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BOUNDARY INTEGRAL EQUATIONS IN THE THEORY OF THIN PLATES

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1. Introduction

The method of boundary integral equations, connected with the classical boundary value problems of potential theory, due to the progress of modern computational techniques, has became an effective tool to solve the boundary value problems.

It allows to reduce a three-dimensional problem to a two-dimensional one or a two-dimensional problem to a one-dimensional one, what is advantageous when solving the problem numerically.

In this paper, different possible formulations of the problem of a thin plate, with simply supported or clamped edges, are presented. They lead to a different systems of boundary integral equations.

One of the possible approaches is through the Rayleigh — Green formula, which represents a biharmonic function as a superposition of four biharmonic potentials. It leads to the system of weakly singular integral equations of the first kind.

Besides that, a biharmonic function can be represented in several ways by different combinations of two biharmonic potentials, what leads to different systems of boundary integral equations of the first and second kind for boundary functions which in general do not have a physical interpretation.

2. Basic relationships and equation of equilibrium for the thin plate

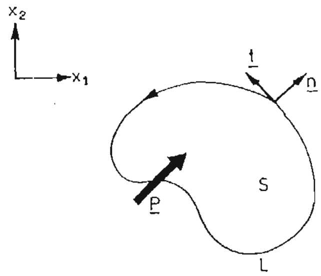
A homogenenus, thin elastic plate, we describe in the following way. The middle surface of the plate is a region S of the plane (x_1, x_2), its boundary constitutes a curve L (Fig. 1):

$$\boldsymbol{x} \in S;\ \boldsymbol{x} = [x_1, x_2]. \tag{1}$$

The curve L is given in the parametric form:

$$\boldsymbol{x} \in L: \boldsymbol{x} = [x_1(l), x_2(l)], \tag{2}$$

the parameter l being the arc length along the curve L. We assume that L consists of a finite number of segments of the class C^2. $\boldsymbol{t}$ is the tangent vector of L, $\boldsymbol{n}$ is the normal



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vector directed outwards S:

$$t(l) = \frac{dx(l)}{dl}, \qquad n(l) = [t_2(l), \; -t_1(l)]. \tag{3}$$

By $\varkappa$ we denote the internal curvature of the curve L.

$$\varkappa(l) = -\frac{dt(l)}{dl} n(l), \qquad n = -\frac{1}{\varkappa}\frac{dt}{dl}, \tag{4}$$

the reciprocal of $\varkappa$ is the curvature radius ϱ:

$$\varrho = \frac{1}{\varkappa}. \tag{5}$$

The deflection of the middle surface of the plate in the direction of the x_3 axis, describes the function $w = w(x)$, its derivatives have the meaning of the deflection slopes of the middle surface of the bent plate with respect to the surface (x_1, x_2).

We consider now an element of the transversal cross-section of the plate, of unit length, having the normal vector n and the tangent vector t. The derivatives in the directions n and t we denote respectively:

$$\frac{\partial}{\partial n} = n\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial t} = t\frac{\partial}{\partial x}, \tag{6}$$

$\frac{\partial w}{\partial n}$ and $\frac{\partial w}{\partial t}$ represent respectively the slopes of the middle surface in the directions n and t, $\frac{\partial^2 w}{\partial n^2}$ and $\frac{\partial^2 w}{\partial t^2}$ represent the curvatures of the middle surface.

The bending moment M_{nn} and the twisting moment M_{nt}, acting on the element of the cross-section, are given respectively:

$$M_{nn} = -D\left[\frac{\partial^2 w}{\partial n^2} + \nu\frac{\partial^2 w}{\partial t^2}\right] = -D\left[\Delta w - (1-\nu)\frac{\partial^2 w}{\partial t^2}\right], \tag{7}$$

$$M_{nt} = -D(1-\nu)\frac{\partial^2 w}{\partial n \partial t}, \tag{8}$$

Δ is the Laplace operator.

The transverse force Q_n s given as:

$$Q_n = -D\frac{\partial}{\partial n}\Delta w. \tag{9}$$

The limits of the above quantities at the boundary L of S, we consider as the functions of the parameter l.

