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DYNAMIC OF THE MATERIAL BODY WITH VARIABLE MASS

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1. Introduction

The paper contains the analysis of the material deformable body of with the mass is growing with time. Such a body may be the model of the real constructions being in the course of assembly. This study is an attempt of description of the dynamic process in the aforementioned material system. For example, in the figure 1 are shown three engine-

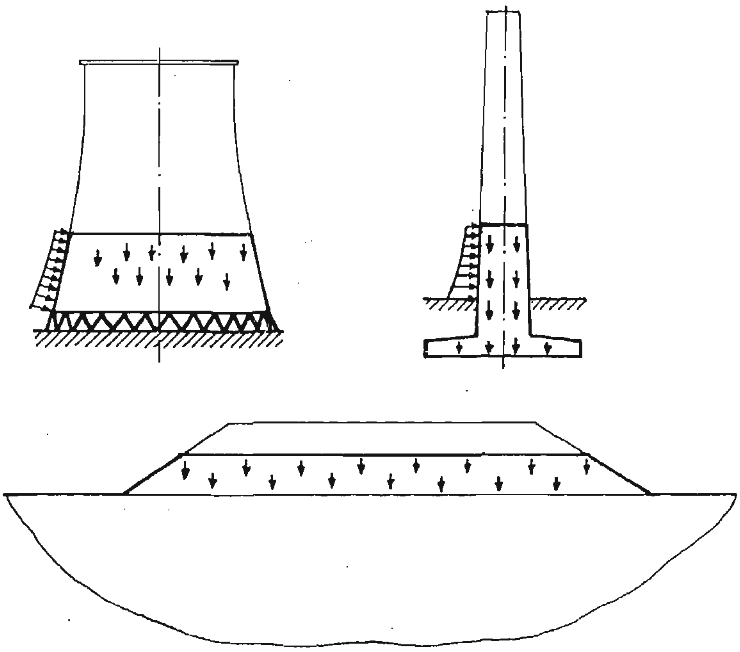

Fig. 1.

ering objects in the course of erection. It is easy to notice, that there are two factors, which are not usually taken into consideration in the classical analysis of the construction:

a) the load as well as the deformation of the construction take place already in the course of erection of the object,

b) using the material which undergoes the ageing (for example: concrete) we deal with the anisotropic body.

In the literature of the subject there is small amount of the papers devoted to that problem. There is only monograph of Arutunian and Kolmanowski [1] where one can find an extensive discussion of that problem together with numerical examples. In principle, the authors confined themselves to the discussion of the linear, viscoelastic problem, concentrating on the models of the ageing body. This monograph is a recapitulation of the earlier works of these authors as well as others [2, 3, 4, 5, 6, 7].

This study is an attempt of the construction of the model of the material body with variable mass treated as nonlinear one, at the same time, the starting point is classical problem, i.e. dynamic process of the material body with constant mass.

The paper presents the definition of the body model, there is description of the motion, the measures of strain and stress as well as forces acting on the body. Next there is proposal of the modified principled of behaviour, which supplemented with constutive relations create the basic system of the model relations. It was proved, that all relations to be derived coincide with classical equations for dynamic process in the case of limiting transition from the body with growing mass to the body with constant mass.

2. Basic definition and assumptions

In Euclidean space $\boldsymbol{E}$ was defined an arbitrary system of coordinates $\{x^i\}$ interrelated with Cartesian coordinates $\{z^i\}$ by transformation:

$$x^i = x^i(z^1, z^2, z^3), \quad z^a = z^a(x^1, x^2, x^3), \tag{1}$$

which is further called the system of the spatial coordinates.

The natural basic of the system $\{x^i\}$ are vectors:

$$g_i(x) = z^a_{,i}(x) e_a, \tag{2}$$

metric tensor is in the form:

$$g_{ij}(x) = g_i(x) g_j(x) = z^a_{,i} z^b_{,j} \delta_{ab}. \tag{3}$$

In the same space was defined second system of coordinates $\{X^I\}$, which is called the system of the material coordinates:

$$X^I = X^I(Z^1, Z^2, Z^3), \quad Z^A = Z^A(X^1, X^2, X^3), \tag{4}$$

with the natural basic:

$$G_I(X) = Z^A_{,I}(X)\, e_A, \tag{5}$$

and the metric tensor:

$$G_{IJ}(X) = G_I(X)\, G_J(X) = Z^A_{,I} Z^B_{,J}\, \delta_{AB}. \tag{6}$$

Cartesian systems $\{z^a\}$ and $\{Z^A\}$ are identical.

The following definition of the material system (material body) with increasing mass have been introduced:

Def. 1. The material system is a set B composed of the bodies $A_I, A_J, A_K, \ldots$, being Boolean algebra with relations of the alternation $\vee$, conjuction $\wedge$ and arrangement $<$.

Def. 2. The body with increasing mass is called a material subsystem B^* if it is the subset of the material system $B^* \subset B$ and if the elements of B^* are the bodies $B_t \in B^*$ with the following structure:

a) B_t body is the set of elements P called material points. Subscript t refers to the moment t under consideration, and for every $t_2 > t_1, B_{t1} \subset B_{t2}$.

b) Every element $B_t \in B^*$ can be mapped onto Euclidean space E, it means that at every moment t the configuration $\varkappa_t$ of the body B_t in E is given.

c) In the space E to each body configuration B_t is assigned Borel measure on all subsets of B_t set.

d) For every material point $P \in B_t$ the specified system of the constitutive equations defining the material is fulfilled.

