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COMPARISON OF THE BOUNDARY COLLOCATION AND FINITE ELEMENT METHODS FOR SOME HARMONIC 2D PROBLEMS

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1. Introduction

The analytical solutions to boundary value problems typical of mechanics of continuous media are as a rule possible for simple geometries only, such as circular or rectangular regions. Thus, numerical methods are often the only means for solving boundary value problems of engineering significance. The most widely used methods are the finite difference (FDM) and the finite element (FEM) methods. However, in the recent years we have also witnessed the fast development of the so-called boundary methods, [1 - 4]. Thus, in view if the different approaches now available, it seems necessary to work out procedures for effective comparison of them in order to facilitate their optimal choice in a given situation.

A special case of the boundary methods which will be referred to in the present paper is the boundary collocation method (BCM). The method is not very popular in comparison with other boundary methods (such as the boundary integral method) as it is applicable only to linear sets of differential equations for which some general solutions satisfying the equations inside the region considered are known. An extensive review of BCM as used in linear continuous mechanics is given in [5].

There exist a number of papers which attempt to compare the accuracy of results obtained by BCM against the exact solutions obtained analytically, see [6 - 11], for instance. On the other hand, the performance comparisons of BCM and other approximate methods are not numerous. Shuleshko [12] made comparisons for three different versions of the collocation procedure: (a) the BCM in which the equations are exactly satisfied inside the region but only approximately on its boundary, (b) the internal collocation method in which we satisfy exactly the boundary conditions whereas the equations inside the region are fulfilled approximately, and (c) the mixed collocation method in which all the equations are satisfied in an approximate way only. The comparison was carried out for a torsion problem of a prismatic beam with a rectangular cross-section. The conclusion was that the method (a) was superior to the other approaches.

A comparison of nine approximate methods including BCM was presented in [13] for some thin plate bending problems for which exact solutions were available. Unfortunately, FEM and FDM were not included. As a conclusion the authors classified each of the methods as good, fair or poor depending on eleven selected technical criteria. France [14] compared two versions of BCM in the form of the straightforward boundary collocation method and the overdetermined boundary collocation method with least squares for the case of 2D Laplace equation in the rectangular region. The latter version yielded slightly better results.

The results reported in [15] may be interpreted in favor of BCM as well. For the case of the exact solution to the Laplace equation in the square region with discontinuous boundary conditions five different methods were compared in that paper, including the standard FEM approach and the method of "large singular finite elements", the latter being just a version of BCM based on large elements. This method yielded the most accurace results whereas the FEM performance was very poor.

In [16] the application of "large singular finite elements" to the solution of a torsion problem for a quadrangle, for which no exact solution existed, was proposed. The results were again superior with respect to those obtained by using FEM.

The comparison of BEM and BCM with a special choise of trial functions called by the authors the superposition method was performed in [17]. Nine exact solutions to some plane elasto-static problem were used for comparison. BCM turned out again to yield better results. In [18] some objections as to the results of the paper [17] were raised, but no definite conclusions were formulated.

To the best of authors' knowledge, no paper specifically devoted to the comparison of FEM and BCM has ever been published. Taking into account the popularity of the former method and the simplicity of the latter one, such a comparison seems to desirable. The more so that the current tendency to combine different methods by exploiting their virtues and eliminating the faults, cf. [3 - 4], [19 - 20], may in this way be given an additional perspective.

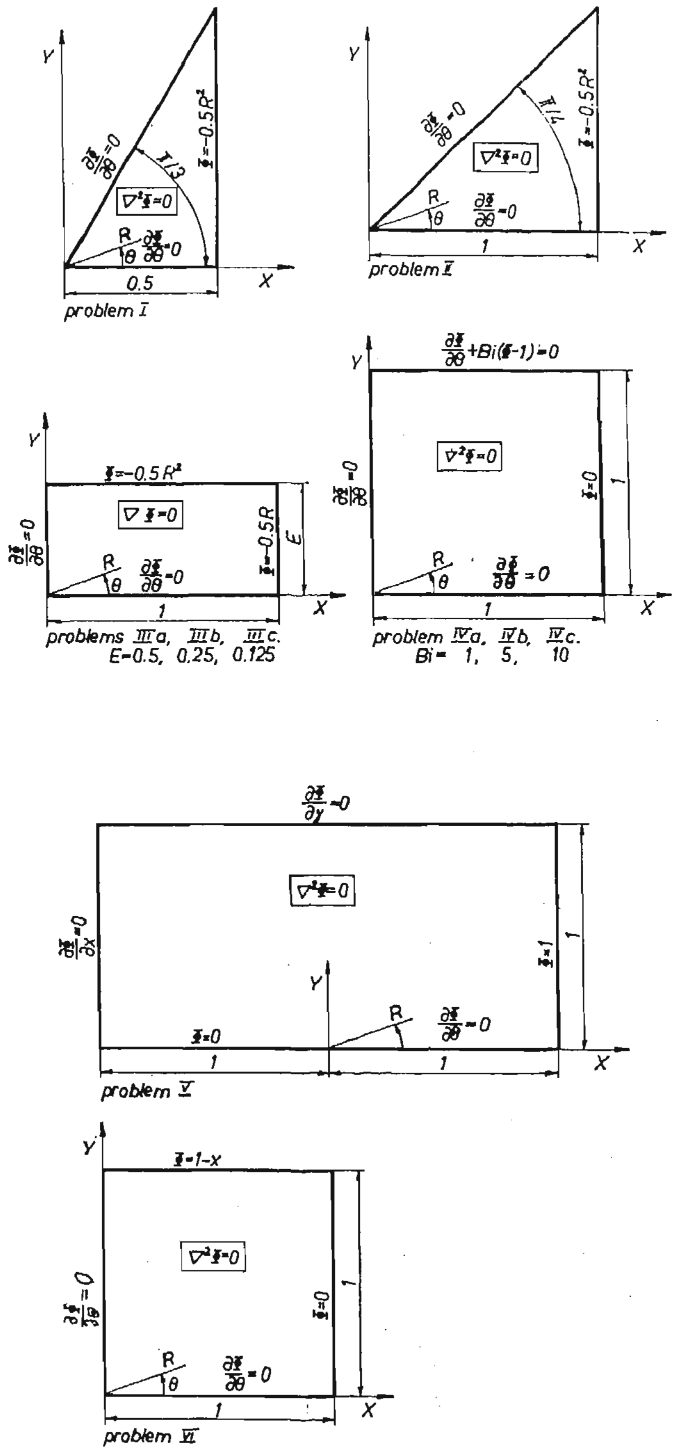
The purpose of this paper is to carry out a through comparison of BCM and FEM. Some harmonic 2D boundary value problems are considered, for which the exact solutions are available. The key question to be posed below reads: which of the two methods yields more accurate results given the same "level of discretization" measured by the number of assumed degrees of freedom.