The effective transverse force acting on the element of the boundary of unit length:

$$V_n = Q_n+\frac{d}{dl}M_{nt} = -D\left[\frac{\partial}{\partial n}\Delta w+(1-\nu)\frac{d}{dl}\frac{\partial^2 w}{\partial n\partial t}\right]; \quad x\in L. \tag{10}$$

The equation of equilibrium of a thin, homogeneous elastic plate, loaded by the lateral load, described by the intensity $p(x)$, has the form:

$$\Delta\Delta w = \frac{P}{D}, \tag{11}$$

D is the bending stiffness of the plate:

$$D = \frac{Eh^3}{12(1-\nu^2)}, \tag{12}$$

where h the plate thickness, E the elasticity modulus and ν the Poisson ratio of the material of the plate (see [1, 2]).

3. The integral formula for the function w

For a function w satisfying in the region S, bounded by the curve L, the differential equation (11), we can derive the following integral formula.

Let G be a Green function of the equation (11). In the sense of the theory of generalized functions, G is the solution of the equation:

$$\Delta\Delta G(\boldsymbol{x}, \boldsymbol{x}') = \delta_{(2)}(\boldsymbol{x}-\boldsymbol{x}'), \tag{13}$$

where $\delta_{(2)}(\boldsymbol{x}-\boldsymbol{x}')$ is the two-dimensional Dirac delta function. In general G is of the form:

$$G = G_0+G_1, \tag{14}$$

where G_0 is a particular solution of the equation (13), while G_1 is a solution of the homogeneous biharmonic equation and satisfies appropriate boundary conditions. We may take G_0 in the form:

$$G_0(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{8\pi}[r^2\ln r-r^2], \quad \boldsymbol{r} = \boldsymbol{x}-\boldsymbol{x}'. \tag{15}$$

For G_0 we can take also the function $\frac{1}{8\pi}r^2\ln r$ or $\frac{1}{8\pi}r^2\ln\frac{r}{a}$; the expression (15) has the advantage, that its laplasian is proportional to $\ln r$.

In the forthcoming formulae $\boldsymbol{x}$ will be a fixed point of the area S or the boundary L, whereas $\boldsymbol{x}'$ will be varying. To the fixed point of the boundary L corresponds a fixed value of the parameter l, to the varying point of L corresponds the varying value of the parameter l':

$$x \in L: x = x(l); \quad x' \in L: x' = x'(l'). \tag{16}$$

With primes we shall distinguish the quantities corresponding to the varying point x'. $\frac{\partial}{\partial n}$ will denote the derivative in the direction normal to the boundary L at the point l with respect to the variable x, whereas $\frac{\partial}{\partial n'}$ will denote the normal derivative at the point l with respect to the variable x':

$$\frac{\partial}{\partial n} = n(l)\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial n'} = n'(l')\frac{\partial}{\partial x'}. \tag{17}$$

For $x \in S$ we have the following identity:

$$w(x) \equiv \int_S ds' w(x')\delta_2(x-x') = \int_S ds' w(x')\Delta\Delta G(x, x') =$$

$$= \int_S ds' w(x')\frac{\partial}{\partial x'_\alpha}\frac{\partial}{\partial x'_\alpha}\frac{\partial}{\partial x'_\beta}\frac{\partial}{\partial x'_\beta} G(x, x'); \quad \alpha, \beta = 1, 2, \quad x, x' \in S. \tag{18}$$

Performing in the above formula four times integration by parts, we obtain the following formula — called the Rayleigh — Green identity — for the function w satisfying in the region S Eq. (11):

$$w(x) = \oint_L dl' w(x'(l'))\frac{\partial}{\partial n'}\Delta G(x, x'(l')) - \oint_L dl' \frac{\partial}{\partial n'} w(x'(l'))\Delta G(x, x'(l')) +$$

$$+ \oint_L dl' \Delta w(x'(l'))\frac{\partial}{\partial n'} G(x, x'(l')) - \oint dl' \frac{\partial}{\partial n'}\Delta w(x'(l'))G(x, x'(l')) +$$

$$+ \frac{1}{D}\int_S ds' p(x')G(x, x'). \tag{19}$$

The formula (19) has the twofold meaning. On the one hand, when the function G satisfying suitable boundary conditions is known, it allows for the derivation of the function w satisfying given boundary conditions, with the given loadings p. On the other hand, using the simplest G, for example G_0, it constitutes the base to formulate the boundary equation; for the unknown functions. The solution can then be found by simple integration. In particular, the boundary equations method allows for the derivation of G itself, assuming p to be the concentrate force (see [3]).