The above definition is an extension of the known definition of the material system [8]. According with the postulate b) there is possibility of the assignment to the elements $P \in B_t$ of the definite place in the space E, what was written:

$$X = \varkappa_t(P), \qquad X \in E. \tag{7}$$

It is assumed that the mapping (7) is so choosen, that for the same element $P \in B_{t1}$ and $P \in B_{t_2}$, $\varkappa_{t1}(P)$ should equal to $\varkappa_{t2}(P)$.

The aforementioned definition of the body with increasing mass one can interpret twofold:

— as a set of bodies B_t with constant mass, at the same time, the mapping of the points of these bodies $X = \varkappa_t(P)$ is so choosen that for the successive moments $t_1 < t_2 < < t_3 \ldots$ their pictures in E overlap respectively $\varkappa_{t1}(P) = \varkappa_{t2}(P) = \varkappa_{t3}(P) \ldots$

— as a one body, whose picture in E evolves in time, i.e. it constitutes differentiable manifold in the space with variable boundaries.

In the later passage of the paper the second interpretation was used as being much closer to the physical interpretation of the problem.

Evolution of the material body is a determined process, i.e. there is known function defining the instant when the element P becomes an element of the material body. This function is called the function of the mass increment $\tau(\varkappa_t(P)) = \tau(X)$ and was described as follows:

Def. 3.:

a) Function $\tau(X)$ takes the values from the time interval $[t_0, T]$ and it means the instant t of the particle joining with the coordinate X to the material body. It the instant $t = T$ the body reaches its nominal mass.

b) Function $\tau(X)$ is the continuous function of the class C^1 on the finite number of the subdomains Ω_t. On the boundaries of these subdomains there can be discontuities of the function $\tau(X)$ itself or its derivatives. At the same time it is assumed that the boundaries of the subdomains may be only the equiscalar surfaces of the function $\tau(X)$.

c) Function $\tau(X)$ has been choosen in such a way that at every moment of time

$t \in [t_0, \infty)$ there is ensured material continuity of the body — there are no relative extremum at each of the subdomains.

d) Equiscalar surfaces $\tau(X)$ are smooth ones, coinciding with some part of the domain boundary Ω_t.

The definition written above enables us to formulate the problem. From the engineering point of view, the goal is to build-up and to describe the motion of the material body taking up determined shape in the space. It is assumed that the body shape to be designed in the space E assumes the configuration $\mathcal{K}$ and occupies the domain Ω (fig. 2). On this space there is defined function $\tau(X)$ which allows for explicit determination of that part of the domain $\Omega_t \subset \Omega$ which is occupied by the body for given instant t. At instant t for every $X \in \Omega_t$ the relation $\tau(X) \leqslant t$ is fulfilled. Configuration $\mathcal{K}$ (further called the initial reference configuration) and the function $\tau(X)$ in the explicit way determine each of the configurations $\mathcal{K}_t$ of the body B_t in E.

The body domain Ω_t is limited by the surface S_t and

$$S_t = S_t^1 \cup S_t^2 \cup S_t^3, \tag{8}$$

where S_t^1 is the surface with assigned kinematic boundary conditions, S_t^2 is the surface with assigned kinetic boundary conditions. Surface S_t^3 is uncloaded, equiscalar surface of the function $\tau(X)$, the evolving surface.

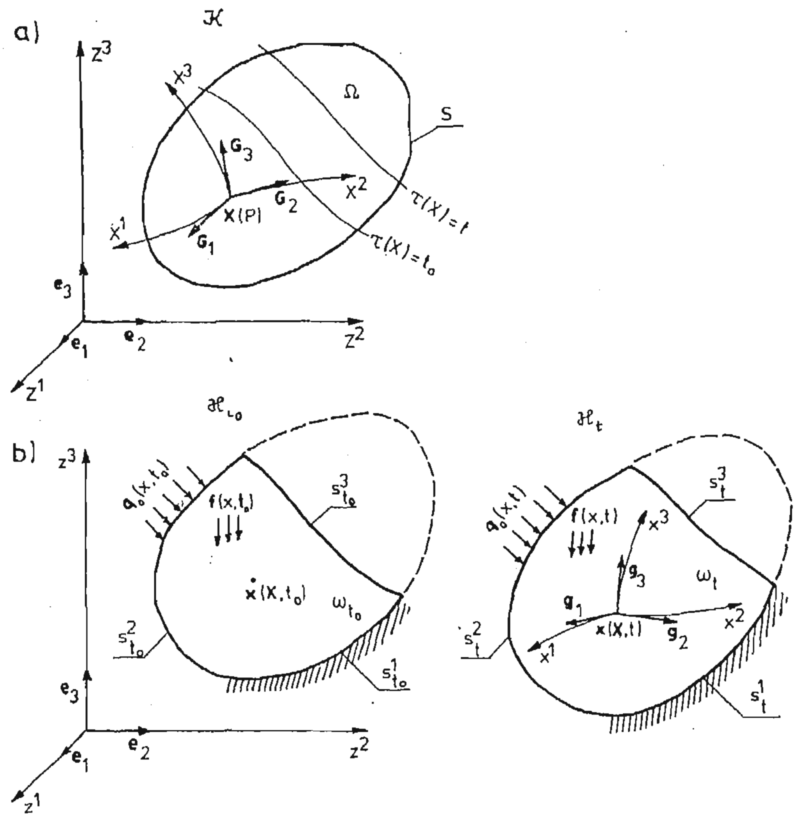

Fig. 2.

The body is loaded by the body forces $f(x, t)$ as well as by the surface forces $q_0(x, t)$ on S_t^2. Every element with coordinate X at instant t has attributed: mass density $\varrho_0(X)$ and the initial speed $v_0(X)$. It is assumed that at the moment $t = \tau(X)$ at point X the deformation equals to zero.