2. Test problems and the analytical solutions

The problem chosen for this study are as follows, cf. Fig. 1:

Problem I.

$$\nabla^2\Phi = 0 \quad \text{in} \quad 0 < \Theta < \frac{\pi}{3}, \quad 0 < R < \frac{0.5}{\cos\Theta},$$



problem I
0.5
π/3
problem II
1
π/4
problems IIIa, IIIb, IIIc.
E=0.5, 0.25, 0.125
problem IVa, IVb, IVc.
Bi= 1, 5, 10
problem V
problem VI

Fig. 1.

with the boundary conditions:

$$\frac{\partial \Phi}{\partial \Theta} = 0 \quad \text{for} \quad \Theta = \begin{cases} 0, & 0 \leqslant R \leqslant 0.5 \\ \pi/3, & 0 \leqslant R \leqslant 1 \end{cases},$$

$$\Phi = -0.5R^2 \quad \text{for} \quad X = 0.5, \quad 0 \leqslant Y \leqslant \sqrt{3}/2.$$

Problem II.

$$\nabla^2 \Phi = 0 \quad \text{in} \quad 0 < \Theta < \pi/4, \quad 0 < R < 1/\cos\Theta$$

with the boundary conditions:

$$\frac{\partial \Phi}{\partial \Theta} = 0 \quad \text{for} \quad \Theta = \begin{cases} 0, & 0 \leqslant R \leqslant 1 \\ \pi/4, & 0 \leqslant R \leqslant \sqrt{2} \end{cases},$$

$$\Phi = -0.5R^2 \quad \text{for} \quad X = 1, \quad 0 \leqslant Y \leqslant 1.$$

Problem III.

$$\nabla^2 \Phi = 0 \quad \text{in} \quad 0 < x < 1, \quad 0 < Y < E,$$

$$\frac{\partial \Phi}{\partial \Theta} = 0 \quad \text{for} \quad \Theta = \begin{cases} 0, & 0 \leqslant R \leqslant 1 \\ \pi/2, & 0 \leqslant R \leqslant E \end{cases},$$

$$\Phi = -0.5R^2 \quad \text{for} \quad X = 1, \quad 0 \leqslant Y \leqslant E,$$

$$\Phi = -0.5R^2 \quad \text{for} \quad Y = E, \quad 0 \leqslant X \leqslant 1.$$

The values of $E = 0.5$, $E = 0.25$, $E = 0.125$ correspond to subproblems IIIa, IIIb, IIIc respectively.

Problem IV.

$$\nabla^2 \Phi = 0 \quad \text{in} \quad 0 < X < 1, \quad 0 < Y < 1$$

with the boundary conditions:

$$\frac{\partial \Phi}{\partial \Theta} = 0 \quad \text{for} \quad \Theta = \begin{cases} 0, & 0 \leqslant R \leqslant 1 \\ \pi/2, & 0 \leqslant R \leqslant 1 \end{cases},$$

$$\Phi = 0 \quad \text{for} \quad X = 1, \quad 0 \leqslant Y \leqslant 1,$$

$$\frac{\partial \Phi}{\partial Y} + Bi(\Phi - 1) = 0 \quad \text{for} \quad Y = 1, \quad 0 \leqslant X \leqslant 1.$$

The values of $Bi = 1$, $Bi = 5$, $Bi = 10$ correspond to subproblems IVa, IVb, IVc respectively.

Problem V.

$$\nabla^2 \Phi = 0 \quad \text{in} \quad -1 < X < 1, \quad 0 < Y < 1$$

with the boundary conditions:

$$\Phi = 0 \quad \text{for} \quad \Theta = \pi, \quad 0 \leqslant R \leqslant 1,$$

$$\frac{\partial \Phi}{\partial \Theta} = 0 \quad \text{for} \quad \Theta = 0, \quad 0 \leqslant R \leqslant 1,$$

$$\Phi = 1 \quad \text{for} \quad X = 1, \quad 0 \leqslant Y \leqslant 1,$$

$$\frac{\partial \Phi}{\partial Y} = 0 \quad \text{for} \quad Y = 1, \quad -1 \leqslant X \leqslant 1,$$

$$\frac{\partial \Phi}{\partial X} = 0 \quad \text{for} \quad X = -1, \quad 0 \leqslant Y \leqslant 1.$$

Problem VI.

$$\nabla^2 \Phi = 0 \quad \text{in} \quad 0 < X < 1, \quad 0 < Y < 1$$

with the boundary conditions:

$$\frac{\partial \Phi}{\partial \Theta} = 0 \quad \text{for} \quad \Theta = \begin{cases} 0, & 0 \leqslant R \leqslant 1 \\ \pi/4, & 0 \leqslant R \leqslant 1' \end{cases}$$

$$\Phi = 0 \quad \text{for} \quad X = 1, \quad 0 \leqslant Y \leqslant 1,$$

$$\Phi = 1 - X \quad \text{for} \quad Y = 1, \quad 0 \leqslant X \leqslant 1.$$

Problems I, II and III may be referred to some solutions of the Saint-Venant torsion problem, cf. [21], problem IV to some steady state temperature problem, cf. [22], problem V is the so called Motz problem, [23], and problem VI was employed in [25] for comparing FEM and BEM. The exact solutions to all the above problems are given in Tabl. I. The derivatives $\partial\Phi/\partial X$ and $\partial\Phi/\partial Y$ may easily be obtained, if necessary.