4. The simply supported plate

We call the plate simply supported, if the deflection and the bending moment are zero at its boundary:

$$x \in L: w = 0 \quad M_{nn} = -D\left[\Delta w - (1-\nu)\frac{\partial^2 w}{\partial t^2}\right] = 0. \tag{20}$$

If the deflection w is equal to zero at the boundary, its derivatives along the boundary are equal to zero, in particular:

$$x \in L: \quad 0 = \frac{d^2 w}{dl^2} = \frac{d}{dl} t \frac{\partial w}{\partial x} = \frac{\partial^2 w}{\partial t^2} + \frac{dt}{dl} \frac{\partial w}{\partial x} =$$
$$= \frac{\partial^2 w}{\partial t^2} + \frac{dt}{dl} n \frac{\partial w}{\partial n} = \frac{\partial^2 w}{\partial t^2} - \varkappa \frac{\partial w}{\partial n}. \tag{21}$$

Hence:

$$x \in L: \quad w = 0 \Rightarrow \frac{\partial^2 w}{\partial t^2} = \varkappa \frac{\partial w}{\partial n}. \tag{22}$$

For the rectilinear boundary $\varkappa = 0$, the boundary conditions (20) have thus the form:

$$x \in L: \quad w = 0, \quad \frac{\partial^2 w}{\partial n^2} = 0, \tag{23}$$

or

$$x \in L: \; w = 0, \quad \Delta w = 0. \tag{24}$$

For the curvilinear boundary:

$$x \in L: \; w = 0, \quad \frac{\partial^2 w}{\partial n^2} + \nu\varkappa \frac{\partial w}{\partial n} = 0, \tag{25}$$

or

$$x \in L: \; w = 0, \quad \Delta w - (1-\nu)\varkappa \frac{\partial w}{\partial n} = 0. \tag{26}$$

5. The boundary integral equations for the simply supported plate resulting from the boundary formula

The integral formula (19) gives the representation of the function w in terms of the boundary values of the function itself and its derivatives. We shall call them boundary functions. The boundary conditions for the function w determine immediately some boundary functions, being at the same time the integral equations for the remaining functions, treated as the unknowns in these equations.

We consider first the simply supported plate in the form of a polygon, satisfying the boundary conditions (24). From the formula (19) it follows, that the deflection function of such a plate may be represented in the form:

$$w = -\oint_L dl' \frac{\partial w}{\partial n} \Delta G - \oint_L dl' \frac{\partial}{\partial n'} \Delta w G + \frac{1}{D} \int_S ds' p G. \tag{27}$$

At the same time, from the boundary conditions (24), we obtain the following set of equations for the functions $\frac{\partial w}{\partial n}$ and $\frac{\partial}{\partial n} \Delta w$ which we accept as the unknown boundary functions, depending on the parameter l:

$$0 = -\oint_L dl' \frac{\partial w}{\partial n'} \Delta G - \oint_L dl' \frac{\partial}{\partial n'} \Delta w G + \frac{1}{D} \int_S ds' p G,$$
$$0 = -\oint_L dl' \frac{\partial}{\partial n'} \Delta w \Delta G + \frac{1}{D} \int_S ds' p \Delta G, \qquad x, x' \in L. \tag{28}$$

For the function G we take the function G_0, given by the formula (15). (If we have used in the formula (27) not the function G_0 but the function G satisfying the boundary conditions $G = 0$, $\Delta G = 0$, only the last term would remain the this formula, and the boundary conditions would be satisfied automatically).

Deriving the second equation, we have assumed the limit of the contour integral containing the term $\Delta\Delta G$ to be equal to zero, because from (13):

$$\Delta\Delta G(x, x') = 0 \quad \text{for} \quad x \in S,\ x' \in L,\ x \neq x'. \tag{29}$$

The function $\frac{\partial w}{\partial n}$ and $\frac{\partial}{\partial n} \Delta w$ disappear in the corners of the plate. Let us consider a rectangular corner. If the quantities w and Δw disappear at the boundary then its tangent derivatives along the boundary disappear also. But at the corner the tangent derivatives along one edge are equal to the normal derivatives along the other edge, $\frac{\partial w}{\partial n}$ and $\frac{\partial}{\partial n} \Delta w$ disappear thus in the corner. The similar reasoning can be carried out for the corner with arbitrary internal angle.

