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# THE CAYLEY VARIATIONAL PRINCIPLE FOR CONTINUOUS-IMPACT PROBLEMS: A CONTINUUM MECHANICS BASED VERSION IN THE PRESENCE OF A SINGULAR SURFACE

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In 1857, Arthur Cayley presented a variational principle for a class of dynamical problems, which he designated as continuous-impact problems. Cayley exemplified this class by means of a chain hanging over the edge of a table and being set into motion by its own weight. In the following, we present a continuum mechanics based version of the Cayley principle. The moving portion of the chain mentioned by Cayley represents a variable-mass system, and he assumed that the particles of the chain experience a jump in their velocity when being taken into connection with the moving part. We accordingly study a body containing a non-material region of transition, within which certain entities suffer considerable changes of their spatial distribution, and which we replace by equivalent surface growth terms at a singular surface, in order to derive our version of the principle. The falling chain mentioned by Cayley is used as an example problem.

*Key words:* variable-mass systems, singular surface, falling chains

## 1. Introduction

In a communication transmitted to the Royal Society of London, Arthur Cayley (1857) wrote: "There are a class of dynamical problems which, so far as I am aware, have not been considered in a general manner. The problems referred to (which might be designated as continuous-impact problems) are those in which the system is continually taking into connexion with itself particles of infinitesimal mass ..., so as not itself to undergo any abrupt change of velocity, but to subject to abrupt changes of velocity the particles so taken into connexion. For instance, a problem of the sort arises when a portion of a chain hangs over the edge of the table, the remainder of the chain being coiled or heaped up close to the edge of the table, the part hanging over constitutes the moving system, and in each element of time the system takes into connexion with itself, and sets into motion with a finite velocity an infinitesimal length of the chain." For this class of dynamical problems, Cayley presented the following variational principle

$$
\begin{aligned}
&\mathit{\Sigma}\Big[\Big(\frac{d^2x}{dt^2}-X\Big)\,\delta x+\Big(\frac{d^2y}{dt^2}-Y\Big)\,\delta y+\Big(\frac{d^2z}{dt^2}-Z\Big)\,\delta z\Big]\;dm\\
&\mathit{\Sigma}(\mathit{\Delta} u\,\delta\xi+\mathit{\Delta} v\,\delta\eta+\mathit{\Delta} w\,\delta\zeta)\frac{1}{dt}d\mu=0
\end{aligned}
\tag{1.1}
$$

Cayley noted: "... the first line requires no explanation, in the second line $\xi$, $\eta$, $\zeta$ are the coordinates at the time $t$ of the particle $d\mu$ which then comes into connexion with the system; $\mathit{\Delta} u$, $\mathit{\Delta} v$, $\mathit{\Delta} w$ are the finite increments of velocity (or, if the particles is originally at rest, then the finite velocities) of the particles $d\mu$ the instant it has come into connexion with the system; ... The summation extends to the several particles $d\mu$ which come into connexion with the system at time $t$...". After he had formulated his principle (1.1), Cayley cast it into a form similar to (but different from) the classical Lagrange equations of analytical dynamics. Arthur Cayley was one of the leading mathematicians of the 19th century. As was noted by Crelly (1998), "he dealt with the mathematics at hand, without much commentary or clues to its development – much to the consternation of his contemporaries." Indeed, Cayley's presentation does not attempt to derive principle (1.1) from other fundamental relations of mechanics; it obviously was meant as a fundamental principle of its own right. Due to the resulting lack of clue, Cayley's principle (1.1) until recently was seldom anticipated in the mechanics literature. However, since in Cayley's formulations the mass of the dynamical system in hand, the moving part of the chain, appears to be variable, the contribution of Cayley (1857) represents a fundamental step in the theory of variable-mass systems. A review on the dynamics of variable-mass systems was presented by Irschik and Holl (2004). In the following, we only refer to some selected works that appear to be relevant in the present context. Cayley's work was mentioned in the thesis by I.V. Meshchersky on variable-mass systems in 1897. In the introduction to his thesis, Meshcherky gave an overview on variable-mass problems in astronomy and rocketry. As was noted by Meshcherky, the equation of motion of a planet with a variable mass obeying continuous impacts and separations was obtained by Seeliger (1890). Tait (1895) solved the problem of a rocket fired vertically, where he attributed the motive power of the rocket to the continual detachment of a portion of mass of the rocket. Meshscherky noted a relationship between the latter two solutions and Cayley formulation (1.1) for continuous-impact problems. However, the details of the thesis of Meshcherky became known to a wider audience only through the collection of his papers on the mechanics of bodies, Meshcherky (1949). Throughout the twentieth century, the study of bodies with variable mass formed an active field of research in various fields, however mostly without reference to Cayley (1857). Several continuous-impact type problems of the rectilinear motion of chains, ropes, cables and strings were treated in the twentieth-century literature by means of the classical equation of conservation of energy, as well as by the Lagrange equations in their classical form, without considering the corrections suggested by Cayley (1857). The fall of a hanging folded chain and the whip served as model problems in these studies. Paradoxical effects such as infinite velocities followed as the result of the latter treatments, which were reviewed by Steiner and Troger (1995). In extension, the problem of a body deploying along a cable was studied by Crellin *et al.* (1997). This interest in the motion of chains, ropes, cables and strings was motivated by applications in the field of tethered satellites. It was pointed out by Steiner and Troger (1995) and by Crellin *et al.* (1997) that the fall of a hanging folded chain, the whip and the body deploying along a cable do not represent problems in which the sum of the kinetic energy and the potential energy of the gravity force can be assumed to be conserved in general. This was demonstrated by applying the equation of balance of momentum, and by comparing the respective outcomes with the results of the classical forms of the equation of balance of kinetic energy and the classical forms of the Lagrange equations, which assume the system to be conservative. In order to account for the apparent difference in these results, Steiner and Troger (1995) introduced a Carnot energy loss in the equation of balance of kinetic energy and in the Lagrange equations. However, numerical and experimental studies with a tendency towards confirming the assumption of conservative behaviour of falling folded chains were published afterwards, see Schagerl *et al.* (1997), and Tomaszewski *et al.* (2006). Cayley's work (1857) was explicitly brought into these discussions on the behaviour of falling chains by Wong and Yasui (2006), who even claimed that the solution presented by Cayley for the motion of a chain hanging over the edge of a table would be incorrect, since it is non-conservative. Their considerations focused on the related problem of a hanging folded chain. More recently, the discussion turned again towards the possible correctness of non-conservative solutions, such as the one given by Cayley (1857), see Wong *et al.* (2007), and the recent considerations by Grewal *et al.* (2011). Among other arguments, the latter authors gave attention to fact that O'Reilly and Varadi (1999), in a concise study of shocks in one-dimensional thermomechanical media, also had discussed the example of a falling folded chain from a thermodynamic perspective. From the work of O'Reilly and Varadi (1999) it is clear that energy conservation in falling chain problems represents only one limit of a whole solution spectrum. In a recent study on the chain hanging from a table and set into motion, de Sousa *et al.* (2011) presented experimental results, which indeed do fit better to the non-conservative solution.

