JOURNAL OF THEORETICAL AND APPLIED MECHANICS
50, 3, pp. 807-817, Warsaw 2012
50th Anniversary of JTAM

OBJECTIVITY AND FRAME INDIFFERENCE OF ACCELERATION-SENSITIVE MATERIALS

Wolfgang Muschik
Technische Universität Berlin, Institut für Theoretische Physik, Berlin, Germany
e-mail: muschik@physik.tu-berlin.de

Constitutive equations have to be compatible with material axioms. Here, the axioms of material frame indifference and material motion dependence, that are the transformation properties of the constitutive equations by changing observer and by changing the motion of the material, are discussed. Starting out with the observer-independent material mapping, its domain – the state space – is extended by a second entry describing the motion of the material with regard to a fixed standard frame of reference. This results in observer-independent tensorial constitutive equations. Different component representations of these equations belong to different observers. Finally, two examples –anisotropic heat conduction and Maxwell's equations – are considered.

Key words: material frame indifference, material motion dependence, accelaration-sensitive material

1. Introduction

It is nearly impossible to cite all publications on the objectivity and frame indifference. Therefore, we refer to the literature which is cited in two papers: Muschik (1998), Muschik and Restuccia (2008), and here we discuss the concepts of objectivity and material frame indifference (MFI) which were developed during the last two decades.

These concepts are slightly different from the classical ones except from one item: We have rigorously to distinguish between the motion of the material with respect to a freely, but fixed chosen standard frame of reference liable for all materials and the relative motion of an arbitrary frame (synonymously an observer) with regard to the material. The observer can be changed freely, resulting in changing the relative motion between the observer and material, but not affecting the motion of the material with respect to the fixed standard frame of reference. This distinction requires the introduction of a second entry in the domain of the constitutive mapping. The missing of this second entry excludes acceleration-sensitive materials[1].

The second item which is slightly different from the classical procedure is the presupposed observer-invariance of the balance equations: No observer is distinguished[2]. The fields of the balance equations transform by changing the observer in such a way that their observer-invariance holds[3]. A consequence is for example that the force density is not a tensor under changing the observer (Muschik and Restuccia, 2002)[4].

The material is described by constitutive equations which are represented by the constitutive mapping, its domain consisting of the state space and the 2nd entry, and its range consisting

[1] A fact which was ignored for about more than 30 years during the development of the material theory
[2] Also not the inertial one
[3] The class of the admitted observers must be defined and depends on the considered realm: Classical Mechanics, Electrodynamics, Special and General Relativity Theory, see example in Section 5.2
[4] Inertial forces appear

of the constitutive properties, for example the heat flux density. This distinction is necessary because one part of MFI – the observer-independence of the material mapping – requires this concept.

The last item which is slightly different from the classical procedure is the distinction of tensor equations and component equations. Component equations are generated by choice of a frame (observer), whereas the corresponding tensor equation is observer-independent and serves for describing the material independently of any observer.

The paper is organized as follows: after having introduced balance equations, constitutive equations are discussed in detail. Material axioms and the constitutive mapping will be considered in connection with objectivity and changing the observer. Because this changing must not influence material properties, we need a recipe for describing them by changing the observer: MFI. Finally, the examples of a rigid anisotropic heat conductor and of Maxwell's equations are shortly discussed for elucidation.

2. Balance equations

We denote an arbitrary but fixed observer $\mathbf{B}^*$ as the *standard frame of reference* and another one with respect to $\mathbf{B}^*$ arbitrarily moving frame $\mathbf{B}$ as the chosen observer. A change of the frame in classical mechanics is described by a proper orthogonal time dependent rotation $Q^{j\cdot}_{\cdot l}(t)$ and by a time dependent translation $c^j(t)$ of the position coordinates (Einstein's sum convention)

$$\begin{aligned}&\mathbf{B} \to \mathbf{B}^*: \qquad x^{*j} = Q^{j\cdot}_{\cdot l}(t)[x^l - c^l(t)]\\ &Q^{\cdot l}_{i\cdot}Q^{m\cdot}_{\cdot l} = \delta^m_i = Q^{l\cdot}_{\cdot i}Q^{\cdot m}_{l\cdot} \qquad \det(Q^{j\cdot}_{\cdot l})(t) = 1\end{aligned} \tag{2.1}$$

This changing of the observer[5] is achieved by a *Euclidean transformation*.

Balance equations belonging to the observer $\mathbf{B}$ are of the shape[6]

$$\mathbf{B}: \qquad \partial_t(\rho\varPsi) + \nabla\cdot(\rho\mathbf{v}\varPsi + \mathbf{\Phi}) + \varSigma = 0 \tag{2.2}$$

Here $\varPsi$ is the balanced quantity, $\mathbf{\Phi}$ its flux, ρ the mass density, $\mathbf{v}$ the material velocity and $\varSigma$ production and supply of $\varPsi$. Because no observer is distinguished, we accept

Axiom I:

- Balance equations are observer-invariant[7].