3. Motion. State of strain

In the course of the motion the material body at instant t occupies the actural configuration $\varkappa_t$. Each of the material points $P = \mathscr{K}^{-1}(X)$ takes up the position $x(P)$. The set of all these configurations we call the body motion, and it is written:

$$x^i = \varkappa_t^i(P) = \chi^i(X, t), \tag{9}$$

where $\chi(...)$ is the deformation function, $X \in \Omega_t$, $t \in [t_0, \infty)$.

Equation (9) describes the motion with respect to the reference configuration $\mathscr{K}$. The mapping $x \leftrightarrow X$ is explicit ($j \equiv \det(\partial x^i/\partial X^I) \neq 0$), thanks to that there is inverse relation:

$$X^I = \chi^{-1I}(x, t). \tag{10}$$

Speed and accleration of the point X was written in the form

$$\begin{aligned} v^i(x, t) &= \dot{x}^i(x, t) = \frac{d}{dt}\chi^i(X, t). \\ a^i(x, t) &= \ddot{x}^i(x, t) = \frac{d^2}{dt^2}\chi^i(X, t). \end{aligned} \tag{11}$$

The basic measure of the deformation is deformation gradient. From definition we have [8]:

$$F_I^i(X, t) = \frac{\partial}{\partial X^I}\chi^i(X, t) \quad \text{and} \quad F_i^{-1I}(x, t) = \frac{\partial}{\partial x^i}\chi^{-1I}(x, t). \tag{12}$$

In general, configuration $\mathscr{K}$ is not the configuration of the state of the natural body, thus, the deformation (12) doesn't express the real deformation of the element at point X. At instant $t = \tau(X)$ the deformation gradients equal to:

$$F_I^i(X, \tau) \equiv \mathring{F}_I^i(X), \quad F_i^{-1I}(x, \tau) \equiv \mathring{F}_i^{-1I}(X), \tag{13}$$

and in general are different from g_I^i and $g_i^I(g_I^i(x, X) = g^i G_I, g_i^I(x, X) = G^I g_i)$.

On the other hand, there is not such configuration, where all elements (particles) would be in natural state. To overcome this difficulty it was necessary to introduce the concept of the local reference configuration.

Def. 4. The local reference configuration $\mathscr{K}_\tau$ in the neighborhood of the element X is tangent to the actual configuration $\varkappa_{\tau(x)}$ at the place $x(X, \tau)$.

Definition 4 allows for the construction of the configuration, in which all elements of the body are — according with the previous assumption — in the natural state (undeformed). At instant $t = \tau(X)$ the position of the element X equals to:

$$x^i = \chi^i(X, \tau(X)). \tag{14}$$

Let:

$$\Theta^\alpha(X) = \delta^\alpha_i x^i(X, \tau(X)) = \varkappa^\alpha_{(\tau)}(X),$$
$$X^I(\Theta) = \varkappa^{-1I}_{(\tau)}(\Theta). \tag{15}$$

The system of coordinates $\{\Theta^\alpha\}$ determined by means of eq. (15) will be distinguished by introduction of the local reference configuration $\mathscr{K}_\tau$. To mark off, all quantities described with respect to local reference configuration will possess dash over the letter.

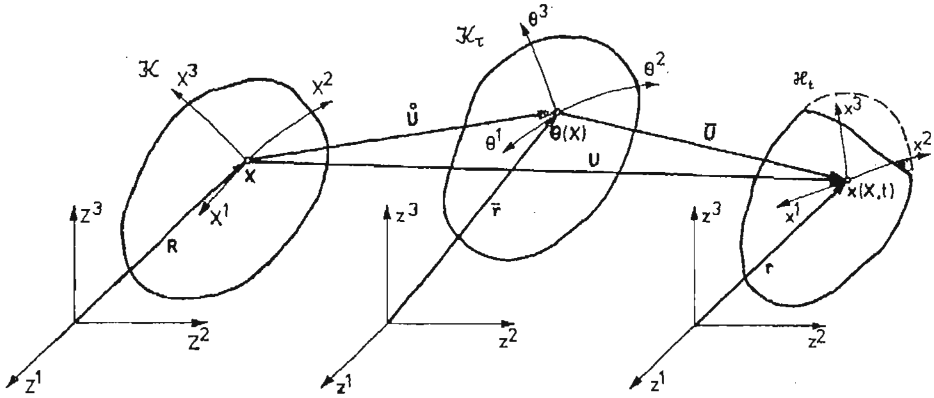

Fig. 3.

In the fig. 3 is shown the material body in configurations $\mathscr{K}$, $\mathscr{K}_\tau$ and $\varkappa_t$. The radiuses-vectors of the point X in the particular configurations are equal to:

$$R(X) = R^I(X) G_I(X),$$
$$r(X, t) = r^i(X, t) g_i(x), \tag{16}$$
$$\bar{r}(\Theta, t) \equiv r(\varkappa^{-1}_{(\tau)}(\Theta), t).$$

Vector of the natural base of the system $\{\Theta^\alpha\}$ are equal to:

$$\bar{G}_\alpha(\Theta) = \left[\frac{\partial}{\partial\Theta^\alpha} r(x, t)\right]_{t=\tau(X)} = \left[\frac{\partial}{\partial\Theta^\alpha} r(\varkappa^{-1}_{(\tau)}(\Theta), t)\right]_{t=\tau(X)} = \mathring{F}^i_I(X)\varkappa^{-1}_{(\tau),\alpha}(\Theta)\mathring{g}_i(X). \tag{17}$$

Putting $K^I_\alpha(\Theta) \equiv \varkappa^{-1I}_{(\tau),\alpha}(\Theta)$ we have:

$$\bar{G}_\alpha(\Theta) = \mathring{F}^i_I(X) K^I_\alpha(\Theta)\mathring{g}_i(X), \tag{18}$$

where $\mathring{g}_i(X) \equiv g_i(x, \tau(X))$, $\mathring{F}^i_I(X) \equiv F^i_I(X, \tau(X))$.