It is the scope of the present contribution to bring Cayley's principle (1.1) into a contemporary light by presenting a corresponding continuum mechanics based version. The following plan of our work is motivated by the definition of the class of continuous-impact problems stated by Cayley (1857): A system of this type "is continually taking into connexion with itself particles of infinitesimal mass..., so as not itself to undergo any abrupt change of velocity, but to subject to abrupt changes of velocity the particles so taken into connexion." A region of transition is to be observed in problems of this type. Within the region of transition, the material particles are subjected to considerable changes of their velocity. The region of transition represents a non-material volume, such as the small spatial volume between the coiled and the moving part of the chain. Correspondingly, a three-dimensional continuum mechanics based formulation suitable for bodies with a non-material region of transition is developed first. In the second step, the region of transition is replaced by an equivalent singular surface, similar to formulations known from the field of interfacial transport phenomena. The present considerations are complementary to our earlier work on rational treatments of the laws of balance and jump, Irschik (2003, 2007). Particularly in the latter paper, we started from the fundamental local equations of balance and jump, e.g. for mass and momentum, in order to obtain manipulated forms, such as for the kinetic energy, or for configurational forms of balance, while in the following we start from the general balance law in an integral form, which we use to derive continuum mechanics based formulations of the Cayley variational principle for continuous impact problems, as well as corresponding relations for balance of momentum and kinetic energy, respectively. Our exposition ends with concluding remarks.

## 2. A continuum mechanics based version of the principle of Cayley

In order to derive a continuum mechanics based extension of Cayley principle (1.1), we begin with the general balance law in the form given in Section 157 of Truesdell and Toupin (1960), see Irschik and Holl (2004) for a review. The general balance law asserts how the rate of change of the total of some entity $\rho\Psi$ contained in a material volume $\mathcal{V}$ with a closed surface $\mathcal{S}$ is to be balanced by an appropriate combination of the supply $s[\Psi]$ of $\Psi$ within $\mathcal{V}$ and the influx $\mathbf{i}[\Psi]$ of $\Psi$ on $\mathcal{S}$

$$\frac{d}{dt}\int\limits_{\mathcal{V}} \rho\Psi \, dv = \int\limits_{\mathcal{V}} \rho s[\Psi] \, dv - \oint\limits_{\mathcal{V}} d\mathbf{a} \cdot \mathbf{i}[\Psi] \tag{2.1}$$

The density of mass is denoted by $\rho$. The oriented area element $d\mathbf{a}$ of the material surface $\mathcal{S}$ is a vector pointing outwards of $\mathcal{V}$. The material surface is a surface moving at the velocity of the material particles located on it, and a material volume is defined correspondingly. The dot product utilised in the surface integral at the right hand side of (2.1) is defined and explained in the exposition on tensor fields by Ericksen (1960). In the present paper, we restrict the entity $\Psi$ to scalars and vectors. Hence, the influx $\mathbf{i}[\Psi]$ stands for a vector or a tensor of second order, respectively. The following continuum mechanics derivations are motivated by the fact that the particles of the chain studied in the exposition of Cayley (1857) are subjected to a considerable change of velocity within a small region, the latter being instantaneously located between the coiled part, which is at rest, and the hanging part, which is moving. This region of transition forms a non-material volume. A non-material volume has a closed surface moving at a velocity different from the material particles located on it. Cayley (1857) and Routh (1898) approximated the presence of a region of transition in the chain by assuming that a jump in the velocity of the particles would take place between the coiled and the hanging part. In a refined modelling, it may be necessary to model the region of transition in more detail, at first as a finite part of the body. In the general balance law (2.1), we therefore need to introduce the rate of change of the quantity contained in a non-material volume of finite extension. We thus decompose the material volume $\mathcal{V}$ in (2.1) instantaneously into three parts, where $t$ denotes the current time

$$\mathcal{V} = v^+(t) \cup v_\Sigma(t) \cup v^-(t) \tag{2.2}$$

From these three regions, let $v_\Sigma(t)$ be a finite three-dimensional volume that separates the other two partial volumes $v^+(t)$ and $v^-(t)$, see Fig. 1.