The set of equations we have obtained, is the set of the Fredholm's equations of the first kind, as the unknown functions appear only under the integral sign. The integral kernels posesses only weak, logarithmic singularities. In the second equation only the unknown function $\frac{\partial}{\partial n'} \Delta w$ appears.

Let us consider now the simply supported plate with curvilinear boundary. From the integral formula (19) and the boundary conditions (26) results the following representation for the function w:

$$w = -\oint_L dl' \frac{\partial w}{\partial n'} \left[\Delta - (1-\nu)\varkappa' \frac{\partial}{\partial n'}\right] G - \oint_L dl' \frac{\partial}{\partial n'} \Delta w G + \frac{1}{D} \int_S ds' p G. \tag{30}$$

Let us mention, that in the boundary conditions the boundary function $\frac{\partial w}{\partial n}$ appears. At the same time, from the boundary conditions (26) we obtain the following set of equations for the unknown boundary functions:

$$0 = -\oint_L dl' \frac{\partial w}{\partial n'} \left[\Delta - (1-\nu)\varkappa' \frac{\partial}{\partial n'}\right] G - \oint_L dl' \frac{\partial}{\partial n'} \Delta w G + \frac{1}{D} \int_S ds' p G,$$
$$0 = -\frac{1}{2}(1-\nu)\varkappa \frac{\partial w}{\partial n} + (1-\nu) \oint_L dl' \frac{\partial w}{\partial n'} \varkappa' \frac{\partial}{\partial n'} \Delta G - \oint_L dl' \frac{\partial}{\partial n'} \Delta w \Delta G +$$
$$+ \frac{1}{D} \left[\Delta - (1-\nu)\varkappa \frac{\partial}{\partial n}\right] \int_S ds' p G. \tag{31}$$

The appropriate derivatives of the function $G = G_0$ are given by the formulae $A1$ (2, 3, 5, 7). When deriving the second equation, we have taken into account (29) and the limiting properties of the two-dimensional harmonic potential of a double layer, which constitutes the integral with the kernel $\frac{\partial}{\partial n'}\Delta G$ (see $A2$ (1, 2)). For $\varkappa = 0$ equations (31) turn into equation (28). We assume also that the limiting transition to the plate with corners ($\varrho \Rightarrow 0$) exists.

The second equation contains the unknown function outside the integral sign also, it is thus the integral equation of the second kind.

6. Different integral representation and the boundary equations for the function w

The integral formula (19) is not the only possible representation of the function w, which we can use to solve the boundary value problems. In general, we can seek for the function w in the form:

$$w = w_0 + w_1, \tag{32}$$

where w_0 is a certain particular solution of the equation (11) whereas w_1 satisfies in S homogeneous biharmonic equation. For the constant load p we can take for instance w_0 in the form:

$$w_0 = \frac{p}{D}\frac{r^4}{64}. \tag{33}$$

To the biharmonic function w_1 corresponds the harmonic function Δw_1, which can be looked for in the form of a harmonic potential of a simple or double layer or their combination. It suggests different possible representations of the function w_1.

Let us consider a polygonal, simply supported plate.

a) Let us try to seek for the function w in the following form:

$$w = \oint_L dl' f(l') \frac{\partial}{\partial n'}\Delta G + \oint_L dl' g(l') \frac{\partial G}{\partial n'} + w_0. \tag{34}$$

In the above and in the following formulae, for the function G we shall accept the function G_0, given by the formula (15). From the boundary conditions (24) we obtain the following set of equations for the unknown functions f and g (for points located not at the corners of the boundary):

$$\begin{aligned} 0 &= \frac{1}{2}f(l) + \oint_L dl' f(l') \frac{\partial}{\partial n'}\Delta G + \oint_L dl' g(l') \frac{\partial G}{\partial n'} + w_0, \\ 0 &= \frac{1}{2}g(l) + \oint_L dl' g(l') \frac{\partial}{\partial n'}\Delta G + \Delta w_0. \end{aligned} \tag{35}$$

We have made use of $A2$ (1, 2) and (29).

We have obtained the set of the Fredholm's equations of the second kind. The kernels of the equations are non-singular. At a rectilinear segment of the boundary, to which belongs a point l, after performing the limitary transition $A2(2)$, the expressions $\frac{\partial G}{\partial n'}$ and $\frac{\partial}{\partial n'} \Delta G$ are equal to zero.