Tensor K is in reality transformation tensor of the configuration $\mathscr{K}$ on configuration $\mathscr{K}_\tau$. The inverse tensor K^{-1} has the form:

$$K^{-1\alpha}_I(X) = \frac{\partial\varkappa^\alpha_{(\tau)}(X)}{\partial X^I_{\varkappa_i}} = \delta^\alpha_i[\mathring{F}^i_I(X) + \mathring{v}^i(X)\tau_{,I}(X)]. \tag{19}$$

The motion and deformation gradient with respect to the local reference configuration are equal to:

$$x^i = \chi^i(\varkappa^{-1}_{(\tau)}(\Theta), t) \equiv \bar{\chi}^i(\Theta, t), \tag{20}$$

$$\bar{F}^i_\alpha(\Theta, t) \equiv \frac{\partial}{\partial\Theta^\alpha}\bar{\chi}^i(\Theta, t). \tag{21}$$

Deformation gradient $\bar{F}$ which is called further the gradient of the relative deformation expresses the real deformation of the material point. It is interrelated with gradient F by the transformation expression:

$$\bar{F}^i_\alpha(\Theta, t) = F^i_I(X, t)K^I_\alpha(\Theta), \tag{22}$$

and at moment $t = \tau(X)$ equals to 1.

$$\bar{F}^i_\alpha(\Theta, \tau) = \mathring{F}^i_I(X)K^I_\alpha(\Theta) = \mathring{g}^i_\alpha(\Theta), \tag{23}$$

where $\mathring{g}^i_\alpha(\Theta) = g^i\bar{G}_\alpha$.

Transformation (15) is continuous, if the function $\tau(X)$ is of the class at least C^1. Otherwise, tensor K as well as the quantities connected with local reference configuration have the surfaces of discontinuities. Discontinuity of function $\tau(X)$ in the real process corresponds with the pause in the erection of the material body while the discontinuity of the gradient $\nabla\tau(X)$ refers to the sudden change of the speed of increment of the body mass, Other measures of the strain are in the form:

— tensors of Green's deformation:

$$\begin{aligned} C_{IJ}(X, t) &= F_{jI}(X, t)F^j_J(X, t),\\ \bar{C}_{\alpha\alpha}(\Theta, t) &= \bar{F}_{j\alpha}(\Theta, t)\bar{F}^j_\beta(\Theta, t) = K^I_\alpha C_{IJ}K^J_\beta, \end{aligned} \tag{24}$$

— tensors of Green-Saint Venant's strain:

$$\begin{aligned} 2E_{IJ}(X, t) &= C_{IJ}(X, t) - G_{IJ}(X, t),\\ 2\bar{E}_{\alpha\beta}(\Theta, t) &= \bar{C}_{\alpha\beta}(\Theta, t) - \bar{G}_{\alpha\beta}(\Theta) = 2K^I_\alpha(E_{IJ} - \mathring{E}_{IJ})K^J_\beta. \end{aligned} \tag{25}$$

4. Principles of conservation

4.1. Principle of the mass conservation. According with the formulated problem the whole mass of the material body changes with time. It is assumed, that at any instant t at the domain ω_t there exists the scalar function $\varrho(x, t)$, which is interpreted as a mass density. At instant $t = \tau(X)$ the particle with coordinate X has assigned the known mass density equal to $\varrho_0(X)$:

$$\varrho(x, \tau(X)) = \varrho_0(X). \tag{26}$$

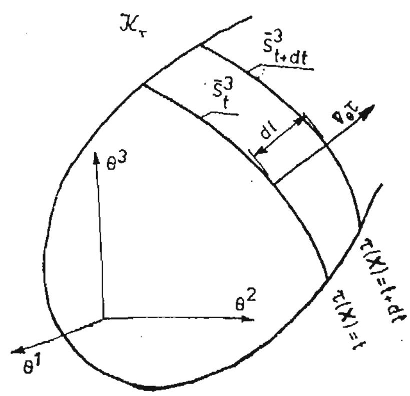

Fig. 4.

The body in local reference configuration $\mathscr{K}_\tau$ occupies at the moment t the domain $\bar{\Omega}_t$ (fig. 4) and the mass increment takes place on the surface $\bar{S}_t^3$. At instant $t+dt$ the domain occupied by the body equals to $\bar{\Omega}_{t+dt}$. The total mass of the body in the time interval $[t, t+dt]$ increases by the mass of the material particles being contained between surfaces $\bar{S}_t^3$ and $\bar{S}_{t+dt}^3$. The measure of the distance between these two surfaces is the segment dl collinear with $\mathrm{grad}_\Theta \tau$.

