MECHANIKA
TEORETYCZNA
I STOSOWANA
3-4, 22 (1984)

TWO VERSIONS OF WOŹNIAK'S CONTINUUM MODEL OF HEXAGONAL-TYPE GRID PLATES

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1. Introduction

The subject of the considerations are plane-stress statical problems of dense, elastic, hexagonal grid plates, constructed from bars, Fig. 1. The structures of this type are widey used in civil engineering, cf. [1] as well as in aerospace technology. Diffuculties occuring, when exact solutions of statical problems of lattice-type plates are being sought, justify attempts endeavouring to formulate approximate continuum approaches.

The most simple, asymptotic (in Woźniak's meaning, [2]) model has been established by Horvay, cf. [3, 4]. In these papers effective Young modulus and Poisson's ratio for honeycomb plates have been obtained and exhibited by means of the appropriate diagrams.

The aim of the present paper is to discuss continuum descriptions of the analysed plate response by means of the two-dimensional Cosserat's media with fibrous structure, utilized by Woźniak in his lattice-type shell theory, [2]. In the most general among many of Woźniak's concepts, the deformation of the grid surface structure consisted of nodes („elements") and rods („ligaments") is approximated by means of a model of a regular system of bodies, cf. [2], part I. The „elements" of the structure act as the bodies of the system. The interactions between the bodies are transmitted by the „ligaments". One of the basic assumptions of the theory is the existence of the potential of binary interactions. This assumption (see (3.4), p. 39, [2]) restricts the applications of the theory to a certain class of surface structures, that will be further called the structures of simple layout, in which any two directly interacting elements, being joined by one ligament only (cf. [2], p. 50).

The behaviour of a complementary class of structures, which will be called the structures of complex layout, cannot be examined (without additional justifications) by means of the regular system of bodies theory. Continuum approach to the lattice-type plates of complex layout has been presented in the paper [5] of Klemm and Woźniak. The authors assume, that also in the case of complex structure the Woźniak's theory of grid shells and

[1] By means of this term, grid structures constructed from bars connected in rigid nodes are understood in the paper.

plates (based on the regular system of bodies theory) can be applied. The complex geometry implies modifications of constitutive equations only.

Constitutive equations of the theory of complex layout grid plates are not uniquely definite. Several topics resulting from this fact are discussed in the paper. An analysis is exemplified by the case of honeycomb grids which belong to the complex ones.

Thus the internal forces, i.e. stress $p^{\alpha\beta}$ and couple stress m^{α} tensors are not uniquely determined, because of the arbitrariness of the definitions of elastic plate potential σ. Two ways of computing this function will be presented. The first one has been proposed by Klemm and Woźniak, [6]. It is thought appropriate to recall, to correct (an isotropy of the model has not been revealed) and to generalise Klemm and Woźniak's results by taking into account transverse shear deformations of the lattice rods.

In Sec. 4 a new method of defining the plate potential σ leading to the new version of constitutive equations is presented.

Some of effective elastic moduli (so called micropolar moduli) can not be uniquely defined. This has been noted by Woźniak, Pietras and Konieczny in the papers [7 - 9] pertaining to the discrete elasticity theory. This lack of uniqueness follows from an inadequacy of the relatively simple continuum Cosserat's model when deformations of discrete two-dimensional structure are being analysed. Nevertheless such a model is undoubtedly more accurate than Horvay's asymptotic theory.

2. Formulation of the problem

2.1 Basic assumptions. The grid is assumed to be composed of straight bars whose axes constitute a plane, regular, equilateral honeycomb (hexagonal) layout, the internode spacings being equal to l, see Fig. 1. Allthough the lattice bars need not to be prismatic they are required to possess two symmetry axes. The structure is made of an elastic, isotropic and homogeneous material elastic properties of which being characterized by Young modulus E and Poisson's ratio ν. Considerations are confined to the grids constructed by bars sufficiently slender so as to the conventional, improved (by taking

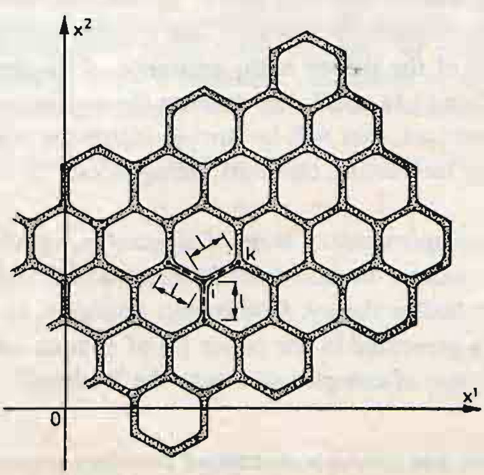

Fig. 1

into account transverse shear deformations of bars) theory of elastic rods can be applied. Moreover the thickness of the grid is assumed to be of unit depth. The loads considered: in-plane tangent forces and moments normal to the mid-surface are concentrated in nodes.