The functions f and g do not have any direct physical meaning, they need not satisfy also any special conditions at the corners.

We can consider the above way of derivation of the boundary integral equations for the boundary conditions (24) as the analogue of the classical solution of Dirichlet's problem.

We shall not use the representation (34) to derive the boundary equations for the plate with a curvilinear contour, because we would obtain in this case the equations with strong singularities, what we want to avoid. For polygonal plates, we can use also the representation with the kernels G and $\frac{\partial}{\partial n'} \Delta G$.

b) The natural representation of the deflection function of a loaded plate is the representation in the form of a sum of particular solution w_0 and the solution due to the distributions along the boundary of the appropriate forces and moments (see [3]):

$$w = \oint_L dl' f(l') \frac{\partial G}{\partial n'} + \oint_L dl' g(l') G + w_0. \tag{36}$$

From the boundary conditions (24), we obtain for the polygonal plate the following set of equations (for points located not in the corners):

$$\begin{aligned} 0 &= \oint_L dl' f(l') \frac{\partial G}{\partial n'} + \oint_L dl' g(l') G + w_0, \\ 0 &= \frac{1}{2} f(l) + \oint_L dl' f(l') \frac{\partial}{\partial n'} \Delta G + \oint_L dl' g(l') \Delta G + \Delta w_0, \end{aligned} \tag{37}$$

and for the plate with a curvilinear boundary, from the boundary conditions (26), the set of equations:

$$\begin{aligned} 0 &= \oint_L dl' f(l') \frac{\partial G}{\partial n'} + \oint_L dl' g(l') G + w_0, \\ 0 &= \frac{1}{2} f(l) + \oint_L dl' f(l') \left[\frac{\partial}{\partial n'} \Delta G - (1-\nu)\varkappa \frac{\partial}{\partial n} \frac{\partial G}{\partial n'} \right] + \\ &\quad + \oint_L dl' g(l') \left[\Delta G - (1-\nu)\varkappa \frac{\partial}{\partial n} G \right] + \left[\Delta - (1-\nu)\varkappa \frac{\partial}{\partial n} \right] w_0. \end{aligned} \tag{38}$$

The appropriate derivatives of the function G are given by the formula $A1$ (2, 3, 4, 5, 7).

We have obtained the set of equations, where the second equation containes the boundary function outside the integral sign, is thus the integral equation of the second type. The equations for the polygonal plate differ from those for the plate with curvilinear boundary relatively little. The second equation contains the weakly singular integral with the logarithmic kernel.

c) We can also look for the function w in the form:

$$w = \oint_L dl' f(l') \Delta G + \oint_L dl' g(l') \frac{\partial G}{\partial n'} + w_0. \tag{39}$$

For a polygonal plate, from the boundary conditions (24), we obtain the equations:

$$\begin{aligned} 0 &= \oint_L dl' f(l') \Delta G + \oint_L dl' g(l') \frac{\partial G}{\partial n'} + w_0, \\ 0 &= \frac{1}{2} g(l) + \oint_L dl' g(l') \frac{\partial}{\partial n'} \Delta G + \Delta w_0. \end{aligned} \tag{40}$$

whereas for the plate with a curvilinear boundary, from the conditions (26), the equations:

$$\begin{aligned} 0 &= \oint_L dl' f(l') \Delta G + \oint_L dl' g(l') \frac{\partial G}{\partial n'} + w_0, \\ 0 &= \frac{1}{2}(1-\nu)\varkappa(l) f(l) - (1-\nu)\varkappa(l) \oint_L dl' f(l') \frac{\partial}{\partial n} \Delta G + \frac{1}{2} g(l) + \\ &+ \oint dl' g(l') \frac{\partial}{\partial n'} \Delta G - (1-\nu)\varkappa(l) \oint_L dl' g(l') \frac{\partial}{\partial n} \frac{\partial G}{\partial n'} + [\Delta - (1-\nu)\varkappa(l)] w_0. \end{aligned} \tag{41}$$

The appropriate derivatives of the function G are given by the formulae $A1$ (3, 4, 5, 6, 7).

The second equation is of the second kind, the first equation is of the first kind. For $\varkappa = 0$ the set of equations (41) turns into the set of equations (40).