Consequently, we obtain for

$$\mathbf{B}^*: \qquad \partial^*_t(\rho^*\varPsi^*) + \nabla^*\cdot(\rho^*\mathbf{v}^*\varPsi^* + \mathbf{\Phi}^*) + \varSigma^* = 0 \tag{2.3}$$

This axiom has a very clear interpretation: Because no observer is distinguished, they all "see" their balance equations – having the same component form – which belong to different bases. If two equations are of the same shape for two different observers, we call them *observer-invariant*. The axiom, that balance equations are observer-invariant, is not a new one (Noll, 1958; Bertram, 1989). The advantage of this axiom is, that it uses only quantities which are measurable for the corresponding observers independently of their mutual motion to each other.

[5]Only valid in non-relativistic physics

[6]That is a component equation belonging to the basis $\{\mathbf{e}^j\}$ of $\mathbf{B}$ written down symbolically for shortness

[7]Also called form-invariant

As an example, we consider the balance of momentum which writes for the two observers in components

$$\begin{aligned} \mathbf{B}: &\qquad \partial_t(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\mathbf{v} - \mathbf{T}^{\mathrm{T}}) - \mathbf{f} = 0 \\ \mathbf{B}^*: &\qquad \partial_t^*(\rho^*\mathbf{v}^*) + \nabla^*\cdot(\rho^*\mathbf{v}^*\mathbf{v}^* - \mathbf{T}^{*\mathrm{T}}) - \mathbf{f}^* = 0 \end{aligned} \tag{2.4}$$

For elucidation, we are here only interested in the transformation properties of the components of the force density, because historically the inertal frame was of great importance. For defining this frame, one needs a rule or another recipe by which we can decompose $\mathbf{f}$ and $\mathbf{f}^*$ into their so-called imposed and inertial parts ($_{imp}$ and $_{in}$)

$$f^{*k} = f^{*k}_{imp} + f^{*k}_{in} = Q^{k\cdot}_{\cdot j}(f^j + f^j_{rel}) = Q^{k\cdot}_{\cdot j}(f^j_{imp} + f^j_{in} + f^j_{rel}) \tag{2.5}$$

Here, $\mathbf{f}_{rel}$ is the relative part of the force density which prevents that $\mathbf{f}$ and $\mathbf{f}^*$ transforms as tensors of the first order by changing the observer according to $(2.1)_1$. The decomposition into the imposed and inertial parts cannot be achieved by measuring devices. The imposed part is now defined as a tensor of the first order

$$f^{*k}_{imp} = Q^{k\cdot}_{\cdot j} f^j_{imp} \qquad \longrightarrow \qquad f^{*k}_{in} = Q^{k\cdot}_{\cdot j}(f^j_{in} + f^j_{rel}) \tag{2.6}$$

according to (2.5). A frame $\mathbf{B}^*_{\odot}$ is denoted as an *inertial frame*, if

$$\mathbf{f}^*_{\odot in} \equiv \mathbf{0} \qquad \longrightarrow \qquad \mathbf{f}_{rel} = -\mathbf{f}_{in} \tag{2.7}$$

Because in $(2.6)_2$ the $Q^{k\cdot}_{\cdot j}$ is arbitrarily orthogonal, $(2.7)_2$ is valid for all non-inertial observers.

3. Constitutive equations

For solving balance (2.2), we need variables $\mathbf{z}$ on which the fields $\{\Psi, \mathbf{\Psi}, \Sigma\}$ are defined and on which ∂_t and ∇ operate. The variables $\mathbf{z}$, to which ρ and $\mathbf{v}$ belong, span the *state space*, and $\{\Psi, \mathbf{\Psi}, \Sigma\}(t)$ are the *constitutive properties* which describe together with the chosen state space the material under consideration. The constitutive properties depend on the state space: this connection, the *constitutive equations*, are not arbitrary. They have to satisfy material axioms which are considered in the next section.

3.1. Material axioms

Material frame indifference belong to the *material axioms* (Muschik *et al.*, 2001) which are

1. the 2nd law of thermodynamics
2. transformation properties by changing the observer
   - of the constitutive mapping
   - of its domain, the state space
   - of its range, the constitutive properties
3. transformation properties by changing the motion of the material
4. material symmetry
5. finite speed of wave propagation.

Here we are concerned with axioms #2 and #3.