Consider a bar i-k, cf. Fig. 2. Generalized forces and displacements at both nodes i and k are given in Fig. 3; slope deflection equations, cf. [10], read

$$M_{ik} = -\frac{EJ}{l}\left[s\varphi_i + r\varphi_k - (s+r)\Psi_{ik}\right], \qquad M_i = \frac{EJ}{l}\left(\frac{s-r}{2}\right)(\varphi_k - \varphi_i),$$

$$T_{ik} = -T_i = 2(s+r)\frac{EJ}{l^2}\left[\Psi_{ik} - \frac{1}{2}(\varphi_i + \varphi_k)\right], \tag{2.1}$$

$$N_{ik} = N_i = 12\eta p\frac{EJ}{l^2}\cdot\gamma_{ik} = 2(s+r)\bar{\eta}\frac{EJ}{l^2}\cdot\gamma_{ik},$$

where
$$\eta = Al^2/12J, \qquad \bar{\eta} = 6\eta\cdot p/(s+r), \tag{2.2}$$

$$s = \varphi_{ii}/\Delta, \qquad r = -\varphi_{ik}/\Delta, \qquad \Delta = \varphi_{ii}^2 - \varphi_{ik}^2,$$

$$\varphi_{ii} = 2c_1 + \frac{1}{2}c_2 + \frac{\varkappa(1+\nu)}{3\eta}c_3,$$
$$\varphi_{ik} = 2c_1 - \frac{1}{2}c_2 + \frac{\varkappa\cdot(1+\nu)}{3\eta}c_3, \tag{2.3}$$

$$c_1 = \int_0^{1/2}\frac{\bar{\xi}^2 J}{\bar{J}(\bar{\xi})}d\bar{\xi}, \qquad c_2 = \int_0^{1/2}\frac{J d\bar{\xi}}{\bar{J}(\bar{\xi})}, \qquad c_3 = \frac{1}{2}\cdot p = \int_0^{1/2}\frac{A d\bar{\xi}}{\bar{A}(\bar{\xi})}.$$

Functions $\bar{A}(\bar{\xi})$ and $\bar{J}(\bar{\xi})$ express cross section area and moment of inertia whereas A and J denote auxiliary effective quantities. In the considered case of rectangular cross sections

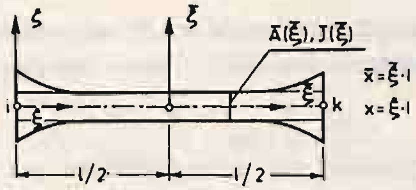

Fig. 2

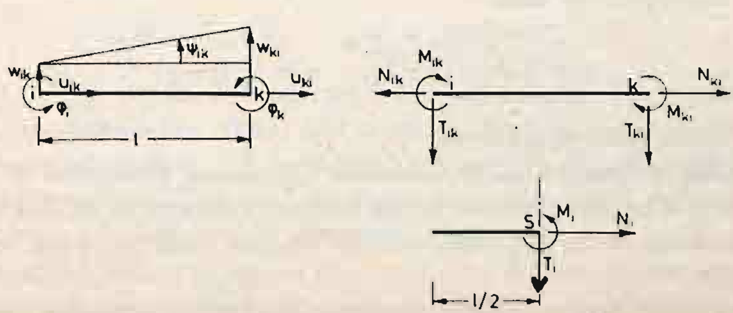

Fig. 3

of the rods, the coefficient $\varkappa$ is equal to 1.2, cf. [10]. The slope deflection ψ_{ik} and the extension γ_{ik} of the member i-k are defined as follows