7. The plate with clamped edges

A plate is called clamped (or with clamped edges) if the deflection and the deflection slope are equal to zero at the boundary:

$$x \in L: \quad w = 0, \quad \frac{\partial w}{\partial n} = 0. \tag{42}$$

From the integral formula (19), the following representation results for the deflection function of the plate under the lateral load $p(x)$:

$$w = \oint_L dl' \Delta w \frac{\partial}{\partial n'} G - \oint_L dl' \frac{\partial}{\partial n'} \Delta w G + \frac{1}{D} \int_S ds' p G. \tag{43}$$

From the boundary conditions (42), we obtain the following set of equations for the function Δw and $\frac{\partial}{\partial n'} \Delta w$ on L, which we accept as the unknown boundary functions:

$$\begin{aligned} 0 &= \oint_L dl' \Delta w \frac{\partial}{\partial n'} G - \oint_L dl' \frac{\partial}{\partial n'} \Delta w G + \frac{1}{D} \int_S ds' p G, \\ 0 &= \oint_L dl' \Delta w \frac{\partial}{\partial n} \frac{\partial}{\partial n'} G - \oint_L dl' \frac{\partial}{\partial n'} \Delta w \frac{\partial G}{\partial n} + \frac{1}{D} \int_S ds' p \frac{\partial G}{\partial n}. \end{aligned} \tag{44}$$

The appropriate derivatives of the function G are given by the formulae $A1(2, 3, 4)$. We have obtained the set of equations of the first kind, the first one being non-singular, the second one being weakly singular. The form of the equations does not depend on the shape of the plate. In the case of a polygonal plate, the boundary functions disappear in the corners. $\Big($From the disappearance of w and $\frac{\partial w}{\partial n}$ at the boundary follows the disappearance of $\frac{\partial^2 w}{\partial t^2}$, $\frac{\partial^3 \omega}{\partial t^3}$ and $\frac{\partial^2}{\partial t^2}\cdot\frac{\partial w}{\partial n}$, from here follows the disappearance of Δw and $\frac{\partial}{\partial n}\Delta w\Big)$.

We can consider the representation (43) as the particular case of the representation:

$$w = \oint_L dl' f\frac{\partial G}{\partial n'} + \oint_L dl' gG + w_0, \tag{45}$$

where w_0 is the particular solution of the equation (11).

Besides the representation (43) or (45) for the function w, we can make use of the other integral representations.

a) Let us consider the representation:

$$w = \oint_L dl' f\Delta G + \oint_L dl' gG + w_0. \tag{46}$$

From the boundary conditions (42), the following boundary equations for the boundary functions f and g result:

$$\begin{aligned} 0 &= \oint_L dl' f\Delta G + \oint_L dl' gG + w_0, \\ 0 &= -\frac{1}{2}f(l) + \oint_L dl' f\frac{\partial}{\partial n}\Delta G + \oint_L dl' g\frac{\partial G}{\partial n} + \frac{\partial w_0}{\partial n}. \end{aligned} \tag{47}$$

b) On the other hand the representation:

$$w = \oint_L dl' f\Delta G + \oint_L dl' g\frac{\partial G}{\partial n'} + w_0, \tag{48}$$

leads to the boundary equations:

$$\begin{aligned} 0 &= \oint_L dl' f\Delta G + \oint_L dl' g\frac{\partial G}{\partial n'} + w_0, \\ 0 &= -\frac{1}{2}f(l) + \oint_L dl' f\frac{\partial}{\partial n}\Delta G + \oint_L dl' g\frac{\partial}{\partial n}\frac{\partial G}{\partial n'} + \frac{\partial w_0}{\partial n}. \end{aligned} \tag{49}$$

In the sets of equations (47) and (49) we find the equation of the second kind with respect to the function f and the equation of the first kind with respect to the function g. The functions f and g need not satisfy any special conditions in the possible corners of the boundary.

8. Conclusions

The deflection function of a thin, homogeneous, isotropic plate, laterally loaded, as a solution of the nonhomogeneous biharmonic equation, can be represented in several ways with the help of appropriate biharmonic potentials, with densities playing the role of boundary functions. The boundary value problem is reduced in this way to the solution of a set of the boundary integral equations for the unknown boundary functions.