Table I. Exact solutions of Problem I - VI

Problem	Function Φ	Reference
I	$\Phi = \frac{1}{3}(X^3 - 3XY^2) - \frac{1}{6}$	[21]
II	$\Phi = -\frac{1}{2}(X^2+Y^2) - \frac{32}{\pi^3}\sum_{n=1,3,}^{\infty} \frac{1}{n^3}(-1)^{\frac{n-1}{2}}\left[1 - \frac{\cosh[n\pi Y/2]}{\cosh[n\pi/2]}\right]\cos(n\pi X/2)$	[21]
III	$\Phi = -\frac{1}{2}(X^2+Y^2) - \frac{32}{\pi^3}\sum_{n=1,3,}^{\infty} \frac{1}{n^3}(-1)^{\frac{n-1}{2}}\left[1 - \frac{\cosh[n\pi Y/2]}{\cosh[n\pi E/2]}\right]\cos(n\pi X/2)$ $E = 0.5;\ 0.25;\ 0,125$	[21]
IV	$\Phi = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2Bi \cos(\mu_n X[\exp(\mu_n Y) + \exp(-\mu_n Y)]}{\mu_n[\exp(\mu_n)(\mu_n + Bi) + \exp(-\mu_n)(Bi - \mu_n)]}$ $Bi = 1;\ 5;\ 10;$	[22] p. 317
V	$\Phi = \sum_{n=1}^{20} a_n R^{(2n-1)/2} \cos[(2n-1)\Theta/2]$ coefficients a_n are given in Table Ia	[24]
VI	$\Phi = \sum_{n=0}^{\infty} \frac{8\cos[(2n+1)\pi X/2]\cosh[(2n+1)\pi Y/2]}{(2n+1)^2\pi^2\cosh[(2n+1)\pi/2]}$	[25]

Table Ia. Coefficients a_n in solution of Problem V

n	a_n
1	0.80232490749016884047
2	0.17531184039017583405
3	0.03447583015889361 87
4	−0.01614243051939626 87
5	0.002880545434045715
6	0.000662109771814473
7	0.00055087468901836
8	−0.00017386598905107
9	0.0000672097568531
10	0.0000307687489651
11	0.000014604603348
12	−0.000006368227833
13	0.00000244129222
14	0.00000106193096
15	0.0000005430244
16	−0.0000002400927
17	0.000000101080
18	0.000000046334
19	0.00000002307
20	−0.00000001059

3. The boundary collocation method

The BCM can be summarized as consisting in using the exact solutions to the governing differential equation(s) of the problem and satisfying the given boundary conditions at a finite number of discrete points along the boundary. The solutions to boundary value problems are used by assuming:

$$\Phi = \sum_{k=1}^{N} X_k \varphi_k(R, \Theta),$$

where $\varphi_k(R, \Theta)$ are trial functions exactly satisfying the 2D Laplace equation and X_k are unknown parameters to be determined from the boundary conditions.

The selection of the trial functions is a crusial factor in using the method. For each b.v. problem we may find trial functions in the literature of differential equations. In this paper, the selection is made on the basis of the general solutions to the Laplace equation expressed in polar coordinates, so that we take:

$$\Phi = A_0 + B_0 \ln R + \sum_{k=1}^{\infty} [(A_k R^{\lambda_k} + B_k R^{-\lambda_k})\cos(\lambda_k \Theta) + (C_k R^{\lambda_k} + D_k R^{-\lambda_k})\sin(\lambda_k \Theta)], \quad (1)$$

where A_0, B_0, A_k, B_k, C_k, D_k and λ_k are unknown constants. Some of the constans will be determined from the boundary conditions.

After introducing the polar coordinate system for each of the problems, there holds the condition:

$$\frac{\partial \Phi}{\partial \Theta} = 0 \quad \text{for} \quad \Theta = 0.$$

This condition is satisfied for:

$$C_k = D_k = 0 \quad \text{for} \quad k = 1, 2, \ldots$$

In all the problems the solution at the origin of the coordinate system has a finite value. Thus:

$$B_k = 0 \quad \text{for} \quad k = 0, 1, 2, \ldots$$

The value of the coefficients λ_k may be found from the boundary condition at Θ = const provided $\Theta \neq 0$, which reads each for particular problems as:

Problem I

$$\frac{\partial \Phi}{\partial \Theta} = 0 \quad \text{for} \quad \Theta = \pi/3 \text{ which yields } \lambda_k = 3k.$$

Problem II

$$\frac{\partial \Phi}{\partial \Theta} = 0 \quad \text{for} \quad \Theta = \pi/4 \text{ which yields } \lambda_k = 4k.$$

Problems III, IV and VI

$$\frac{\partial \Phi}{\partial \Theta} = 0 \quad \text{for} \quad \Theta = \pi/2 \text{ which yields } \lambda_k = 2k.$$

Problem V

$$\Phi = 0 \quad \text{for} \quad \Theta = \pi \text{ which yields } \lambda_k = (2k-1)/2.$$

Using the above results in eq. (1) and confining ourselves to a certain number N of the expansion terms in the solution (1), we proceed by assuming the solution in each particular problem as:

Problem I

$$\Phi = \sum_{k=1}^{N} X_k R^{3(k-1)} \cos[3(k-1)\Theta].$$

Problem II

$$\Phi = \sum_{k=1}^{N} X_k R^{4(k-1)} \cos[4(k-1)\Theta].$$

Problems III, IV and VI

$$\Phi = \sum_{k=1}^{N} X_k R^{2(k-1)} \cos[2(k-1)\Theta].$$

Problem V

$$\Phi = \sum_{k=1}^{N} X_k R^{(2k-1)/2} \cos[(2k-1)\Theta/2]$$

with X , $k = 1, \ldots, N$ being parameters to be determined from the collocation conditions imposed on that part of the boundary, along which the boundary conditions are not yet exactly satisfied. We assume that the collocations points are equally spaced along the boundary, cf. Fig. 2. Imposing the collocation results in a set of linear algebraic equations