The principle of the mass conservation can be formulated as follows: the time derivative of the mass increment of the material body equals to the speed of increase of the body mass on the surface $\bar{S}_t^3$, what was written:

$$\frac{D}{Dt}\int_{\omega_t} \varrho(x, t)\,d\omega(x) = \int_{\bar{S}_t^3} \varrho_0(X)\frac{dl}{dt}\,dS. \tag{27}$$

After transformation of the left hand part of eq. (27) we have:

$$\begin{aligned}
\frac{D}{Dt}\int_{\omega_t} \varrho(x, t)\,d\omega &= \frac{D}{Dt}\int_{\bar{\Omega}_t} \varrho(x, t)\bar{J}(\Theta, t)\,d\Omega(\Theta) = \\
&= \frac{D}{Dt}\int_{\bar{\Omega}_t=\mathrm{const}} \varrho(x, t)\bar{J}(\Theta, t)\,d\Omega(\Theta) + \int_{\bar{S}_t^3} \varrho(x, t)\bar{J}(\Theta, t)\frac{dl}{dt}\,dS(\Theta) = \\
&= \int_{\bar{\Omega}_t=\mathrm{const}} \frac{D}{Dt}[\varrho(x, t)\bar{J}(\Theta, t)]\,d\Omega(\Theta) + \int_{\bar{S}_t^3} \varrho_0(X)\frac{dl}{dt}\,dS(\Theta),
\end{aligned} \tag{28}$$

where $\bar{J}(\Theta, t) \equiv \det(\bar{F}_\alpha^i)\sqrt{\det(g_{ij})/\det(\bar{G}_{\alpha\beta})}$.

After substitution (28) to (27) one can get the local form of the principle of the mass conservation:

$$\frac{D}{Dt}[\varrho(x, t)\bar{J}(\Theta, t)] = 0. \tag{29}$$

4.2. Principle of balance of momentum. Momentum $\mathscr{P}$ of the material body occupying at instant t the domain ω_t can be expressed as:

$$\mathscr{P} \equiv \int_{\omega_t} \boldsymbol{v}(x, t)\varrho(x, t)\,d\omega(x). \tag{30}$$

The principle of balance of momentum postulates that instantenous material derivative of momentum equals to the sum of the forces acting on this body, hence:

$$\begin{aligned}
\frac{D}{Dt}\mathscr{P} = \frac{D}{Dt}\int_{\omega_t} \boldsymbol{v}(x, t)\varrho(x, t)\,d\omega(x) &= \int_{\omega_t} \boldsymbol{f}(x, t)\varrho(x, t)\,d\omega(x) + \\
&+ \int_{S_t^1 \cup S_t^2} \boldsymbol{q}_0(x, t)\,ds(x) + \frac{D}{Dt}\int_{\bar{S}_t^3} \boldsymbol{p}_0(X)\,dS(X).
\end{aligned} \tag{31}$$

According with assumption the material particles with cordinates X have assiggned the vector function $\boldsymbol{v}_0(X)$, what is interpret in the real process as a initial speed at the moment, when the particle "joins" the body, i.e. at moment $t = \tau(X)$. The quantity $\boldsymbol{p}_0(X)$

should be interpreted as a density of momentum falling on the surface S_t^3 resulting from "the joining" material particles. After transformation of the left hand of eq. (31) we have:

$$\frac{D}{Dt}\int_{\omega_t} v(x,t)\varrho(x,t)d\omega(x) = \frac{D}{Dt}\int_{\bar{\Omega}_t} v(x,t)\frac{\varrho_0(X)}{\bar{J}(\Theta)}\bar{J}(\Theta)d\bar{\Omega}(\Theta) =$$
$$= \frac{D}{Dt}\int_{\bar{\Omega}_t=\text{const}} v(x,t)\varrho_0(X)d\Omega(\Theta) + \int_{\bar{S}_t^3} \mathring{v}(X)\varrho_0(X)\frac{dl}{dt}dS(\Theta). \tag{32}$$

Segment dl is connected with the speed of displacement of the boundary $\bar{S}_t^3$ by the dependence:

$$\frac{dl}{dt} = \frac{1}{|\text{grad}_\Theta \tau|} = \frac{1}{\sqrt{\tau_{,I}\tau_{,J}\mathring{C}^{-1IJ}}} \equiv \nu_{(n)}(\Theta). \tag{33}$$

After substituting (32) and (33) to (31) and further transformations we arrive at:

$$\int_{\omega_t} a(x,t)\varrho(x,t)d\omega(x) = \int_{\omega_t} f(x,t)\varrho(x,t)d\omega(x) + \int_{S_t^1\cup S_t^2} q_0(x,t)ds(x) +$$
$$+ \int_{S_t^3} [v_0(X) - \mathring{v}(X)]\varrho_0(X)\nu_n(\Theta)ds(x). \tag{34}$$

The integrand in the last integral one can interpret as the load intensity of the surface S_t^3. The difference $v_0(X) - \mathring{v}(X)$ is the diference of the particle speed with coordinate X and the particle of the surface $x(X, \tau) \in s_t^3$ at the instant, when the particle has joined the material body. After denotation:

$$q(x,t) = \begin{cases} q_0(x,t) & \text{for} \quad x \in s_t^1 \cup s_t^2 \\ [v_0(X) - \mathring{v}(X)]\varrho_0(X)\nu_{(n)}(\Theta) & \text{for} \quad x \in s_t^3 \end{cases} \tag{35}$$

the principle of balance of momentum conservation can be written in the standard way:

$$\int_{\omega_t} a(x,t)\varrho(x,t)d\omega(x) = \int_{\omega_t} f(x,t)\varrho(x,t)d\omega(x) + \int_{S_t} q(x,t)ds(x). \tag{36}$$

After analogous as above transformations one can get the equation expressing the principle of balance of moment of momentum conservation in the form:

$$\int_{\omega_t} \frac{D}{Dt} r(x,t)\times v(x,t)\varrho(x,t)]d\omega = \int_{\omega_t} r(x,t)\times f(x,t)\varrho(x,t)d\omega +$$
$$+ \int_{S_t} r(x,t)\times q(x,t)ds(x). \tag{37}$$