3.2. Constitutive mapping and objectivity

We now consider two observers: $\mathbf{B}$ represented by the basis $\{\mathbf{e}^j\}$ and another observer $\mathbf{B}^*$ represented by $\{\mathbf{e}^{*k}\}$, both detecting the same material. The observer change $\mathbf{B} \to \mathbf{B}^*$ is represented by $(2.1)_1$. For describing this material, each observer needs state space variables and a set which includes the constitutive properties. For the example of heat conduction, the gradient of temperature is a state space variable, and the value of the heat flux is in the set of the constitutive properties. Thus, we have:

— state space variables

$$\text{in } \mathbf{B}: \ \{z^j\} \qquad \text{in } \mathbf{B}^*: \ \{z^{*k}\} \tag{3.1}$$

— sets of constitutive properties

$$\text{in } \mathbf{B}: \ \{M^j\} \qquad \text{in } \mathbf{B}^*: \ \{M^{*k}\} \tag{3.2}$$

The fact that $\mathbf{B}$ and $\mathbf{B}^*$ watch the same material, results in the transformations with respect to $(2.1)_1$

$$z^{*k} = Q^{k\cdot}_{\cdot j} z^j \qquad M^{*k} = Q^{k\cdot}_{\cdot j} M^j \tag{3.3}$$

that means, with respect to changing the observer, the z^j and the M^j are tensor components[8] of the observer-independent tensors $\mathbf{z}(\mathbf{x}, t)$ and $\mathbf{M}(\mathbf{x}, t)$, they are called *objective*

$$z^{*k}(\mathbf{x}, t) = \mathbf{e}^{*k} \cdot \mathbf{z}(\mathbf{x}, t) \qquad z^j(\mathbf{x}, t) = \mathbf{e}^j \cdot \mathbf{z}(\mathbf{x}, t) \tag{3.4}$$

and

$$M^{*k}(\mathbf{x}, t) = \mathbf{e}^{*k} \cdot \mathbf{M}(\mathbf{x}, t) \qquad M^j(\mathbf{x}, t) = \mathbf{e}^j \cdot \mathbf{M}(\mathbf{x}, t) \tag{3.5}$$

Mathematically, we have a double-fibre bundle consisting of $\mathbf{z}$ and $\mathbf{M}$ on $(\mathbf{x}, t)$.

Presupposing, we are in large state spaces (Muschik, 2007), we introduce a constitutive mapping $\mathcal{M}$ (Muschik, 1990) whose domain is spanned by the set of the variables $\mathbf{z}$ and which maps into the constitutive properties, the range of $\mathcal{M}$

$$\mathbf{M}(\mathbf{x}, t) = \mathcal{M}(\mathbf{z}(\mathbf{x}, t)) \tag{3.6}$$

Inserting (3.6) into (3.5), we obtain the constitutive component equations belonging to $\mathbf{B}^*$ and $\mathbf{B}$ by taking (3.4) into account

$$M^{*k}(\mathbf{x}, t) = \mathbf{e}^{*k} \cdot \mathcal{M}(z^{*m}(\mathbf{x}, t)\mathbf{e}_{*m}) \qquad M^j(\mathbf{x}, t) = \mathbf{e}^j \cdot \mathcal{M}(z^m(\mathbf{x}, t)\mathbf{e}_m) \tag{3.7}$$

or in another form

$$M^{*k}(\mathbf{x}, t) = \mathcal{M}^{*k}(z^{*m}(\mathbf{x}, t)) \qquad M^j(\mathbf{x}, t) = \mathcal{M}^j(z^m(\mathbf{x}, t)) \tag{3.8}$$

The procedure in this section can be described by the following

Axiom II:

- The constitutive mapping is observer-independent
- Domain and range of the constitutive mapping are spanned by Euclidean tensors[9].

that means, the state space variables $\mathbf{z}(\mathbf{x}, t)$ and the constitutive properties $\mathbf{M}(\mathbf{x}, t)$ are Euclidean tensors by changing the observer: they are objective. This represents the situation of the famous "frame indifference" considered by a lot of authors (Svendsen and Bertram, 1999; Bertram and Svendsen, 2001). As we will see in the next section, we have to extend the domain of material mapping (3.6).

[8]Not necessarily of the first order: here a symbolic representation is used

[9]Later on, the domain must be extended by additional non-tensorial quantities in connection with the motion of the material

4. Material frame indifference

Constitutive mapping (3.6) does not contain any information about the motion of the material: $\mathbf{z}$ and $\mathbf{M}$ are observer-independent Euclidean tensors which do not refer to any observer. Up to now, there is no place in the material mapping for describing the motion of the material with respect to a chosen frame[10].

We now introduce the so-called *local co-rotational rest frame*, $\mathbf{B^0}$, which is fixed at a material point $\mathbf{X}$ at $(\mathbf{x},t)^0(t)$. Consequently, we have for all times in

$$\mathbf{B}^0: \qquad \mathbf{v}^0(\mathbf{x}^0(t),t) \equiv \mathbf{0} \qquad \boldsymbol{\omega}^0(\mathbf{x}^0(t),t) \equiv \mathbf{0} \tag{4.1}$$