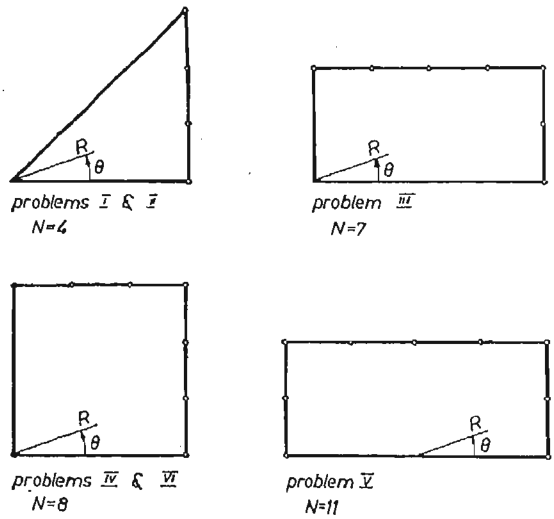
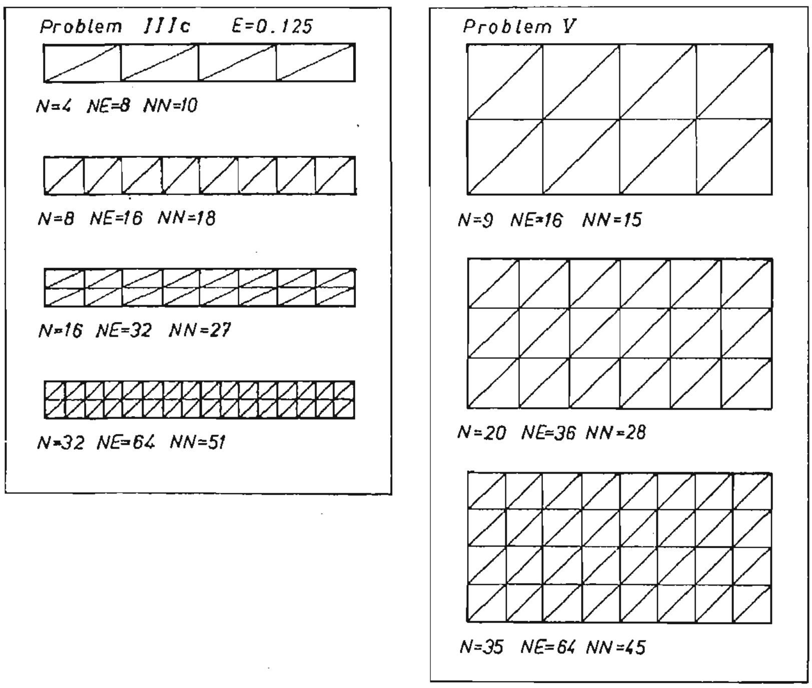
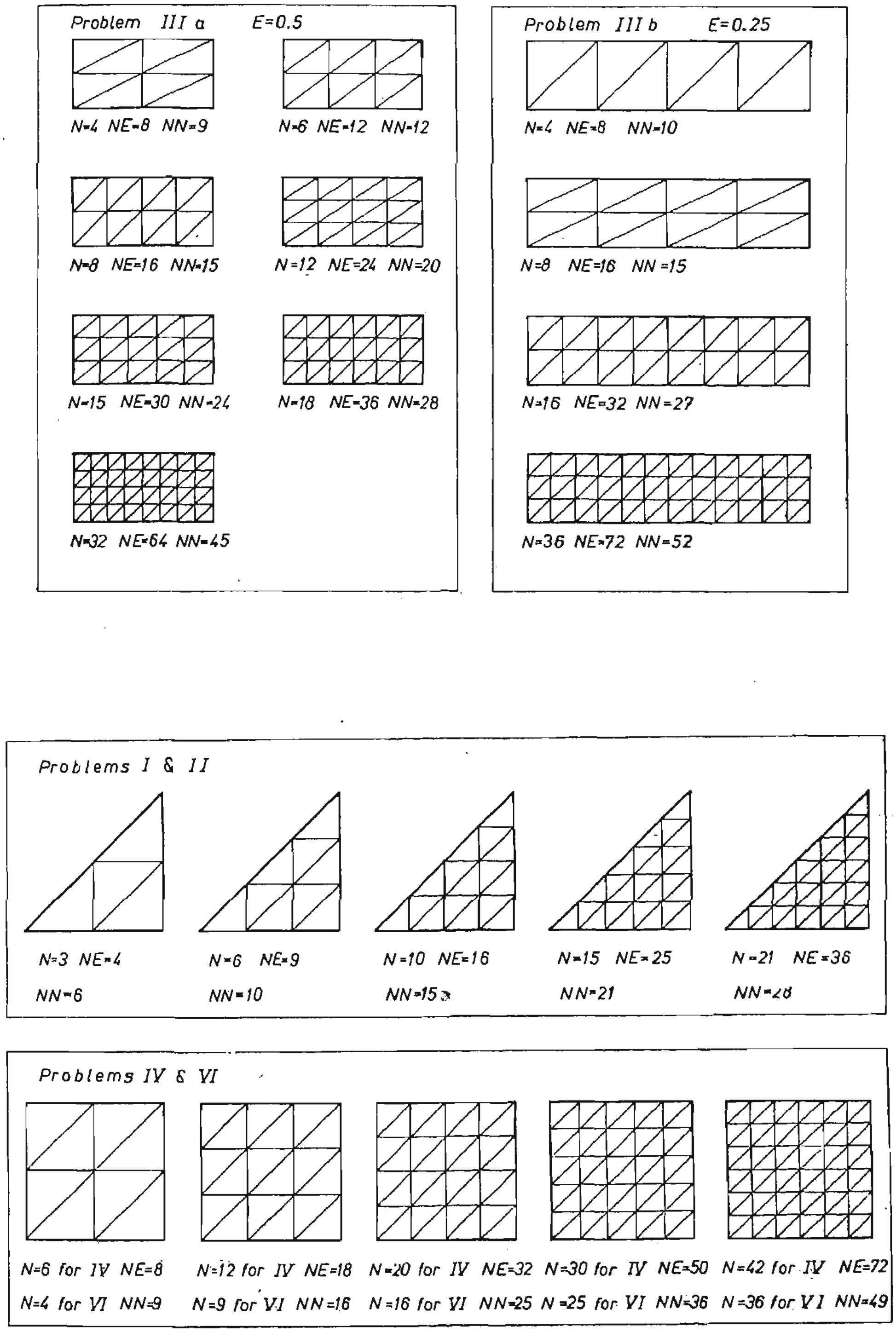


Fig. 2.