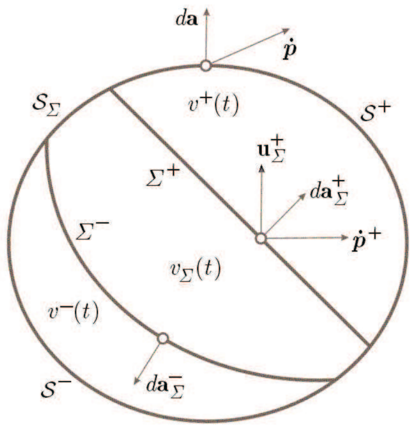


Fig. 1. Material volume $\mathcal{V}$ in the presence of a three-dimensional region of transition, $v_\Sigma(t)$

We characterise the location of $v_\Sigma(t)$ by assuming that the spatial distribution of some of the physical entities under consideration do suffer considerable changes within that region of transition. The left hand side of (2.1) now may be written as

$$\frac{d}{dt}\int\limits_{\mathcal{V}} \rho\Psi\, dv = \frac{d}{dt}\int\limits_{\mathcal{V}^+} \rho\Psi\, dv + \frac{d}{dt}\int\limits_{\mathcal{V}_\Sigma} \rho\Psi\, dv + \frac{d}{dt}\int\limits_{\mathcal{V}^-} \rho\Psi\, dv \tag{2.3}$$

where the material volume $\mathcal{V}_\Sigma$ instantaneously coincides with the non-material region of transition $v_\Sigma(t)$ in (2.2), and $\mathcal{V}^+$ and $\mathcal{V}^-$ are analogously defined. In order to relate the rates of change of the total of $\rho\Psi$ contained in the material volume $\mathcal{V}_\Sigma$ to the rate of change contained in the corresponding non-material volume $v_\Sigma(t)$, the transport theorem is applied. In the following, we use a valuable form first stated in the exposition on the sub-mechanics of the universe by Reynolds (1903), see again Irschik and Holl (2004) for a review. According to Fig. 1, the non-material region of transition $v_\Sigma(t)$ is enclosed by the portion $\mathcal{S}_\Sigma$ of the material surface $\mathcal{S}$, as well as by the two non-material surfaces $\Sigma^+$ and $\Sigma^-$, which separate $v_\Sigma(t)$ from $v^+(t)$ and $v^-(t)$, respectively. The oriented area element of $v_\Sigma(t)$ on $\Sigma^+$ is a vector $d\mathbf{a}^+_\Sigma$ pointing outwards to $v_\Sigma(t)$. The non-material surface $\Sigma^+$ moves at the velocity $\mathbf{u}^+_\Sigma$. Analogously, $d\mathbf{a}^-_\Sigma$ is the oriented area element of $v_\Sigma(t)$ at $\Sigma^-$, the latter surface moving at the velocity $\mathbf{u}^-_\Sigma$. The Reynolds transport theorem then yields

$$\frac{d}{dt}\int\limits_{\mathcal{V}_\Sigma} \rho\Psi\, dv = \frac{d\mathbf{u}_\Sigma}{dt}\int\limits_{v_\Sigma(t)} \rho\Psi\, dv + \int\limits_{\Sigma^+} d\mathbf{a}^+_\Sigma \cdot (\dot{\mathbf{p}} - \mathbf{u}^+_\Sigma)\rho\Psi + \int\limits_{\Sigma^-} d\mathbf{a}^-_\Sigma \cdot (\dot{\mathbf{p}} - \mathbf{u}^-_\Sigma)\rho\Psi \tag{2.4}$$

The notation $d\mathbf{u}_\Sigma/dt$ indicates that the differentiation refers to the motion of the region of transition $v_\Sigma(t)$, see Sect. 81 of Truesdell and Toupin (1960). The relation presented in (2.4) may be derived by subtracting equation (81.4) from equation (81.3) of Truesdell and Toupin (1960). The relation stated in (2.4) dates back to the 1903 exposition of Reynolds (1903), but it was seldom utilised in this form in the subsequent literature. For a contemporary presentation containing formulation (2.4), see Chapter 1.12 of the book on fluid dynamics by Warsi (1999). For balance of mass, linear and angular momentum, see Chapter 7 of the book on the mechanics of solids and fluids by Ziegler (1998). We now turn to the rate of change of the quantities contained in the remaining two material volumes in (2.3), $\mathcal{V}^+$ and $\mathcal{V}^-$. Note that some less well specified regions of the body may be encountered in continuous-impact problems, or that there may be some parts of the body that are of minor interest for the problems in hand. Such a part, for instance, is represented by the coiled part of the hanging chain, the latter part being not precisely specified in the above cited problem statement by Cayley (1857). It may be desirable to exclude unspecified configurations like the coiled part of the chain from the formulation. This is performed as follows. Let the volume to be excluded be represented by the partial volume $v^-(t)$ in the above formulas. For the material volume $\mathcal{V}^-$ instantaneously coinciding with $v^-(t)$, we write down the general equation of balance as