Here $\mathbf{x}^0$ and $\boldsymbol{\omega}^0$ are the velocity and the angular velocity of the material at $\mathbf{X}$ which both vanish by definition for the local co-rotational rest frame $\mathbf{B^0}$. The motion of the material can now be described by a Euclidean tranformation between $\mathbf{B}^0$ and a so-called *standard frame of reference* $\mathbf{B}^\diamond$

$$\mathbf{B}^\diamond \longrightarrow \mathbf{B^0}: \qquad x^{0j} = Q^{\diamond j\cdot}_{\cdot l}(t)[x^{\diamond l} - c^{\diamond l}(t)] \tag{4.2}$$

which is characterized by $Q^{\diamond j\cdot}_{\cdot l}(t)$ and $c^{\diamond l}(t)$. These quantities creating a Euclidean transformation are not Euclidean tensors. Consequently as non-Euclidean quantities, they cannot appear neither in the domain nor in the range of constitutive mapping (3.6). Thus, the domain of the constitutive mapping must be supplemented by a "second entry $\Box^\diamond$" (Muschik, 1998). This is done by introducing it as a functional $\mathcal{F}$ of the quantities which generate Euclidean transformation (4.2)

$$\Box^\diamond(t) \equiv \mathcal{F}[Q^{\diamond j\cdot}_{\cdot l}(t), c^{\diamond l}(t)] \tag{4.3}$$

and which extend the domain of the constitutive mapping.

Axiom III:

- Material Motion Dependence (MMD)

$$\mathbf{M}^\diamond(\mathbf{x},t) = \mathcal{M}(\mathbf{z}(\mathbf{x},t), \Box^\diamond) \tag{4.4}$$

A *constitutive family* is created by different motions of the material, the family parameter is (4.3), and (4.4) are observer-independent tensor equations including the motion of the material with respect to a fixed chosen standard frame of reference $\mathbf{B}^\diamond$. Consequently, the constitutive family consists of identical materials – all defined by the same material mapping $\mathcal{M}$ – that are moving to each other. The material properties $\mathbf{M}^\diamond(\mathbf{x},t)$ depend on the individual motion of the considered element of the constitutive family according to (4.4).

A special time independent second entry is that of differential type which is given by a function of time derivatives of the quantities generating Euclidean transformation (4.2)

$$^d\Box^\diamond = F[\dot{Q}^{\diamond k\cdot}_{\cdot j}, \dot{c}^{\diamond j}, \ddot{Q}^{\diamond k\cdot}_{\cdot j}, \ddot{c}^{\diamond j}, \ldots] \tag{4.5}$$

Materials described by MMD (4.4) are acceleration-sensitiv according to (4.5). The existence of such materials is well known in physics[11]. From the definition of the second entry follows by (4.3) and (4.2) that $\Box^\diamond(t)$ is observer-independent: $\mathbf{B}^\diamond$ and $\mathbf{B^0}$ are two fixed frames, and $\mathcal{M}$ is the observer-independent tensorial constitutive mapping.

---

[10] Transformation $(2.1)_1$ describes only the relative motion of an arbitrary observer with respect to the rest frame of the material

[11] E.g. Barnett effect 1914, *Lexikon der Physik*, CD-ROM Version, Spektrum Akademischer Verlag, Heidelberg, 1998/99

We now consider two identical materials in different states of motion, one resting in $\mathbf{B^0}$ according to (4.1), the other one resting in the standard frame of reference $\mathbf{B}^\diamond$. The second entry for the material resting in $\mathbf{B}^\diamond$ becomes analogously to (4.3) and (4.5)

$$\Box^0 = \mathcal{F}[\delta_j^k, 0^j] \qquad {}^d\Box^0 = F[0_j^k, 0^j, 0_j^k, 0^j, \ldots] \tag{4.6}$$

and its observer-independent material equation is

$$\mathbf{M}^0(\mathbf{x}, t) = \mathcal{M}(\mathbf{z}(\mathbf{x}, t), \Box^0) \tag{4.7}$$

The transition from (4.4) to (4.7) is an observer-independent active transformation of identical materials – described by the same material mapping $\mathcal{M}$ – moving to each other. For this case, we accept the following

Axiom IV:

- A uniform (inaccelerated) motion with respect to two identical materials does not influence their material properties.

Consequently, (4.5) does not contain the constant relative translation velocity

$$^d\Box^\diamond = F[\dot{Q}^{\diamond k\cdot}_{\cdot j}, \ddot{Q}^{\diamond k\cdot}_{\cdot j}, \ddot{c}^{\diamond j}, \ldots] \tag{4.8}$$

The second entry allows one to compare materials in different states of motion. If this motion is uniform, ${}^d\Box^\diamond ={}^d \Box^0$, materials of the same constitutive family are identical. The standard frame of reference is often chosen as an inertial frame $\mathbf{B}^\diamond_\odot$ which was shortly discussed in Section 2. The constitutive equations of the two identical materials resting in $\mathbf{B}^0$ and $\mathbf{B}^\diamond$ which belong to two observers $\mathbf{B}^*$ and $\mathbf{B}^+$ are according to (3.8)

$$\begin{array}{lll} \mathbf{B}^*: & M_*^{\diamond k} = \mathcal{M}_*^k(z_*^m, \Box^\diamond) & M_*^{0k} = \mathcal{M}_*^k(z_*^m, \Box^0) \\ \mathbf{B}^+: & M_+^{\diamond k} = \mathcal{M}_+^k(z_+^m, \Box^\diamond) & M_+^{0k} = \mathcal{M}_+^k(z_+^m, \Box^0) \end{array} \tag{4.9}$$

Changing the observers $\mathbf{B}^* \to \mathbf{B}^+$ does not change the second entries $\Box^\diamond$ and $\Box^0$ describing the motion of the materials with respect to the standard frame of reference.