The most natural representation is one following from the Rayleigh — Green boundary formula, where boundary functions have the meaning of a deflection, deflection slope, bending moment and the transverse force at the boundary. However another representations can lead to „better" sets of integral equations, containing equations of the second kind, whereas from that formula equations of the first kind follow, (see [5]).

Appendix 1

The derivatives of the Green function of the biharmonic equation:

$$G = \frac{1}{8\pi}[r^2 \ln r - r^2]; \qquad \mathbf{r} = \mathbf{x} - \mathbf{x}', \tag{1}$$

$$\frac{\partial G}{\partial n} = \frac{1}{8\pi}(\mathbf{nr})[2\ln r - 1], \tag{2}$$

$$\frac{\partial G}{\partial n'} = -\frac{1}{8\pi}(\mathbf{n'r})[2\ln r - 1], \tag{3}$$

$$\frac{\partial}{\partial n}\frac{\partial G}{\partial n'} = -\frac{1}{8\pi}[(\mathbf{nn'})(2\ln r - 1)] + 2\frac{(\mathbf{nr})(\mathbf{n'r})}{r^2}, \tag{4}$$

$$\Delta G = \frac{1}{2\pi}\ln r, \tag{5}$$

$$\frac{\partial}{\partial n}\Delta G = \frac{1}{2\pi}\frac{(\mathbf{nr})}{r^2}, \tag{6}$$

$$\frac{\partial}{\partial n'}\Delta G = -\frac{1}{2\pi}\frac{(\mathbf{n'r})}{r^2}. \tag{7}$$

Appendix 2

Properties of the two-dimensional double-layer potential. Let u be the potential:

$$u(\mathbf{x}) = \frac{1}{2\pi}\oint_L dl'\mu(l')\frac{\partial}{\partial n'}\ln r \equiv \frac{1}{8\pi}\oint_L dl'\mu(l')\frac{\partial}{\partial n'}\Delta[r^2\ln r - r^2],$$
$$\mathbf{r} = \mathbf{x} - \mathbf{x}'(l'); \quad \mathbf{x} \in S, \quad \mathbf{x}'(l') \in L, \tag{1}$$

L being the boundary of the region S. Denote by u_i and u_e the interior and exterior limits of u at the point x belonging to the smooth boundary $L \cdot u_i$ and u_e are given respectively:

$$u_i(x(l)) = \frac{1}{2\pi} \oint_L dl' \mu(l') \frac{\partial}{\partial n'} \ln r + \frac{1}{2} \mu(l), \tag{2}$$

$$u_e(x(l)) = \frac{1}{2\pi} \oint_L dl' \mu(l') \frac{\partial}{\partial n'} \ln r - \frac{1}{2} \mu(l); \quad x(l) \in L, \quad x'(l') \in L. \tag{3}$$

where the integrals are to be understood in the sense of a principal value.

More generally, relaxing the restriction to a smooth curve, if x is located at a corner of the boundary having an interior angle $\Omega(x)$, then (see[4]):

$$u_i(x(l)) = \frac{1}{2\pi} \oint_L dl' \mu(l') \frac{\partial}{\partial n'} \ln r + \left[1 - \frac{\Omega}{2\pi}\right] \mu(l), \tag{4}$$

$$u_e(x(l)) = \frac{1}{2\pi} \oint_L dl' \mu(l') \frac{\partial}{\partial n'} \ln r - \frac{\Omega}{2\pi} \mu(l). \tag{5}$$

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Резюме

КРАЕВЫЕ ИНТЕГРАЛЬНЫЕ УРАВНЕНИЕ В ТЕОРИИ ТОНКИХ ПЛАСТИНОК

Краевая задача для дифференциального уравнения малых изгибов тонкой пластинки, поперечно нагруженной, может быть приведена к системе интегральных уравнений, соответствующих данному представлению функции изгиба. Возможны различные интегральные представления через бигармонические потенциалы, приводящие к разным системам интегральных уравнений.

Streszczenie

BRZEGOWE RÓWNANIA CAŁKOWE W TEORII CIENKICH PŁYT

Problem brzegowy dla równania różniczkowego opisującego małe ugięcia cienkiej płyty, obciążonej poprzecznie, może być sprowadzony do układu równań całkowych dla określonych na brzegu płyty funkcji brzegowych, odpowiadających danej representacji funkcji ugięcia. Możliwe są różne representacje całkowe poprzez potencjały biharmoniczne, prowadzące do różnych układów równań całkowych.

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