$$\frac{d}{dt}\int\limits_{\mathcal{V}^-} \rho\Psi\, dv = \int\limits_{\mathcal{V}^-} \rho\, s[\Psi]\, dv - \int\limits_{\mathcal{S}^-} d\mathbf{a}\cdot\mathbf{i}[\Psi] + \int\limits_{\Sigma^-} d\mathbf{a}_\Sigma^- \cdot \mathbf{i}[\Psi]^- \tag{2.5}$$

see Fig. 1. The volume $\mathcal{V}^-$ is enclosed by the portion $\mathcal{S}^-$ of the material surface $\mathcal{S}$ and by the part $\Sigma^-$ of the surface of the region of transition $v_\Sigma(t)$. Note that $(-d\mathbf{a}_\Sigma^-)$ is the oriented area element of $v^-(t)$ at $\Sigma^-$. The relation given in (2.5) now is subtracted from general balance law (2.1), thus omitting the volume $\mathcal{V}^-$ from our formulation. In order to derive a counterpart to Cayley variational statement (1.1), we may furthermore state that, for the material volume $\mathcal{V}^+$ in (2.3), there is

$$\frac{d}{dt}\int\limits_{\mathcal{V}^+} \rho\Psi\, dv = \int\limits_{\mathcal{V}^+} \dot{\Psi}\rho\, dv \tag{2.6}$$

since we do not take into account distributed sources of mass in our present formulation. Putting (2.3)-(2.6) into general balance law (2.1), we obtain

$$\begin{aligned}\int\limits_{\mathcal{V}^+} \dot{\Psi}\rho\, dv &= \frac{d\mathbf{u}_\Sigma}{dt}\int\limits_{v_\Sigma(t)} \rho\Psi\, dv + \int\limits_{\Sigma^+} d\mathbf{a}_\Sigma^+\cdot(\dot{\mathbf{p}} - \mathbf{u}_\Sigma^+)\rho\Psi + \int\limits_{\Sigma^-} d\mathbf{a}_\Sigma^-\cdot(\dot{\mathbf{p}} - \mathbf{u}_\Sigma^-)\rho\Psi \\ &= \int\limits_{\mathcal{V}^+} \rho\, s[\Psi]\, dv + \int\limits_{\mathcal{V}_\Sigma} \rho\, s[\Psi]\, dv - \int\limits_{\mathcal{S}^+} d\mathbf{a}\cdot\mathbf{i}[\Psi] - \int\limits_{\mathcal{S}_\Sigma} d\mathbf{a}\cdot\mathbf{i}[\Psi] + \int\limits_{\Sigma^-} d\mathbf{a}_\Sigma^-\cdot\mathbf{i}[\Psi]^-\end{aligned} \tag{2.7}$$

where $\mathcal{S}^+$ denotes that part of the surface of $\mathcal{V}^+$ which is located on the material surface $\mathcal{S}$, see Fig. 1. In the Cayley problem of the hanging moving chain, this region $\mathcal{V}^+$ is represented by the moving part of the chain. For subsequent use, it appears to be convenient to represent the rate of change contained in the non-material region of transition $v_\Sigma(t)$ and its supply in (2.7) by some quantity attributed to a certain non-material surface. Subsequently, we call this singular surface a surface of transition. Without loss of generality, let this surface of transition be chosen as the surface $\Sigma^+$ of $v_\Sigma(t)$. Accordingly, we introduce an equivalent influx $\mathbf{j}[\Psi]$ across the surface $\Sigma^+$. We do this such that the integrals over $\Sigma^-$ formally do not appear in our expressions any more. This yields

$$\begin{aligned}\int\limits_{\mathcal{V}^+} \dot{\Psi}\rho\, dv &+ \int\limits_{\Sigma^+} d\mathbf{a}_\Sigma^+\cdot[(\dot{\mathbf{p}}^+ - \mathbf{u}_\Sigma^+)\rho^+\Psi^+ - (\dot{\mathbf{p}} - \mathbf{u}_\Sigma^+)\rho^-\Psi^-] \\ &= \int\limits_{\mathcal{V}^+} \rho\, s[\Psi]\, dv - \int\limits_{\mathcal{S}^+} d\mathbf{a}\cdot\mathbf{i}[\Psi] - \int\limits_{\mathcal{S}_\Sigma} d\mathbf{a}\cdot\mathbf{i}[\Psi] - \int\limits_{\Sigma^+} d\mathbf{a}_\Sigma^+\cdot(\mathbf{j}[\Psi] + \mathbf{i}[\Psi]^-)\end{aligned} \tag{2.8}$$

The equivalent influx $\mathbf{j}[\Psi]$ thus also attributes certain entities defined at $\Sigma^-$ to the equivalent singular surface $\Sigma^+$. In order to perform this strategy in practice, more information about the problem in hand is necessary. Particularly, formulation (2.8) appears to be straightforward in the case of a region of transition having parallel surfaces $\Sigma^+$ and $\Sigma^-$, with the oriented surface elements pointing in opposite directions. The region of transition then has the geometrical form of a shell. For the corresponding treatment of a three-dimensional thin interfacial region, we refer to the comprehensive exposition on interfacial transport phenomena by Slattery (1990). In fluid mechanics, one generally talks about a shock layer when discussing a thin shell-type region within which considerable changes of some of the field variables take place. A historical exposition and a review on contemporary developments on shock layers was presented by Kluwick (2000). For the classical treatment of singular surfaces, see Sect. 192 of Truesdell and Toupin (1960). It is evident from the example of shock layers in gas dynamics that the constitutive behaviour of the specific material under consideration must be taken into account in the framework of thermodynamics in order to estimate, whether a layer of transition may be treated by the classical formulation for singular surfaces. Since such a detailed study is beyond our present considerations, we have introduced the equivalent influx $\mathbf{j}[\Psi]$ in our subsequent formulations. The equivalent surface influx $\mathbf{j}[\Psi]$ is denoted as a surface growth term in the following. By means of surface growth terms, the present formulation can include both, a region of transition with a finite extent, and a region of transition with a vanishing extension in the thickness direction, and it enables one to consider non-vanishing surface growth terms also in the latter case. General balance law (2.8) now is written as

