, we are able to formulate thermo-mechanical constraints, [13].

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Р е з ю м е

СВЯЗИ В МЕХАНИКЕ ТВЁРДОГО ТЕЛА. ПРИМЕНЕНИЕ НЕСТАНДАРТНОГО АНАЛИЗА

В статьи предложено метод формулировки новых конститутивных соотношении механики, которого исходным положением являются известные определяющие (конститутивные) соотношения. Метод основан на понятиях нестандартного анализа. Применяя предложенный метод доказано, что принцип детерминизма для конститутивных соотношений со связами можна вывести из классического принципа детерминизма, в котором связи отсуствуют.

S t r e s z c z e n i e

WIĘZY W MECHANICE CIAŁA STAŁEGO. ZASTOSOWANIE ANALIZY NIESTANDARDOWEJ

W pracy zaproponowano metodę formułowania nowych relacji konstytutywnych ze znanych relacji konstytutywnych. Wykorzystano w niej efektywnie pojęcia analizy niestandardowej. Stosując powyższą metodę wykazano, że zasada determinizmu dla relacji konstytutywnych z więzami może być otrzymana z zasady determinizmu dla relacji konstytutywnych bez więzów.

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