We now consider the observer change

$$\mathbf{B}^+ \longrightarrow \mathbf{B}^*: \qquad z_*^m = Q^{m\cdot}_{\cdot j} z_+^j \qquad M_*^{\sharp k} = Q^{k\cdot}_{\cdot j} M_+^{\sharp j} \qquad \sharp = \diamond, 0 \tag{4.10}$$

Inserting (4.10) into $(4.9)_1$ results in

$$\begin{aligned} & M_+^{\sharp p} = Q^{\cdot p}_{k\cdot} \mathcal{M}_*^k(Q^{m\cdot}_{\cdot j} z_+^j, \Box^\sharp) = \mathcal{M}_+^p(z_+^m, \Box^\sharp) \\ & Q^{\cdot p}_{k\cdot} \mathcal{M}_*^k(Q^{q\cdot}_{\cdot m} \bullet, \bullet) = \mathcal{M}_+^p(\bullet, \bullet) \end{aligned} \tag{4.11}$$

The last equation describes the connection of the components of the material mapping belonging to different observers. If the material mapping is linear, we obtain from (4.11)

$$M_+^{\sharp p} = L^p_{+m}(\Box^\sharp) z_+^m \qquad Q^{\cdot p}_{k\cdot} L^k_{*q}(\Box^\sharp) Q^{q\cdot}_{\cdot m} = L^p_{+m}(\Box^\sharp) \tag{4.12}$$

These equations do not represent an isotropic function neither for the components of the material mapping nor for the material mapping itself. This fact is clear, because changing observer $(2.1)_1$ has nothing to do with symmetry properties of the material. The observer components of the material mapping are observer-independent for isotropic materials[12] (Muschik, 2012):
— isotropic material

$$\mathcal{M}_*^k = \mathcal{M}_+^k =: \mathcal{M} \qquad L^k_{*q}(\Box^\sharp) = L^k_{+q}(\Box^\sharp) =: L^k_q(\Box^\sharp) \tag{4.13}$$

Consequently, $(4.11)_2$ and $(4.12)_2$ become isotropic function for isotropic material.

For elucidation, a simple, not very realistic example is discussed in the next section.

[12]Which is out of scope of this paper, because we did not introduce a symmetry group

5. Examples

5.1. Anisotropic heat conduction

We consider a simple example of Fourier heat conduction in rigid rotating media. We have two anisotropic materials which are identical when they are resting with respect to each other[13]. There is a standard frame of reference $\mathbf{B}^\diamond$, one of the two materials (marked by $^\diamond$) is resting with respect to this frame, and the other one (marked by ^{0}) is rotating with respect to $\mathbf{B}^\diamond$. The constitutive equations of the two materials moving to each other are (4.4) and (4.7). The state space variables and the MMD-data are

$$\mathbf{z}(\mathbf{x},t) = (\Theta, \nabla\Theta)(\mathbf{x},t) \qquad \Box^\diamond := \mathbf{0} \qquad \Box^0 := \mathbf{\Omega} \tag{5.1}$$

Here Θ is the temperature, and $\mathbf{\Omega}$ describes the constant rotation of the ^{0}-material with respect to $\mathbf{B}^\diamond$. Consequently, two constitutive equations (4.4) and (4.7) transform with (5.1) to the following Fourier equations

$$-\mathbf{q}^\diamond(\mathbf{x},t) = \boldsymbol{\kappa}(\Theta,\mathbf{0})\cdot\nabla\Theta \qquad -\mathbf{q}^0(\mathbf{x},t) = \boldsymbol{\kappa}(\Theta,\mathbf{\Omega})\cdot\nabla\Theta \tag{5.2}$$

which both are observer-independent.