$$\begin{aligned}\int_{\mathcal{V}^+} \dot{\Psi}\rho \, dv + \int_{\Sigma^+} d\mathbf{a}_\Sigma^+ \cdot [\![(\dot{\mathbf{p}} - \mathbf{u}_\Sigma^+)\rho\Psi]\!] \\ = \int_{\mathcal{V}^+} \rho\, s[\Psi]\, dv - \int_{\mathcal{S}^+} d\mathbf{a}\cdot\mathbf{i}[\Psi] - \int_{\mathcal{S}_\Sigma} d\mathbf{a}\cdot\mathbf{i}[\Psi] - \int_{\Sigma^+} d\mathbf{a}_\Sigma^+ \cdot (\mathbf{j}[\Psi] - \mathbf{i}[\Psi]^-)\end{aligned} \tag{2.9}$$

where we have introduced the jump term

$$[\![(\dot{\mathbf{p}} - \mathbf{u}_\Sigma^+)\rho\Psi]\!] = (\dot{\mathbf{p}}^+ - \mathbf{u}_\Sigma^+)\rho^+\Psi^+ - (\dot{\mathbf{p}} - \mathbf{u}_\Sigma^+)\rho^-\Psi^- \tag{2.10}$$

which is well known from the classical Kotchine theorem for a singular surface moving at the non-material velocity $\mathbf{u}_\Sigma^+$, see Section 193 of Truesdell and Toupin (1960). When we let the material volume $\mathcal{V}^+$ shrink to zero, such that $\mathcal{S}^+ \to \Sigma^+$, we obtain the generalised jump condition

$$\int_{\Sigma^+} d\mathbf{a}_\Sigma^+ \cdot \Big([\![(\dot{\mathbf{p}} - \mathbf{u}_\Sigma^+)\rho\Psi]\!] + \mathbf{j}[\Psi] + [\![\mathbf{i}[\Psi]]\!]\Big) = -\int_{\mathcal{S}_\Sigma} d\mathbf{a}\cdot\mathbf{i}[\Psi] \tag{2.11}$$

where we have assumed $s[\Psi]$ and $\dot{\Psi}\rho$ to be bounded within $\mathcal{V}^+$. We have used the abbreviation

$$[\![\mathbf{i}[\Psi]]\!] = \mathbf{i}[\Psi]^+ - \mathbf{i}[\Psi]^- \tag{2.12}$$

Putting $\mathbf{j}[\Psi] = \mathbf{0}$, and setting $\mathbf{i}[\Psi] = \mathbf{0}$ on $\mathcal{S}_\Sigma$ in (2.11), there follows the classical form of the Kotchine jump conditions at a singular surface, see Section 193 of Truesdell and Toupin (1960).

We now study the special case of balance of mass in the presence of a region of transition, and we apply the result to the hanging moving chain afterwards. We thus put $\Psi = 1$ in (2.11), and, since we do not consider distributed sources of mass to be present, we set $\mathbf{i}[\Psi] = \mathbf{0}$ . By a standard argument, this yields the local jump condition

$$d\mathbf{a}_\Sigma^+ \cdot \Big([\![(\dot{\mathbf{p}} - \mathbf{u}_\Sigma^+)\rho]\!] + \mathbf{j}[\Psi = 1]\Big) = 0 \tag{2.13}$$

where $\mathbf{j}[\varPsi = 1]$ denotes the surface mass growth at $\varSigma^+$. Particularly, in the problem of the hanging moving chain introduced by Cayley (1857), we set $\rho^- = \rho^+ = \rho$, since the chain is assumed to be inextensible. Furthermore, we write $\mathbf{u}_\varSigma^+ = \mathbf{0}$, since the location of the table, which forms the upper end of the moving part of the chain, is fixed. Also, we have $\dot{\mathbf{p}}^- = \mathbf{0}$ in the coiled part of the chain. Hence, for the hanging moving chain, we need a non-vanishing equivalent surface mass growth in (2.13) in order to assure the balance of mass, $\mathbf{j}[\varPsi = 1] = -\dot{\mathbf{p}}^+\rho$. Through this equivalent surface mass growth term, we have again found a context between the Cayley class of continuous impact problems and the class of variable-mass problems. When we neglect the equivalent mass influx, $\mathbf{j}[\varPsi] = \mathbf{0}$ in (2.13), we obtain the classical Stokes-Christoffel jump condition for balance of mass at a singular surface moving at the non-material velocity $\mathbf{u}_\varSigma^+$, see Section 189 of Truesdell and Toupin (1960). Putting (2.13) into (2.9), the following formulation of the general balance law is obtained