5. State of stress. Cauchy equations of the motion

In the Fig. 5 there is shown the material body with distinguished element of volume in reference configurations $\mathscr{K}$ and $\mathscr{K}_\tau$ as well as in actual configuration $\varkappa_t$. At the interface of the surfaces cut-off in the thoughts it is postulated the vectors field of stress $t_{(n)}$, which can be presented in the form of the stress tensors:

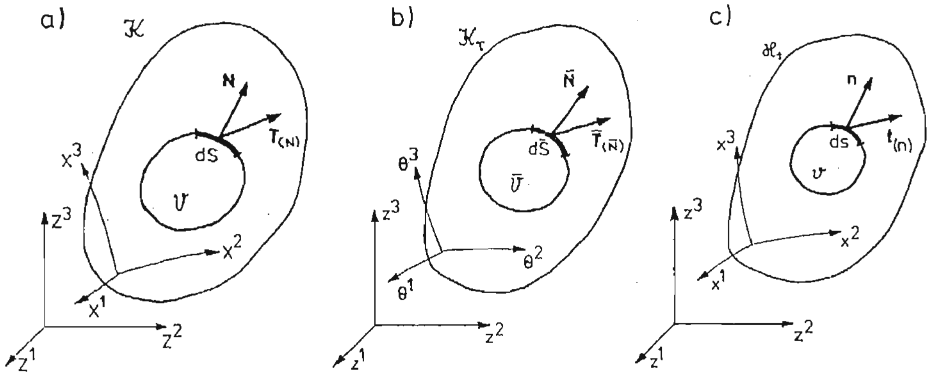

Fig. 5.

$$t_{(n)}ds(x) = t^{kl}(x, t)g_l(x)n_k(x)ds(x) = T^{Il}(X, t)g_l(x)N_I(X)dS(X) = \\ = \bar{T}^{\alpha l}(\Theta, t)g_l(x)\bar{N}_\alpha(\Theta)dS(\Theta), \tag{38}$$

where T^{Il}, $\bar{T}^{\alpha l}$ are Pioli-Kirchhoff stress tensors with respect to $\mathcal{K}$ and $\mathcal{K}_\tau$ configurations respectively.

Cauchy equations of the motion are local forms of the principles of balance of momentum and moment of momentum conservation. Let discuss the domain ω_t limited by the surface s_t. According to def. 3 function $\tau(X)$ or grad $\tau(X)$ can posses definite number of the discontiuties on the surfaces $s_t^1, s_t^2, \ldots, s_t^R$ (Fig. 6). In this way there can take place

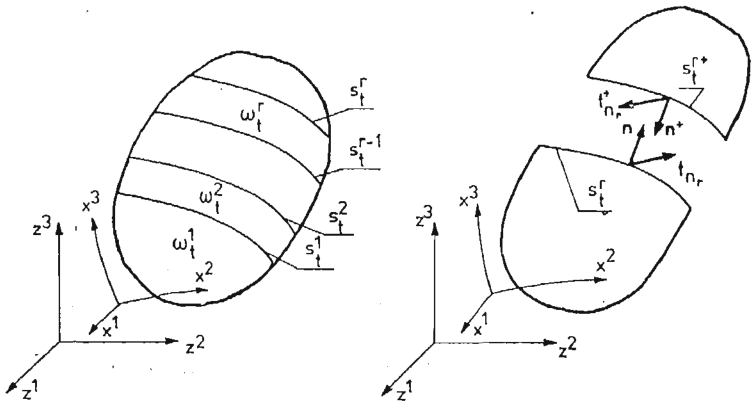

Fig. 6.

the discontiuties of the stress tensor. Dividing the domain ω_t on the separable subdomains ω_t^r as well as taking advantage of the boundary conditions, eq. (36) can be written in the form:

$$\int_{\omega_t} a(x, t)\varrho(x, t)d\omega(x) = \int_{\omega_t} f(x, t)\varrho(x, t)d\omega(x) + \int_{S_t} t_{(n)}ds(x) + \\ + \sum_{r=1}^{R}\Big[\int_{S_t^r} t_{(n_r)}(x, t)ds(x) + \int_{S_t^{r+}} t^+_{(n_r)}(x, t)ds(x)\Big]. \tag{39}$$

The last sum in the eq. (39) is by identity equal to zero, what gives the continuity condition of the stress vector on the surfaces s_t^r:

$$t_{(n_r)}(x, t)+t^{+}_{(n_r)}(x, t) = 0 \quad \text{for} \quad x \in s_t^r. \tag{40}$$

After denoting by $\partial\omega_t^r$ the limiting surface of the domain ω_t^r we have:

$$\int_{\omega_t} a(x, t)\varrho(x, t)d\omega(x) = \int_{\omega_t} f(x, t)\varrho(x, t)d\omega(x)+\sum_{r=1}^{R}\int_{\partial\omega_t^r} t_{(n)}ds(x). \tag{41}$$

Taking advantage of Gauss-Ostrogrodski theorem for each of the terms of sum and carrying-out all necessary transformation, we have:

$$\sum_{r=1}^{R}\int_{\omega_t^r}\{t^{kl}_{;k}(x, t)+\varrho(x, t)\,[f^l(x, t)-a^l(x, t)]\}g_l(x)d\omega(x) = 0. \tag{42}$$

Hence, the first Cauchy equation of the motion together with the continuity condition has the following form:

$$\begin{aligned} &t^{kl}_{;k}(x, t)+\varrho(x, t)\,[f^l(x, t)-a^l(x, t)] = 0 \quad \text{for} \quad x \in \bigcup_{r=1}^{R}\omega_t^r, \\ &(t^{kl}-t^{kl}_{+})n_k = 0 \quad \text{for} \quad x \in \bigcup_{r=1}^{R} s_t^r. \end{aligned} \tag{43}$$

Performing analogous procedure with eq. (37) we get secound Cauchy equation of motion in the form:

$$t^{kl}(x, t) = t^{lk}(x, t) \quad \text{for} \quad x \in \bigcup_{r=1}^{R}\omega_t^r. \tag{44}$$

Cauchy equations of motion (43) and (44) one can get using Pioli-Kirchhoff's stress tensors.