Now two further observers $\mathbf{B}^*$ and $\mathbf{B}^+$ are introduced, and (5.2) results in constitutive component equations

$$\begin{aligned} &\mathbf{B}^*: \quad -q_*^{\diamond j} = \kappa_*^{ji}(\Theta_*,\mathbf{0})\partial_{*i}\Theta \qquad -q_*^{0j} = \kappa_*^{ji}(\Theta_*,\mathbf{\Omega})\partial_{*i}\Theta \\ &\mathbf{B}^+: \quad -q_+^{\diamond j} = \kappa_+^{ji}(\Theta_+,\mathbf{0})\partial_{+i}\Theta \qquad -q_+^{0j} = \kappa_+^{ji}(\Theta_+,\mathbf{\Omega})\partial_{+i}\Theta \end{aligned} \tag{5.3}$$

State space and constitutive properties transform by changing the observer

$$\begin{aligned} &\Theta_* = \Theta_+ =: \Theta && \partial_{*i}\Theta = Q_{i\cdot}^{\cdot k}\partial_{+k}\Theta \\ &q_*^{0j} = Q_{\cdot k}^{j\cdot} q_+^{0k} && q_*^{\diamond j} = Q_{\cdot k}^{j\cdot} q_+^{\diamond k} \\ &\kappa_*^{ji}(\Theta,\bullet) = Q_{\cdot m}^{j\cdot}\kappa_+^{mn}(\Theta,\bullet)Q_{\cdot n}^{i\cdot} && \bullet = \mathbf{0},\mathbf{\Omega} \end{aligned} \tag{5.4}$$

The pairs

$$q_*^{0j} \longleftrightarrow q_*^{\diamond j} \qquad q_+^{0j} \longleftrightarrow q_+^{\diamond j} \qquad q_*^{0j} \longleftrightarrow q_+^{\diamond j} \qquad q_+^{0j} \longleftrightarrow q_*^{\diamond j} \tag{5.5}$$

are not connected by any transformation with regard to changing the observer. The "frame dependence of the heat flux" refers to different local material rest frames moving to each other, and consequently, the frame dependence is caused by different motions of identical materials according to (5.2).

Another example elucidating Axiom I – balance equations are observer-invariant – is discussed in the next section.

5.2. "Non-relativistic" electrodynamics

We start out with the special case of a non-relativistic change of observer $(2.1)_1$, the Galilei transformation[14]

$$\begin{aligned} &\mathbf{B} \longrightarrow \mathbf{B}^*: \qquad (\mathbf{x},t)^\star = (\mathbf{x},t) - \mathbf{w}t \qquad \mathbf{w} = \text{const} \qquad t^\star = t \\ &\nabla^\star = \nabla \qquad \partial_{t^\star} = \partial_t + \mathbf{w}\cdot\nabla \end{aligned} \tag{5.6}$$

[13] They belong to the same constitutive family

[14] As in (2.2) and (2.3), component equations belonging to observers are written symbolically in bold faces; tensor equations will not appear in this section

applied to the general 3-dimensional form of Maxwell's equations valid for arbitrary materials (Muschik, 1999)

$$\begin{aligned} &\nabla \times \mathbf{H}(\mathbf{x},t) = \mathbf{j}(\mathbf{x},t) + \partial_t \mathbf{D}(\mathbf{x},t) &\qquad& \nabla \times \mathbf{E}(\mathbf{x},t) = -\partial_t \mathbf{B}(\mathbf{x},t) \\ &\nabla \cdot \mathbf{B}(\mathbf{x},t) = 0 && \nabla \cdot \mathbf{D}(\mathbf{x},t) = \varrho(\mathbf{x},t) \end{aligned} \tag{5.7}$$

These 12 fields, $\mathbf{E}$ and $\mathbf{D}$, electric field and dielectrical displacement, $\mathbf{H}$ and $\mathbf{B}$, magnetic field and magnetic induction, are coupled to each other by the constitutive equations

$$\mathbf{D}(\mathbf{x},t) = \varepsilon_0 \mathbf{E}(\mathbf{x},t) + \mathbf{P}(\mathbf{x},t) \qquad \mathbf{B}(\mathbf{x},t) = \mu_0 \mathbf{H}(\mathbf{x},t) + \mathbf{M}(\mathbf{x},t) \tag{5.8}$$

The polarization $\mathbf{P}$ and the magnetization $\mathbf{M}$ are generated by the material and vanish in vacuum. The constants ε_0 and μ_0 are the dielectric and the magnetic permeability of vacuum.

According to (2.2) and (2.3), Maxwell's and constitutive equations are presupposed to be observer-invariant

$$\begin{aligned} &\nabla^\star \times \mathbf{H}^\star(\mathbf{x}^\star,t^\star) = \mathbf{j}^\star(\mathbf{x}^\star,t^\star) + \partial_{t^\star} \mathbf{D}^\star(\mathbf{x}^\star,t^\star) &\qquad& \nabla^\star \times \mathbf{E}^\star(\mathbf{x}^\star,t^\star) = -\partial_{t^\star} \mathbf{B}^\star(\mathbf{x}^\star,t^\star) \\ &\nabla^\star \cdot \mathbf{B}^\star(\mathbf{x}^\star,t^\star) = 0 && \nabla^\star \cdot \mathbf{D}^\star(\mathbf{x}^\star,t^\star) = \varrho^\star(\mathbf{x}^\star,t^\star) \end{aligned} \tag{5.9}$$

and

$$\mathbf{D}^\star(\mathbf{x}^\star,t^\star) = \varepsilon_0 \mathbf{E}^\star(\mathbf{x}^\star,t^\star) + \mathbf{P}^\star(\mathbf{x}^\star,t^\star) \qquad \mathbf{B}^\star(\mathbf{x}^\star,t^\star) = \mu_0 \mathbf{H}^\star(\mathbf{x}^\star,t^\star) + \mathbf{M}^\star(\mathbf{x}^\star,t^\star) \tag{5.10}$$

The following statement is well known (Müller, 1985).