$$\begin{aligned}&\int\limits_{\mathcal{V}^+} \dot{\varPsi}\rho\, dv + \int\limits_{\varSigma^+} d\mathbf{a}_\varSigma^+ \cdot (\dot{\mathbf{p}}^+ - \mathbf{u}_\varSigma^+)\rho^+ [\![\varPsi]\!] \\ &\quad = \int\limits_{\mathcal{V}^+} \rho\, s[\varPsi]\, dv - \int\limits_{\mathcal{S}^+} d\mathbf{a} \cdot \mathbf{i}[\varPsi] - \int\limits_{\mathcal{S}_\varSigma} d\mathbf{a} \cdot \mathbf{i}[\varPsi] - \int\limits_{\varSigma^+} d\mathbf{a}_\varSigma^+ \cdot (\mathbf{j}[\varPsi] - \mathbf{j}[\varPsi = 1]\varPsi^- - \mathbf{i}[\varPsi]^-)\end{aligned} \tag{2.14}$$

From (2.14), we now deduce various specialised equations of balance in the presence of a layer of transition. These equations are then applied exemplary to the hanging moving chain. We start with the equation of balance of linear momentum by setting $\boldsymbol{\varPsi} = \dot{\mathbf{p}}$ in (2.14). The supply within $\mathcal{V}^+$ is identified with the assigned body force per unit mass, $s[\dot{\mathbf{p}}] = \mathbf{b}$, and the influx is taken as the negative of the Cauchy stress tensor, $\mathbf{i}[\dot{\mathbf{p}}] = -\mathbf{t}$. We thus obtain

$$\begin{aligned}&\int\limits_{\mathcal{V}^+} \ddot{\mathbf{p}}\rho\, dv + \int\limits_{\varSigma^+} d\mathbf{a}_\varSigma^+ \cdot (\dot{\mathbf{p}}^+ - \mathbf{u}_\varSigma^+)\rho^+ [\![\dot{\mathbf{p}}]\!] \\ &\quad = \int\limits_{\mathcal{V}^+} \rho\mathbf{b}\, dv + \int\limits_{\mathcal{S}^+} d\mathbf{a} \cdot \mathbf{t} + \int\limits_{\mathcal{S}_\varSigma} d\mathbf{a} \cdot \mathbf{t} - \int\limits_{\varSigma^+} d\mathbf{a}_\varSigma^+ \cdot \mathbf{t}^- - \int\limits_{\varSigma^+} d\mathbf{a}_\varSigma^+ \cdot (\mathbf{j}[\dot{\mathbf{p}}] - \mathbf{j}[\varPsi = 1]\dot{\mathbf{p}}^-)\end{aligned} \tag{2.15}$$

This relation now is applied to the Cayley problem of the hanging moving chain. The moving part $\mathcal{V}^+$ of the chain is assumed to have the instantaneous length $s(t)$, such that the particles in $\mathcal{V}^+$ move at the common velocity $\dot{\mathbf{p}} = \dot{s}\mathbf{n}_\varSigma^+$. The assigned body force is the weight per unit mass, $\mathbf{b} = g\mathbf{n}_\varSigma^+$. Furthermore, we set $d\mathbf{a} \cdot \mathbf{t} = \mathbf{0}$ at $\mathcal{S}^+$, since the free end of the chain is not loaded by external forces, and we use $d\mathbf{a} \cdot \mathbf{t} = \mathbf{0}$ at $\mathcal{S}_\varSigma$, since we assume no friction between the table and the chain to be present. Also, there is $d\mathbf{a}_\varSigma^+ \cdot \mathbf{t}^- = \mathbf{0}$, certainly since the chain is assumed to be coiled up loosely. As explained above, we also have $\rho^- = \rho^+ = \rho$, $\mathbf{u}_\varSigma^+ = \mathbf{0}$ and $\dot{\mathbf{p}}^- = \mathbf{0}$. When we neglect the equivalent surface momentum growth in (2.15), $d\mathbf{a}_\varSigma^+ \cdot \mathbf{j}[\dot{\mathbf{p}}] = 0$, we obtain the result derived by Cayley (1857)

$$(\rho A s)\ddot{s} + (\rho A \dot{s})\dot{s} = (\rho A s)g \tag{2.16}$$

where $A$ denotes the cross-section of the chain. However, a whole spectrum of solutions can be obtained when taking into account non-vanishing equivalent surface momentum growth terms. Note also that a further mechanical meaning of the surface momentum growth term can be assigned by comparison of the above methodology with treating the moving part of the chain only, excluding the region of transition. This brings into the play the stress in the moving part of the chain in its upper cross section, see Irschik and Holl (2002) for a corresponding solution.

We now proceed to the equation of balance of kinetic energy. We thus set

$$\varPsi = \frac{1}{2}\dot{\mathbf{p}} \cdot \dot{\mathbf{p}} \tag{2.17}$$

in (2.14). The corresponding supply within $\mathcal{V}^+$ consists of the power of the assigned body forces per unit mass and of the power density of internal forces

$$s\Big[\frac{1}{2}\dot{\mathbf{p}}\cdot\dot{\mathbf{p}}\Big] = \mathbf{b}\cdot\dot{\mathbf{p}} - \frac{1}{\rho}\mathbf{t}\cdot\,\mathrm{grad}\,\dot{\mathbf{p}} \tag{2.18}$$

while the influx term gives the surface power density of the surface tractions

$$d\mathbf{a}\cdot\mathbf{i}\Big[\frac{1}{2}\dot{\mathbf{p}}\cdot\dot{\mathbf{p}}\Big] = -(d\mathbf{a}\cdot\mathbf{t})\cdot\dot{\mathbf{p}} \tag{2.19}$$