Problem III a E=0.5
N=4 NE=8 NN=9
N=6 NE=12 NN=12
N=8 NE=16 NN=15
N=12 NE=24 NN=20
N=15 NE=30 NN=24
N=18 NE=36 NN=28
N=32 NE=64 NN=45
Problem III b E=0.25
N=4 NE=8 NN=10
N=8 NE=16 NN=15
N=16 NE=32 NN=27
N=36 NE=72 NN=52
Problems I & II
N=3 NE=4
NN=6
N=6 NE=9
NN=10
N=10 NE=16
NN=15
N=15 NE=25
NN=21
N=21 NE=36
NN=28
Problems IV & VI
N=6 for IV NE=8
N=4 for VI NN=9
N=12 for IV NE=18
N=9 for VI NN=16
N=20 for IV NE=32
N=16 for VI NN=25
N=30 for IV NE=50
N=25 for VI NN=36
N=42 for IV NE=72
N=36 for VI NN=49

Fig. 3.

for the coefficients X_k. To illustrate this let us just give the explicit form of this equation set for Problem II:

$$\sum_{k=1}^{N} \{R_i^{4(k-1)}\cos[4(k-1)\Theta_i]\}X_k = -0.5R_i^2, \quad i = 1, 2, \ldots, N,$$

where:

$$R_i = \sqrt{1+\frac{(i-1)^2}{(N-1)^2}}, \quad \Theta_i = \mathrm{arctg}\left[\frac{(i-1)}{(N-1)}\right].$$

The number N (i.e. the number of linear equations to be solved) is reffered to for the purpose of comparison with the FEM solutions as the number of degrees of freedom. The linear equation solver used in this study was taken from [26], p. 398 in the form of the Gauss elimination routine.

4. The finite element method

The constant strain triangular elements are used as the basis for the FEM program taken from [26]. The discretization patterns are shown in Fig. 3. The number of degrees of freedom in each case is equal to the number of nodes at which the function Φ is unknown.

5. Error criteria

Two different error criteria have been employed. The first one is based on "global" error measures for Φ and its derivatives which are given by:

$$ER1 = \frac{1}{NP}\sum_{i=1}^{NP} |\Phi_e(X_i, Y_i) - \Phi_a(X_i, Y_i, N)|,$$

$$ER2 = \frac{1}{NP}\sum_{i=1}^{NP} \left|\frac{\partial\Phi_e(X_i, Y_i)}{\partial X} - \frac{\partial\Phi_a(X_i, Y_i, N)}{\partial X}\right|,$$

$$ER3 = \frac{1}{NP}\sum_{i=1}^{NP} \left|\frac{\partial\Phi_e(X_i, Y_i)}{\partial Y} - \frac{\partial\Phi_a(X_i, Y_i, N)}{\partial Y}\right|.$$

The subscripts "e" and "a" above refer to the exact and approximate by means of either BCM or FEM solutions respectively. The points (X_i, Y_i) at which the errors are evaluated are uniformly distributed over the domains considered, cf. Fig. 4. The parameter NP used below stands for the number of such points in specyfic problem.

The second error criterion has a local character and is defined by:

$$PR = \max|\Phi_e - \Phi_a(N)|.$$

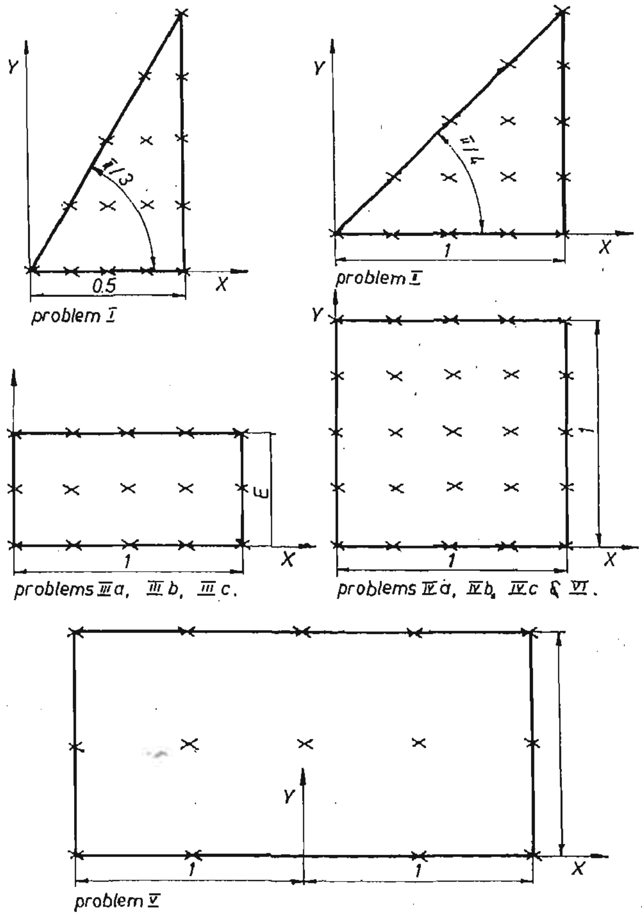


Fig. 4.

To simplify the FEM computations, the maximum is taken over the nodes in the finite element mesh. In BCM the local criterion was applied in the exact way by looking for the maximum of the point error along the boundary.

6. Results and conclusions

As noted before, the way of selecting the trial functions in BCM makes it possible to satisfy exactly not only the differential equation but also the boundary condition on a part of the domain boundary. Moreover, in Problem I we satisfy the boundary condition at the entire boundary by taking $N = 2$. In other words the two first trial functions multiplied by scalar coefficients form the exact solution to this problem. Thus $ER1 = ER2 = ER3 = 0$, cf. Tabl. 2. It is interesting to note that a further increase in the number of expansion terms for this case implies the worsening of the results which is due to the deterioration of the equation set conditioning.