6. Constitutive equations

To describe the dynamic process of the with growing mass it is necessary to formulate the constitutive equations. It is possible to use without any limits the same equations as in the classic problems. Let for example, the body to be built from isotropic elastic material, then the constitutive equations dependences have the form [9]:

$$t^{kl}(x, t) = f^{kl}(\bar{C}_{KL}(\Theta, t)). \tag{45}$$

This study contains the set of equations of the problem of initial-boundary body with growing mass. Full description requires additionally of the formulation of the initial and boundary conditions of the process.

7. Example

An example presented below illustrates the function of the mass increment and the relations between the kinematic quantities for given material body.

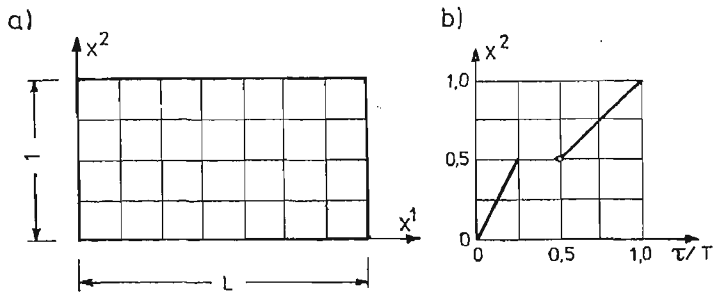

Fig. 7.

A flat rectangular disk of dimensions $L \times 1$ is considered (Fig. 7a). Coordinate systems $\{X^I\}$ and $\{x^i\}$ were assumed as Cartesian ones. Let the function of the mass increment $\tau(X)$ to be in the form:

$$\tau(X^1, X^2) = T_I X^2 = \begin{cases} T_1 X^2 & \text{for} \quad X^2 \leqslant \dfrac{1}{2} \\ T_2 X^2 & \text{for} \quad X^2 > \dfrac{1}{2} \end{cases}, \tag{46}$$

where $T_2 > T_1 > 0$. The initial moment is $t_0 = 0$. It was assumed that $T_2 = T_1 = T$ (see Fig. 7).

As it results from (46) the disk is build in two time intervals $\left[0, \frac{1}{2}T_1\right]$ and $\left[\frac{1}{2}T_2, T_2\right]$. The pause in the course of building of the body corresponds to the time interval $\left(\frac{1}{2}T_1, \frac{1}{2}T_2\right)$. The diagram of function $\tau(X)$ is shown in fig. 7b.

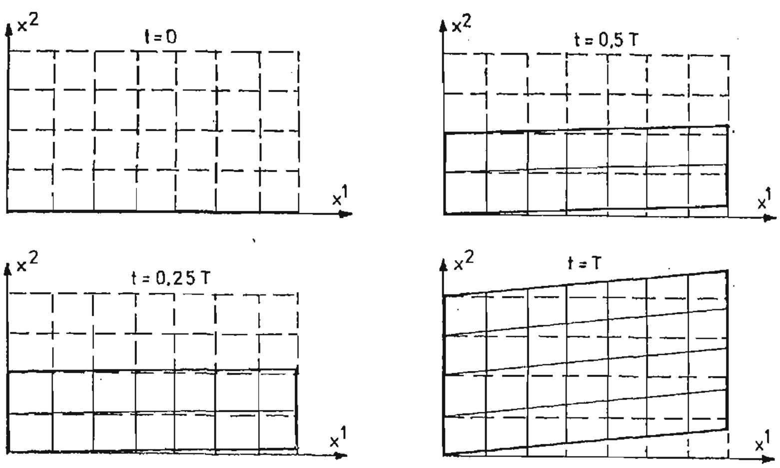

Fig. 8.

The disk is subjected to the "pure" forced shear. Disk motion is described by equations:

$$x = \chi(X, t), \quad \chi^1(X, t) = X^1, \quad \chi^2(X, t) = AtX^1 + X^2, \tag{47}$$

where A — an arbitrary constant.

The successive phases of disk construction as well as its successive configurations are shown in fig. 8.