We thus obtain from the general equation of balance, (2.14), that

$$\begin{aligned}\int\limits_{\mathcal{V}^+}\ddot{\mathbf{p}}\cdot\dot{\mathbf{p}}\rho\,dv &+ \int\limits_{\Sigma^+} d\mathbf{a}_{\Sigma}^+\cdot(\dot{\mathbf{p}}^+ - \mathbf{u}_{\Sigma}^+)\rho^+\Big[\!\!\Big[\frac{1}{2}\dot{\mathbf{p}}\cdot\dot{\mathbf{p}}\Big]\!\!\Big] \\ &= \int\limits_{\mathcal{V}^+}\rho\mathbf{b}\cdot\dot{\mathbf{p}}\,dv + \int\limits_{\mathcal{S}^+}(d\mathbf{a}\cdot\mathbf{t})\cdot\dot{\mathbf{p}} + \int\limits_{\mathcal{S}_{\Sigma}}(d\mathbf{a}\cdot\mathbf{t})\cdot\dot{\mathbf{p}} - \int\limits_{\Sigma^+}(d\mathbf{a}_{\Sigma}^+\cdot\mathbf{t}^-)\cdot\dot{\mathbf{p}}^- \\ &\quad - \int\limits_{\mathcal{V}^+}\mathbf{t}\cdot\,\mathrm{grad}\,\dot{\mathbf{p}}\,dv - \int\limits_{\Sigma^+}(d\mathbf{a}_{\Sigma}^+\cdot\Big(\mathbf{j}\Big[\frac{1}{2}\dot{\mathbf{p}}\cdot\dot{\mathbf{p}}\Big] - \frac{1}{2}\mathbf{j}[\Psi=1]\dot{\mathbf{p}}^-\cdot\dot{\mathbf{p}}^-\Big)\end{aligned} \tag{2.20}$$

where $\mathbf{j}\Big[\frac{1}{2}\dot{\mathbf{p}}\cdot\dot{\mathbf{p}}\Big]$ represents the equivalent influx of kinetic energy across the surface of transition $\Sigma^+$. This represents the equation of balance of kinetic energy in the presence of a region of transition. We now apply (2.20) to the Cayley problem of a hanging moving chain, using the modelling given above for the derivation of the equation of balance of momentum, (2.16). We thus obtain the result

$$(\rho A s)\ddot{s}\dot{s} + (\rho A\dot{s})\frac{1}{2}\dot{s}^2 = (\rho A s)g\dot{s} - A\mathbf{n}_{\Sigma}^+\cdot\mathbf{j}\Big[\frac{1}{2}\dot{\mathbf{p}}\cdot\dot{\mathbf{p}}\Big] \tag{2.21}$$

where we have set the power of the internal forces to zero within the moving part of the chain, $\mathcal{V}^+$, since the chain is assumed to be inextensible and to remain straight in its hanging part. From a comparison of (2.21) and (2.16), we see that an equivalence of the latter relations is assured, when we introduce an equivalent surface influx of kinetic energy of amount

$$\mathbf{n}_{\Sigma}^+\cdot\mathbf{j}\Big[\frac{1}{2}\dot{\mathbf{p}}\cdot\dot{\mathbf{p}}\Big] = \frac{1}{2}\rho\dot{s}^3 \tag{2.22}$$

in (2.21). The non-vanishing surface growth term $\mathbf{j}\Big[\frac{1}{2}\dot{\mathbf{p}}\cdot\dot{\mathbf{p}}\Big]$ of (2.22) is in analogy to the Carnot loss terms that were introduced by Steiner and Troger (1995) and by Crellin *et al.* (1997) in order to explain the differences between the equation of balance of linear momentum and the equation of balance of kinetic energy in the problems of the falling folded chain and of the body deploying along a cable. In our present contribution, $\mathbf{j}\Big[\frac{1}{2}\dot{\mathbf{p}}\cdot\dot{\mathbf{p}}\Big]$ is an outcome of considering the rate of change of the kinetic energy and the supplies contained in the non-material region of transition $v_{\Sigma}(t)$, and of attributing certain entities defined at $\Sigma^-$ to the surface of transition $\Sigma^+$. We note that our strategy does not allow judging whether a non-vanishing equivalent momentum influx $\mathbf{j}[\dot{\mathbf{p}}]$ should be introduced in the problem of the hanging moving chain, in addition to $\mathbf{j}\Big[\frac{1}{2}\dot{\mathbf{p}}\cdot\dot{\mathbf{p}}\Big]$. This question must be answered by experiments, or by a more detailed modelling. What our present three-dimensional formulation does provide, is the form of the equations of balance that have to be used in the problems in hand. How these equations must be brought into coincidence by means of equivalent surface growth terms has been exemplary demonstrated for the hanging moving chain, see (2.16) and (2.21). A justification for the necessity of surface growth terms in order to ensure the consistency of jump relations at a singular surface was given by Irschik (2003), see also Irschik (2007).