$$\Psi_{ik} = (w_k - w_i)/l, \quad \gamma_{ik} = (u_k - u_i)/l. \tag{2.4}$$

Setting the effective quantities A and J so as to

$$p = 2c_3 = 1, \quad c_2 = 1/2, \tag{2.5}$$

the simplified versions of the relations (2.1)

$$\begin{gathered} M_{ik} = -\frac{EJ}{l}\frac{1}{\bar{\eta}}[(3\eta+\bar{\eta})\varphi_i + (3\eta-\bar{\eta})\varphi_k - 6\eta\Psi_{ik}], \\ M_i = -\frac{EJ}{l}(\varphi_k - \varphi_i), \quad N_{ik} = N_i = 12\eta\cdot\frac{EJ}{l^2}\cdot\gamma_{ik}, \\ T_{ik} = -T_i = 12\frac{\eta}{\bar{\eta}}\frac{EJ}{l^2}\cdot\left[\Psi_{ik} - \frac{1}{2}\cdot(\varphi_i+\varphi_k)\right], \end{gathered} \tag{2.6}$$

are found, where the formulae

$$s+r = 6\eta/\bar{\eta}, \quad s-r = 2, \tag{2.7}$$

are used

In the case of $\bar{A}(\xi) = 1\cdot h = \text{const}$, $\bar{J}(\xi) = 1\cdot h^3/12 = \text{const}$ (where h stands for a height of bars) we have

$$c_1 = 1/24, \quad c_2 = 1/2, \quad c_3 = p/2 = 1/2, \tag{2.8}$$

hence

$$\bar{\eta} = \eta + 12\zeta, \quad \zeta = (1+\nu)/5, \quad \eta = \frac{l^2}{h^2}, \quad \frac{EJ}{l^3} = \frac{E}{12\eta\sqrt{\eta}} \tag{2.9}$$

If the lattice bars are sufficiently slender (hl < 1/6, say) and influence of shear deformations of the bars can be neglected thus

$$\bar{\eta} = \eta, \quad s = 4, \quad r = 2. \tag{2.10}$$

In the course of the procedure one more ratio ϱ (defined as a quotient of the diameter of the circle inscribed in the hexagonal opening to the spacing of the centres of neighbouring openings) is employed. We have

$$\varrho = (\sqrt{3\eta}-1)/\sqrt{3\eta}, \quad \eta = \frac{1}{3}(1-\varrho)^{-2}. \tag{2.11}$$

The ratio ϱ varies from zero to one.

2.2 Foundations of Woźniak's continuum approach. Continuum description of a response of the considered grid structure is based on the Woźniak's concept [5, 6]. It is worth recalling here the basic ideas of the approach, exemplifying the methods by the specific case of hexagonal plate.

Proceeding in this way as in [4], the nodes of the lattice are divided into two families of main and intermediate nodes, Fig. 4. The division depends on the observation, i.e.

on the fixed coordinate system. Displacements of main nodes are assumed to be approximated by functions: $x^\alpha \to u^\alpha, \varphi, \alpha = 1,2$, which are supposed to be regular and sufficiently smooth, so as to in the vicinities $r \leqslant l\sqrt{3}$ of the nodes linear approximation can be applied. The grid plate can be devided (by various ways) into repeated segments. Fig. 4 shows two types of hexagonal segments: with the centres in the intermediate joints (type I) or in the main ones (type II). Assuming the function u^α, φ to be linear in the segments' areas, displacements of the main nodes (adjoining the centre of the segment) can be expressed by means of the values of functions u^α, φ and their first derivatives $\partial_\alpha u^\beta$, $\partial^\alpha \varphi$, referred to the segments'centre. Then an energy of the segment (i.e. the energy due to de-

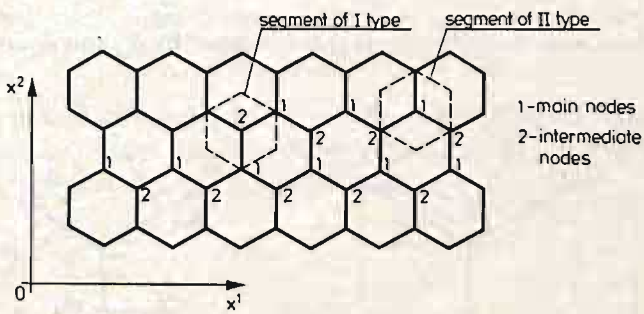

Fig. 4

formations of the rods belonging to the segment) can be found. Dividing this energy by the area P of the segment „i", i = I, II, an energy density $\sigma(i)$ is obtained. The function $\sigma(i)$ can be expressed (as it will be shown further) in terms of components of strain measures

$$\gamma_{\alpha\beta} = \partial_\alpha u_\beta - e_{\alpha\beta}\varphi, \quad \varkappa_\alpha = \partial_\alpha \varphi, \quad \partial_\alpha = \partial/\partial x^\alpha \tag{2.12}$$