The local system of the material coordinates is defined by eq. (15):

$$\begin{aligned} \Theta(X) &= \varkappa_{(\tau)}(X), \quad \varkappa^1_{(\tau)}(X) = X^1, \\ & \qquad \varkappa^2_{(\tau)}(X) = AT_i X^1 X^2 + X^2, \\ X(\Theta) &= \varkappa^{-1}_{(\tau)}(\Theta), \quad \varkappa^{-11}_{(\tau)}(\Theta) = \Theta^1, \\ & \qquad \varkappa^{-12}_{(\tau)}(\Theta) = \frac{\Theta^2}{AT_i\Theta^1 + 1}. \end{aligned} \tag{48}$$

The transformation matrix of system $\{X^I\}$ into $\{\Theta^\alpha\}$ is in the form:

$$K(\Theta) = \nabla_\Theta \varkappa^{-1}_{(\tau)}(\Theta), \quad K^I_\alpha(\Theta) = \begin{bmatrix} 1 & 0 \\ -\dfrac{AT_i\Theta^2}{(AT_i\Theta^1+1)^2} & \dfrac{1}{AT_i\Theta^1+1} \end{bmatrix}. \tag{49}$$

Coordinate system $\{\Theta^\alpha\}$ has the surface of discontinuity for $X^2 = \frac{1}{2}$. It is equiscalar surface of the function $\tau(X)$ at the moment of "the pause of building" of the disk. The vectors of the natural basis of the system $\{\Theta^\alpha\}$ as well as the first metric form equal to:

$$\begin{aligned} \bar{G}_1 &= g_1, \\ \bar{G}_2 &= \frac{A^2T_i^2\Theta^1\Theta^2}{(AT_i\Theta^1+1)^2} g_1 + \frac{1}{AT_i\Theta^1+1} g_2, \\ \bar{G}_{\alpha\beta}(\Theta) &= \begin{bmatrix} 1 + \dfrac{A^4T_i^4(\Theta^1\Theta^2)^2}{(AT_i\Theta^1+1)^4} & \dfrac{A^2T_i^2\Theta^1\Theta^2}{(AT_i\Theta^1+1)^3} \\ \text{sym.} & \dfrac{1}{(AT_i\Theta^1+1)^2} \end{bmatrix}. \end{aligned} \tag{50}$$

The motion in relation to the local configuration expresses the equation:

$$\bar{\chi}(\Theta, t) = \chi(X(\Theta), t), \quad \bar{\chi}^1(\Theta, t) = \Theta^1, \quad \bar{\chi}^2(\Theta, t) = At\Theta^1 + \frac{\Theta^2}{AT_i\Theta^1+1}. \tag{51}$$

The strain measures in relation to the local configuration are equal:

— gradient of the relative deformation:

$$\bar{F}^i_\alpha(\Theta, t) = \begin{bmatrix} 1 & 0 \\ At - \dfrac{AT_i\Theta^2}{(AT_i\Theta^1+1)^2} & \dfrac{1}{AT_i\Theta^1+1} \end{bmatrix}, \tag{52}$$

— tensor of the relative deformation:

$$\bar{C}_{\alpha\beta}(\Theta, t) = \begin{bmatrix} 1+\left[At-\dfrac{AT_t\Theta^2}{(AT_t\Theta^1+1)^2}\right]^2 & \text{sym.} \\ \dfrac{1}{AT_t\Theta^1+1}\left[At-\dfrac{AT_t\Theta^2}{(AT_t\Theta^1+1)^2}\right] & \dfrac{1}{(AT_t\Theta^1+1)^2} \end{bmatrix}, \tag{53}$$

— tensor of the relative strain:

$$2\bar{E}_{\alpha\beta} = \begin{bmatrix} \left[At-\dfrac{AT_t\Theta^2}{(AT_t\Theta^1+1)^2}\right]^2 - \dfrac{A^4T_t^4(\Theta^1\Theta^2)^2}{(AT_t\Theta^1+1)^4} & \text{sym.} \\ \dfrac{1}{AT_t\Theta^1+1}\left[At-\dfrac{AT_t\Theta^2}{(AT_t\Theta^1+1)^2}\right] - \dfrac{A^2T_t^2\Theta^1\Theta^2}{(AT_t\Theta^1+1)^3} & 0 \end{bmatrix}. \tag{54}$$

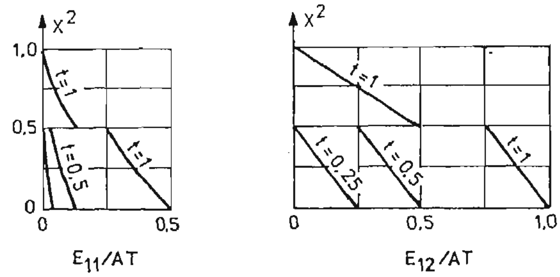

Fig. 9.

All the strain measures in relation to the local configuration on the line $X^2 = \dfrac{1}{2}$ are discrete. For example, fig. 9 presents the diagrams of non-zero values of strain tensor coordinates $\bar{E}$ following their transformation into the system $\{x^i\}$ according to the relation:

$$2\bar{E}_{ij}(\Theta, t) = 2\bar{E}_{\alpha\beta} g_i^\alpha g_j^\beta . \tag{55}$$

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Р е з ю м е

ДИНАМИКА МАТЕРИАЛЬНОГО ТЕЛА С ПЕРЕМЕННОЙ МАССОЙ

В работе сконструировано модель материального тела с переменной массой. Представлено определение такой модели, описано движение, мери деформации, напряжения и силы действующие на тело. Кроме этого предлагаем модифицированные законы сохранения, которые дополненные определяющими уравнениями дают основную систему соотношений модели.

В окончании работы показано пример иллюстрирующий кинематические зависимости модели.

S t r e s z c z e n i e

DYNAMIKA CIAŁA MATERIALNEGO O ZMIENNEJ MASIE

W pracy skonstruowano model ciała materialnego o zmiennej masie. Podano definicję takiego ciała, opisano ruch, miary odkształcenia i naprężenia oraz siły działające na ciało. Następnie postuluje się zmodyfikowane zasady zachowania, które po uzupełnieniu związkami konstytutywnymi dają podstawowy układ relacji modelu. W zakończeniu pracy podano przykład ilustrujący zależności kinematyczne modelu.

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