Proposition: Presupposing the observer invariance of the charge density by Galilean changing observer $(5.6)_1$

$$\varrho^\star = \varrho \tag{5.11}$$

the transformed fields are

$$\begin{aligned} &\mathbf{B}^\star = \mathbf{B} &\qquad& \mathbf{E}^\star = \mathbf{E} + \mathbf{w} \times \mathbf{B} \\ &\mathbf{D}^\star = \mathbf{D} && \mathbf{H}^\star = \mathbf{H} - \mathbf{w} \times \mathbf{D} \\ &\mathbf{j}^\star = \mathbf{j} - \varrho \mathbf{w} \end{aligned} \tag{5.12}$$

■

Inserting transformation formulas (5.12) into (5.10), we obtain the transformation rules for the polarization and for the magnetization

$$\mathbf{P}^\star = \mathbf{P} - \varepsilon_0 \mathbf{w} \times \mathbf{B} \qquad \mathbf{M}^\star = \mathbf{M} + \mu_0 \mathbf{w} \times \mathbf{D} \tag{5.13}$$

Consequently, we obtain the strange result that vacuum of the observer $\mathbf{B}$ ($\mathbf{P} \equiv \mathbf{0}$ and $\mathbf{M} \equiv \mathbf{0}$) is not transformed into vacuum of the observer $\mathbf{B}^*$ ($\mathbf{P}^* \neq \mathbf{0}$ and $\mathbf{M}^* \neq \mathbf{0}$). This means that vacuum is not observer-independent. Consequently, we have to distinguish observers for which polarization and magnetization vanish in vacuum and those for which polarization and magnetization depend on the relative velocity between the observers

$$\mathbf{P}^\star_{vac} = -\varepsilon_0 \mathbf{w} \times \mathbf{B} \qquad \mathbf{M}^\star_{vac} = \mu_0 \mathbf{w} \times \mathbf{D} \tag{5.14}$$

We will not at all accept this strange statement, and therefore we obtain the well known result that the Galilei transformation $(5.6)_1$ is not an allowed observer transformation in electrodynamics. If $(5.6)_1$ is not allowed, also $(2.1)_1$ cannot be allowed in electrodynamics.

We discussed this example, because we want to demonstrate that the observer invariance of the balance equations – here the Maxwell equations – and the material frame indifference of the constitutive equations as basic axioms may single out special observer transformations, here the Galilei and the Euclidean transformation[15].

Although (5.13) is not correct with regard to the Euclidean observer change, we can ask, if (5.13) may be valid approximatively in a special limit. We remember the connection of the vacuum constants

$$\frac{1}{\varepsilon_0\mu_0} = c^2 \tag{5.15}$$

Here c is the velocity of light, and we obtain from (5.8) and (5.15)

$$\varepsilon_0\mathbf{w}\times\mathbf{B} = \frac{1}{\mu_0 c^2}\mathbf{w}\times(\mu_0\mathbf{H}+\mathbf{M}) = \frac{1}{c^2}\mathbf{w}\times\mathbf{H} + \varepsilon_0\mathbf{w}\times\mathbf{M} \tag{5.16}$$

and

$$\mu_0\mathbf{w}\times\mathbf{D} = \frac{1}{\varepsilon_0 c^2}\mathbf{w}\times(\varepsilon_0\mathbf{E}+\mathbf{P}) = \frac{1}{c^2}\mathbf{w}\times\mathbf{E} + \mu_0\mathbf{w}\times\mathbf{P} \tag{5.17}$$

Neglecting the "relativistic terms"

$$\frac{1}{c^2}\mathbf{w}\times\boxtimes \approx \mathbf{0} \tag{5.18}$$

we obtain from (5.13) in the non-relativistic approximation neglecting relativistic terms

$$\mathbf{P}^\star \approx \mathbf{P} - \varepsilon_0\mathbf{w}\times\mathbf{M} \qquad\qquad \mathbf{M}^\star \approx \mathbf{M} + \mu_0\mathbf{w}\times\mathbf{P} \tag{5.19}$$

In this approximation, Galilei transformation $(5.6)_1$ transfers vacuum into vacuum. Thus, the Galilei transformation can be used as a non-relativistic approximation. This can also be proved by taking the non-relativistic limit of the relativistic description of electrodynamics into account (Schmutzer, 1968).