We now turn to Cayley principle (1.1). In order to obtain coincidence, in (2.14) we set

$$\varPsi = \dot{\mathbf{p}} \cdot \delta\mathbf{p} \tag{2.23}$$

where we assume the virtual change of position $\delta\mathbf{p}$ to be a smooth vector field. The corresponding supply within $\mathcal{V}^+$ is given by the sum of the virtual virials of the assigned body forces per unit mass and of the internal forces

$$s[\dot{\mathbf{p}} \cdot \delta\mathbf{p}] = \mathbf{b} \cdot \delta\mathbf{p} - \frac{1}{\rho}\mathbf{t} \cdot \operatorname{grad}\delta\mathbf{p} \tag{2.24}$$

while the influx term is represented by the virtual virial of the surface tractions

$$d\mathbf{a} \cdot \mathbf{i}[\dot{\mathbf{p}} \cdot \delta\mathbf{p}] = -(d\mathbf{a} \cdot \mathbf{t}) \cdot \delta\mathbf{p} \tag{2.25}$$

see Irschik (2000) for the notion of virtual virials. We eventually obtain our counterpart to Cayley principle (1.1), written for a material volume containing a non-material region of transition or a singular surface. The result is

$$\begin{aligned} &\int\limits_{\mathcal{V}^+} (\ddot{\mathbf{p}} - \mathbf{b}) \cdot \delta\mathbf{p}\rho\, dv - \int\limits_{\mathcal{S}^+} (d\mathbf{a} \cdot \mathbf{t}) \cdot \delta\mathbf{p} - \int\limits_{\mathcal{S}_\varSigma} (d\mathbf{a} \cdot \mathbf{t}) \cdot \delta\mathbf{p} + \int\limits_{\varSigma^+} (d\mathbf{a}_\varSigma^+ \cdot \mathbf{t}) \cdot \delta\mathbf{p}^- \\ &\qquad + \int\limits_{\mathcal{V}^+} \mathbf{t} \cdot \operatorname{grad}\delta\mathbf{p}\, dv + \int\limits_{\varSigma^+} d\mathbf{a}_\varSigma^+ \cdot (\dot{\mathbf{p}}^+ - \mathbf{u}_\varSigma^+)\rho^+ [\![\dot{\mathbf{p}}]\!] \cdot \delta\mathbf{p} \\ &\qquad + \int\limits_{\varSigma^+} d\mathbf{a}_\varSigma^+ \cdot (\mathbf{j}[\dot{\mathbf{p}} \cdot \delta\mathbf{p}] - \mathbf{j}[\varPsi = 1]\dot{\mathbf{p}}^- \cdot \delta\mathbf{p}) = 0 \end{aligned} \tag{2.26}$$

Applying (2.26) to the hanging moving chain and setting the equivalent influx term $\mathbf{j}[\dot{\mathbf{p}} \cdot \delta\mathbf{p}]$ to zero, we again arrive at the Cayley result stated in (2.16). It is noted again, however, that this solution is only a limiting case of solutions following from non-vanishing equivalent influx terms.

## 3. Conclusions

With respect to Cayley's original formulation (1.1), we can say the following. Without further need of explanation, the first two lines of (2.26) are seen to be analogous to the first line of (1.1). In the third line of (2.26), the term $(\dot{\mathbf{p}}^+ - \mathbf{u}_\varSigma^+)\rho^+$ represents the influx of mass through the surface element $(-d\mathbf{a}_\varSigma^+)$ of the surface of transition $\varSigma^+$ into the non-material moving region $v^+(t)$, and $[\![\dot{\mathbf{p}}]\!]$ is the jump of velocity across the region of transition. Thus, a direct analogy between the third line of (2.26) and the second line of (1.1) is seen to exist. The last line of (2.26) accounts for the behaviour of the particles within the region of transition, cast into the form of equivalent surface growth terms defined at $\varSigma^+$. As already pointed out, the latter terms must be evaluated in a more detailed and problem-oriented study. The present formulation does not provide more than the general form of the relations to be encountered. In this sense, however, our present contribution can serve as a rational extension of the Cayley principle. Neglecting some or all of the terms in the last line in (2.26) may provide a good first estimate of the solution in some cases, such as in the above modelling of the Cayley chain problem. It is hoped, that extended formulation (2.26) might contribute to the understanding of continuous impact problems in which it is not appropriate to neglect the quantities that are equivalent to the above surface growth terms.

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**Zasada wariacyjna Cayley'a w zagadnieniu ciągło-uderzeniowym: wersja mechaniki kontinuum z obecnością powierzchni osobliwej**

Streszczenie

W 1857 roku Arthur Cayley zaprezentował zasadę wariacyjną dla pewnej klasy zagadnień dynamicznych, którą nazwał ciągło-uderzeniową. Cayley zegzemplifikował tę klasę problemem łańcucha zwisającego z krawędzi stołu i wprowadzanego w ruch własnym ciężarem. W niniejszej pracy zasadę Cayley'a przedstawiono w wersji opartej na mechanice kontinuum. Dawny uczony opisał ruchomą część spadającego łańcucha układem o zmiennej masie i założył, że jego fragmenty spoczywające jeszcze na stole doświadczają skokowego przyrostu prędkości w momencie przyłączania się do części wprawionej już w ruch. Nawiązując do podejścia Cayley'a, w obecnym artykule przeprowadzono analizę ruchu ciała zawierającego niematerialny obszar przejściowy, w którym pewne wielkości podlegają znacznym zmianom konfiguracji przestrzennej i które wyrażono zastępczymi przyrostami na powierzchni osobliwej. Takie sformułowanie pozwoliło wyprowadzić własną wersję zasady Cayley'a, a spadający łańcuch posłużył jako przykład przedstawionej analizy.

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