Table II. Global errors and condition number for BCM; Problem I

N	$ER1$	$ER2$	$ER3$	$\varkappa$
2	0.0	0.0	0.0	0.134 E+1
3	0.931 E−10	0.124 E−9	0.155 E−9	0.457 E+1
4	0.970 E−10	0.206 E−9	0.425 E−9	0.153 E+2
5	0.128 E−9	0.539 E−9	0.135 E−8	0.536 E+2
6	0.145 E−9	0.501 E−9	0.845 E−9	0.196 E+3
7	0.124 E−9	0.542 E−9	0.938 E−9	0.756 E+3
8	0.970 E−10	0.654 E−8	0.125 E−7	0.291 E+4
9	0.186 E−9	0.861 E−9	0.167 E−8	0.116 E+5
10	0.186 E−9	0.685 E−7	0.120 E−6	0.472 E+5
11	0.109 E−9	0.911 E−8	0.159 E−7	0.194 E+6
12	0.299 E−9	0.521 E−6	0.913 E−6	0.807 E+6
13	0.101 E−9	0.434 E−6	0.756 E−6	0.340 E+7
14	0.489 E−9	0.374 E−5	0.648 E−5	0.242 E+8
15	0.640 E−9	0.578 E−5	0.100 E−4	0.612 E+8
20	0.183 E−6	0.138 E−1	0.240 E−1	0.934 E+11

The problem of conditioning for the equation set matrix A for BCM requires special attention. Depending on the relative distribution of the collocation points the matrix may become ill-conditioned or ever singular. For the equally distributed collocation points assumed in this study, the increase in N is always followed by the increase in the condition number defined as [27]:

$$\varkappa = \frac{1}{N} \|\mathsf{A}\|_E \|\mathsf{A}^{-1}\|_E .$$

This clearly means that the conditioning of the governing set of equation becomes worse, cf. Tabls. II and III. This effect allows to formulate a general property of the BCM solutions as obtained in the present study: the increase in N pays off to a certain critical value of the number of collocation points only, beyond which the overall performance of BCM

Table III. Global errors of function Φ and condition numbers; Problem II

N	$ER1$	$\varkappa$
2	0.110399 E−1	0.190000 E+1
3	0.863781 E−3	0.533193 E+1
4	0.178118 E−3	0.151460 E+1
5	0.245563 E−4	0.467881 E+2
6	0.175982 E−4	0.151932 E+3
7	0.791734 E−5	0.507495 E+3
8	0.365082 E−5	0.174014 E+4
9	0.179495 E−5	0.606870 E+4
10	0.195945 E−5	0.217053 E+5
15	**0.189282** E−5	0.148221 E+6
20	0.194735 E−5	0.849611 E+8
25	0.303794 E−5	0.182446 E+10

Table IV. Global errors for straightforward BCM and collocation for unknown function together with ist derivatives; Problem VI

straightforward BCM				collocation for function and deriv.			
N	*ER*1	*ER*2	*ER*3	*N*	*ER*1	*ER*2	*ER*3
3	0.736 E−1	0.105 E+0	0.193 E+0	5	0.432 E−1	0.910 E−1	0.132 E+0
5	0.208 E−1	0.627 E+0	0.151 E+0	9	0.138 E−1	0.310 E−1	0.804 E−1
7	0.122 E−1	**0.764 E−1**	0.146 E+0	13	0.668 E−2	0.259 E−1	0.550 E−1
9	0.282 E−2	0.117 E+0	**0.137 E+0**	17	**0.346** E−2	**0.766 E−2**	0.518 E−1
11	0.674 E−2	0.220 E+0	0.256 E+0	21	0.362 E−2	0.216 E−1	**0.358** E−1
13	0.581 E−2	0.429 E+0	0.412 E+0	25	0.423 E−2	0.224 E−1	0.499 E−1
15	0.584 E−2	0.865 E+0	0.746 E+0	29	0.450 E−2	0.597 E−1	0.582 E−1
17	**0.808** E−3	0.179 E+1	0.145 E+1	33	0.516 E−2	0.289 E−1	0.124 E+0
19	0.680 E−2	0.381 E+1	0.315 E+1				
21	0.879 E−2	0.818 E+1	0.672 E+1				

becomes worse, cf. Tabl. 3 in which the best results are underlined. We may therefore say that despite the success of using BCM for solving Problem I, the way of selecting trial function and imposing the boundary conditions employed in this paper (which may be called the straightforward boundary collocation method) has its inherent weaknesses.

Table V. Comparison of global errors for FEM and BCM; Problem I

N	ER1		ER2		ER3	
	FEM	BCM	FEM	BCM	FEM	BCM
2		0.0		0.0		0.0
3	0.116 E−1	0.931 E−10		0.124 E−9		0.155 E−9
6	0.564 E−2	0.144 E−9	0.630 E−1	0.501 E−9	0.831 E−1	0.845 E−9
10	0.868 E−3	0.186 E−9				
15	0.209 E−2	0.640 E−9	0.488 E−1	0.578 E−5	0.428 E−1	0.100 E−4
21	0.129 E−2	0.232 E−6	0.138 E−1	0.738 E−2	0.381 E−1	0.128 E−1

Table VI. Comparison of global errors for FEM and BCM; Problem II

N	ER1		ER2		ER3	
	FEM	BCM	FEM	BCM	FEM	BCM
3	0.225 E−3	0.864 E−3				
6	0.164 E−1	0.176 E−4	0.108 E+0	0.238 E−2	0.106 E+0	0.245 E−2
10	0.332 E−2	0.180 E−4				
15	0.325 E−2	0.189 E−5	0.893 E−1	0.759 E−3	0.473 E−1	0.757 E−3
21	0.252 E−2	0.195 E−5	0.392 E−1	0.537 E−3	0.419 E−1	0.535 E−3
36	0.178 E−2		0.351 E−1		0.406 E−1	