6. Discussion

The material mapping (3.6) – defined on state space $(3.1)_1$ and mapping to constitutive properties $(3.2)_1$ – does not contain accelerations, because (3.6) is an observer-independent tensorial equation, and accelerations belong always to a special frame. Taking the experimental fact into account, that materials can be acceleration-sensitive, the domain of the constitutive mapping has to be extended by a second entry – (4.3) or (4.5) – which introduces these missing accelerations. This extension establishes tensorial constitutive family (4.4). Each observer generates its constitutive component equation (4.9) by choosing its base. Observer transformations are generated by Euclidean transformation $(2.1)_1$ changing the observer components according to (3.3).

Following up the scheme discussed above, axioms are needed to ensure the procedure:

- Balance equations are observer-invariant
- Constitutive mappings are observer-independent (and therefore tensorial)
- State space and constitutive properties are objective
- The domain of the constitutive mapping is extended by the accelerations of the material with respect to a standard frame of reference excluding the inaccelerated boost (MMD).

[15]Electrodynamics is compatible with the Lorentz tranformation by changing the observer

This four material axioms are shortly denoted as the material frame indifference and material motion dependence.

The material motion dependence can clearly be seen by considering the example of heat conduction. If the second entry would be cancelled, one has instead of two equations (5.2) only one. Because now no accelerations appear in this tensorial heat conduction equation, the heat conduction tensor does not depend on acceleration and the material becomes acceleration-insensitive. The second example of non-relativistic electrodynamics shows that the observer invariance of the basic equations may exclude special observer transformations.

Acknowledgment

Thanks to H.-H. v. Borzeszkowski for valuable and helpful discussions, especially on the relativistic version of the items presented here.

References

1. Bertram A., 1989, *Axiomatische Einführung in die Kontinuumsmechanik*, BI-Wissenschafts-verlag, Mannheim, p. 179
2. Bertram A., Svendsen B., 2001, On material objectivity and reduced constitutive equations, *Archive of Mechanics*, **53**, 653-675
3. Müller I., 1985, *Thermodynamics*, Pitman, London, Sect. 9.6.1./2
4. Muschik W., 1990, *Aspects of Non-Equilibrium Thermodynamics*, World Scientific, Singapore, Sect. 1.2
5. Muschik W., 1998, Objectivity and frame indifference, revisted, *Archive of Mechanics*, **50**, 541-547
6. Muschik W., 1999, Covariance and objectivity in electrodynamics, [In:] *Ciągłe modele zjawisk mechanicznych w układach materialnych i środowisku*, Politechnika Poznańska, Poznań, Polska, ISBN 83-7143-047-7, p. 71-84
7. Muschik W., 2007, Why so many "schools" of thermodynamics?, *Forschung im Ingenieurwesen*, **71**, 149-161, 3.1
8. Muschik W., 2012, Comment on: I-Shih Liu: Constitutive theory of anisotropic rigid heat conductors, *Continuum Mechanics and Thermodynamics*, DOI 10.1007/s00161-011-0224-7, *Continuum Mechanics and Thermodynamics*, **24**, 175-180
9. Muschik W., Papenfuss C., Ehrentraut H., 2001, A sketch of continuum thermodynamics, *Journal of Non-Newtonian Fluid Mechanics* **96**, 25-290, 4.1
10. Muschik W., Restuccia L., 2002, Changing the observer and moving materials in continuum physics: Objectivity and frame indifference, *Technische Mechanik*, **22**, 152-160
11. Muschik W., Restuccia L., 2008, Systematic remarks on objectivity and frame indifference, liquid crystal theory as an example, *Archive of Applied Mechanics*, **78**, 837-854
12. Noll W., 1958, A mathematical theory of the mechanical behavior of continuous media, *Archive of Rational Mechanics and Analysis*, **2**, 197
13. Schmutzer E., 1968, *Relativistische Physik (Klassische Theorie)*, Teubner, Leipzig, Sect. VII §5
14. Svendsen B., Bertram A., 1999, On frame-indifference and form-invariance in constitutive theory, *Acta Mechanica*, **132**, 195-207

Obiektywność i niezmienniczość względem układu odniesienia obiektów wrażliwych na przyspieszenie

Streszczenie

Równania konstytutywne muszą być zgodne z aksjomatami obiektu. W pracy przedyskutowano aksjomat niezmienniczości względem układu współrzędnych oraz zależność od ruchu, które stanowią podstawę transformacji równań konstytutywnych przy zmianie położenia obserwatora oraz przy uwzględnieniu ruchu obiektu. Rozważania rozpoczęto od odwzorowania niezależnego od obserwatora, którego dziedzina – przestrzeń stanu – została rozszerzona o drugi element opisujący ruch obiektu względem ustalonego, standardowego układu odniesienia. To doprowadziło do sformułowania niezależnych równań konstytutywnych w postaci tensorowej. Różne składowe reprezentacji tych równań należą do różnych obserwatorów. Na zakończenie przedstawiono dwa przykłady – problem anizotropowego przewodzenia ciepła i równania Maxwell'a.

Manuscript received December 15, 2011; accepted for print March 6, 2012