Table VII. Comparison of global errors for FEM and BCM; Problem IIIa, E = 0.5

N	ER1		ER2		ER3	
	FEM	BCM	FEM	BCM	FEM	BCM
4	0.903 E−2	0.132 E−2	0.444 E−1	0.480 E−2	0.102 E+0	0.138 E−1
6	0.441 E−2		0.390 E−1		0.849 E−1	
7		0.313 E−4		0.146 E−2		0.623 E−2
8	0.194 E−2		0.368 E−1		0.933 E−1	
10		0.192 E−4		0.832 E−3		0.367 E−2
12	0.212 E−2		0.308 E−1		0.654 E−1	
13		0.154 E−5		0.540 E−3		0.279 E−2
15	0.192 E−2		0.307 E−1		0.949 E−1	
16		0.251 E−5		0.392 E−3		0.230 E−2
18	0.189 E−2		0.309 E−1		0.889 E−1	
19		0.136 E−5		0.301 E−3		0.186 E−2
31		0.239 E−4		0.541 E−2		0.875 E−2
32	0.483 E−3		0.253 E−1		0.557 E−1	
34		0.802 E−5		0.130 E−2		0.361 E−3

As indicated above these are due to sometimes encountered difficulties in making the errors sufficiently small. For Problem VI, for instance, we were not able to obtain the solution better then that having the error of 10%, cf. Tabl. IV. The only way to improve this result

Table VIII. Comparison of global errors for FEM and BCM; Problem IIIb, E = 0.25

N	*ER*1		*ER*2		*ER*3	
	FEM	BCM	FEM	BCM	FEM	BCM
2	0.122 E−1		0.195 E+0		0.109 E+0	
4	0.298 E−2		0.120 E+0		0.942 E−1	
6		0.210 E−3		0.666 E−3		0.839 E−2
8	0.449 E−3		0.105 E+0		0.923 E−1	
11		0.280 E−5		0.182 E−3		0.402 E−2
16	0.195 E−3	0.653 E−5	0.646 E−1	0.863 E−4	0.834 E−1	0.250 E−1
21		**0.163 *E*−5**		0.101 E−3		0.237 E−2
31		0.159 E−2		0.256 E+0		0.917 E−1
36	0.291 E−3	0.552 E−2	0.476 E−1	0.103 E+0	0.527 E−1	0.223 E−1

Table IX. Comparison of global errors for FEM and BCM; Problem IIIc, E = 0.125

N	*ER*1		*ER*2		*ER*3	
	FEM	BCM	FEM	BCM	FEM	BCM
4	0.140 E−2		0.113 E+0		0.416 E−1	
8	0.801 E−3		0.627 E−1		0.395 E−1	
10		0.464 E−4		0.672 E−4		0.449 E−2
16	0.340 E−4		0.593 E−1		0.301 E−1	
18		**0.187 E−5**		**0.436** E−4		0.219 E−2
28		0.231 E−4		0.963 E−4		**0.175 *E*−2**
32	0.220 E−4		0.350 E−1		0.307 E−1	
37		0.517 E−2		0.565 E−1		0.707 E−1

Table X. Comparison of global errors for FEM and BCM; Problem IVa, Bi = 1

N	*ER*1		*ER*2		*ER*3	
	FEM	BCM	FEM	BCM	FEM	BCM
3		0.399 E−1		0.117 E+0		0.498 E−1
4	0.141 E−1		0.208 E+0		0.791 E+0	
5		0.973 E−2		0.711 E−1		0.213 E−1
9	0.631 E−2	0.211 E−2	0.919 E−1	0.459 E−1	0.680 E−1	0.107 E−1
15		0.636 E−3		0.297 E−1		0.909 E−1
16	0.191 E−2		0.493 E−1		0.325 E−1	
17		0.482 E−3		0.260 E−1		0.872 E−2
25	0.146 E−2	0.208 E−3	0.104 E+0	0.151 E−1	0.982 E−1	0.842 E−2
35		**0.168 *E*−3**		**0.865 E−2**		0.623 E−2
36	0.303 E−3		0.113 E−1		0.357 E−1	
37		0.214 E−3		0.878 E−2		0.562 E−2

is to employ the collocation for the unknown function together with its derivatives, which yields the relative error *ER*3 as small as 3%.

Before formulating final conclusions summarizing the findings of this work we note

Table XI. Comparison of global errors for FEM and BCM; Problem IVb, Bi = 5

N	ER1		ER2		ER3	
	FEM	BCM	FEM	BCM	FEM	BCM
3		0.807 E−1		0.382 E+0		0.125 E+0
4	0.374 E−1		0.350 E+0		0.157 E+0	
5		0.206 E−1		0.277 E+0		0.641 E−1
9	0.151 E−1	0.412 E−2	0.327 E+0	0.197 E+0	0.126 E+0	0.432 E−1
15		0.118 E−2				
16	0.541 E−2					
17		0.899 E−3				
25	0.325 E−2	0.384 E−3	0.295 E+0	0.685 E−1	0.112 E+0	0.387 E−1
35		0.233 E−3		0.256 E−1		0.391 E−1
36	0.425 E−2		0.287 E+0		0.649 E−1	
37		0.220 E−3		0.190 E−1		0.387 E−1

Table XII. Comparison of global errors for FEM and BCM; Problem IVc, Bi = 10

N	ER1		ER2		ER3	
	FEM	BCM	FEM	BCM	FEM	BCM
3		0.928 E−1		0.585 E+0		0.152 E+0
4	0.491 E−1		0.566 E+0		0.190 E+0	
5		0.247 E−1		0.459 E+0		0.834 E−1
9	0.192 E−1	0.461 E−2	0.521 E+0	0.344 E+0	0.148 E+0	0.708 E−1
15		0.128 E−2				
16	0.862 E−2					
17		0.976 E−3				
25	0.436 E−2	0.497 E−3	0.478 E+0	0.125 E+0	0.846 E−1	0.704 E−1
35		0.350 E−3		0.464 E−1		0.727 E−1
36	0.379 E−2		0.466 E+0		0.708 E−1	
37		0.334 E−3		0.337 E−1		0.726 E−1

Table XIII. Comparison of global errors for FEM and BCM; Problem V

N	ER1		ER2		ER3	
	FEM	BCM	FEM	BCM	FEM	BCM
7		0.169 E−2		0.277 E−2		0.439 E−2
9	0.262 E−1					
11		**0.900 E−4**		**0.253 E−3**		0.527 E−3
19		0.356 E−3		0.634 E−3		**0.267 E−3**
20	0.210 E−1		0.252 E−1		0.357 E−1	
35	0.130 E−1	0.441 E−3		0.457 E−3		0.100 E−2

that only a limited class of problems has been considered. It seems that the problems selected for the analysis happened to favor BCM rather than FEM, because all the problems allowed to pick out such trial functions which assured the exact satisfaction of the boundary conditions at least on a part of the boundary. In other words, rather than attemp-

Table XIV. Comparison of global errors for FEM and BCM; Problem VI

N	$ER1$		$ER2$		$ER3$	
	FEM	BCM	FEM	BCM	FEM	BCM
4	0.227 E−1		0.241 E+0		0.167 E+0	
5		0.432 E−1		0.910 E−1		0.132 E+0
9	0.110 E−1	0.138 E−1	0.161 E+0	0.310 E−1	0.159 E+0	0.804 E−1
13		0.668 E−2		0.259 E−1		0.550 E−1
16	0.558 E−2					
17		0.346 E−2		0.766 E−2		0.518 E−1
25	0.426 E−2	0.423 E−2	0.112 E+0	0.224 E−1	0.101 E+0	0.499 E−1
33		0.516 E−2		0.289 E−1		0.123 E+0

Table XV. Comparison of global errors and local ones for BCM; Problem II

N	$ER1$	PR
2	0.110 E−1	0.250 E−1
3	0.864 E−3	0.627 E−2
4	0.178 E−3	0.259 E−2
5	0.246 E−4	0.137 E−2
6	0.176 E−4	0.834 E−3
7	0.792 E−5	0.548 E−3
8	0.363 E−5	0.396 E−3
9	0.179 E−5	0.292 E−3
10	0.196 E−5	0.217 E−3
11	0.193 E−5	0.172 E−3
12	0.186 E−5	0.143 E−3
13	0.182 E−5	0.124 E−3
15	0.189 E−5	0.876 E−4

Table XVI. Comparison of global errors and local ones for FEM

Problem III			Problem VI		
N	$ER1$	PR	N	$ER1$	PR
4	0.903 E−2	0.101 E−1	4	0.227 E−1	0.628 E−1
6	0.441 E−2	0.595 E−2	9	0.110 E−1	0.383 E−1
8	0.194 E−2	0.443 E−2	16	0.558 E−2	0.279 E−1
12	0.212 E−2	0.323 E−2	25	0.426 E−2	0.220 E−1
15	0.192 E−2	0.250 E−2			
18	0.189 E−2	0.202 E−2			

ting to draw very general conclusions on performance of the both methods we below characterize class of problems considered specifically in this paper only.

The conclusions:

1. For the same numbers of the degrees of freedom, BCM leads to more exact results than FEM (see Tabls. V - XIV).

2. For the same numbers of the degrees of freedom, the accuracy of the BCM depends heavily on the type of the boundary value problem. For instance for the Saint-Venant torsion problem Tabls. VI - IX accuracy is much higher than in Problems IV and VI which describe the steady heat conduction Tabls. X - XII, XIV.

3. In both methods the values of functions are more exact than the values of their derivatives. However, in the BCM the ratio of the function error to the derivative error is much greater.

4. As expected in both methods the global errors are smaller than the local ones, but in the BCM this difference is significantly smaller, (see. Tabls. XV - XVI).

5. In the BCM problems may arise while increasing the number of the degrees of freedom. This may lead to the ill-conditioning of the problem matrix, quite differently than in the FEM.

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Резюме

СРАВНЕНИЕ МЕТОДА ГРАНИЧНОЙ КОЛЛОКАЦИИ И МЕТОДА КОНЕЧНЫХ ЭЛЕМЕНТОВ ДЛЯ НЕКОТОРЫХ ГАРМОНИЧЕСКИХ ДВУМЕРНЫХ ГРАНИЧНЫХ ЗАДАЧ

Предметом работы является проблема сравнения эффективности и точности вычисления методом граничной коллокации и методом конечных элементов. Исследуются двумерные гармонические краевые задачи. Метод граничной коллокации применяется в прямой версии.

Решения полученные с помощю выше упомянутых методов были сравнены для функций и их производных с точными решениями. С численных исследований можно вывести, что для того же самого числа степеней свободы результаты полученные с помощю метода граничной коллокации являются более точными чем полученные с помощю метода конечных элементов. Однако эта положительна черта может быть уменьшена том фактом, что метод граничной коллокации требует решения системы уравнений с полной матрицей, так как в методе конечных элементов получаем ленточную и хорошо обусловленную матрицу.

Streszczenie

PORÓWNANIE METODY KOLLOKACJI BRZEGOWEJ Z METODĄ ELEMENTÓW SKOŃCZONYCH DLA NIEKTÓRYCH HARMONICZNYCH DWUWYMIAROWYCH PROBLEMÓW BRZEGOWYCH

W pracy porównano efektywność i dokładność obliczeniową metody kollokacji brzegowej i metody elementów skończonych. Rozważano dwuwymiarowe, harmoniczne problemy brzegowe. Metoda kollokacji brzegowej była stosowana w tzw. prostej wersji.

Rozwiązania uzyskane przy pomocy wyżej wymienionych metod były porównywane dla funkcji i ich pochodnych z rozwiązaniami dokładnymi.

Z badań numerycznych można wyciągnąć wniosek, że dla tej samej liczby stopni swobody wyniki uzyskane przy pomocy metody kollokacji brzegowej są dokładniejsze od uzyskanych przy pomocy metody elementów skończonych. Jednakże ta cecha dodatnia może być pomniejszona przez fakt, że metoda kollokacji brzegowej wymaga rozwiązania układu równań liniowych z całkowicie wypełnioną macierzą współczynników gdy tymczasem metoda elementów skończonych daje pasmową i zwykle lepiej uwarunkowaną macierz.

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