($e_{\alpha\beta}$ denote Ricci tensor) and external loads subjected to intermediate nodes. Internal forces $p^{\alpha\beta}_{(i)}$ and $m^\alpha_{(i)}$ i.e. stresses and stress couples, which are defined as follows

$$p^{\alpha\beta}_{(i)} = \frac{\partial \sigma_{(i)}}{\partial \gamma_{\alpha\beta}}, \quad m^\alpha_{(i)} = \frac{\partial \sigma_{(i)}}{\partial \varkappa_\alpha} \tag{2.13}$$

satisfy (see [2]) the equations of equilibrium

$$\partial_\alpha p^{\alpha\beta}_{(i)} + p^\beta_{(i)} = 0, \quad \partial_\alpha m^\alpha_{(i)} + e_{\alpha\beta} p^{\alpha\beta}_{(i)} + Y^3_{(i)} = 0, \tag{2.14}$$

where $p^\beta_{(i)}$, $Y^3_{(i)}$ denote densities of external forces and couples. The equations of equilibrium (2.14), constitutive Eqs. (2.13) and strain — displacements relations (2.12) constitute the system of equations of the lattice-type plate theory. By adding appropriate boundary conditions, (see [2] Ch. IV) the theory is completed and well-established; thus the boundary value problems for finite domains can be examined.

The topics of the present paper are concerned with the constitutive equations (2.13). In the subsequent sections two versions of these equations, resulting from two methods of defining the density of strain energy of the lattice, will be presented.

3. Constitutive equations due to Woźniak and Klemm (variant I)

The derivation presented in [5] will be recalled here; considerations are generalised to the case of deep bars, for which the slope deflection equations (2.6) hold true. The starting point of the procedure is a division of the plate into repeated segments of the type I, the intermediate nodes „a" being the centres of them, Fig. 5. Three main nodes S_i, i = I, II, III lie on the vertices of the hexagon. With the each bar $a-S_i$ a local base $t_{(i)}, \tilde{t}_{(i)}$ is associated, cf. Fig. 5. We have

$$t^1_{(i)} = \tilde{t}^2_{(i)} = \frac{\sqrt{3}}{2}\varepsilon_{i2}, \quad t^2_{(i)} = -\tilde{t}^1_{(i)} = \frac{1}{2}(1-3\cdot\delta_{i2}), \tag{3.1}$$

where Kronecker delta and the difference (i-j) are denoted by δ_{ij} and ε_{ij}, respectively.

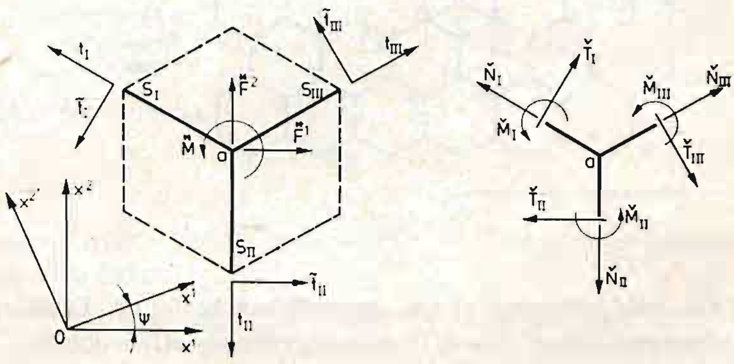

Fig. 5

By using of the assumption of the segment-wise linear behaviour of displacement functions, the displacements of S_i points can be determined by means of the values of u^α and φ functions and their first derivatives computed in the point „a"

$$u^\alpha_{(i)} = u^\alpha|_{(a)} + \partial_\beta u^\alpha|_{(a)}\cdot t^\beta_{(i)}\cdot l, \quad \varphi_{(i)} = \varphi|_{(a)} + \partial_\alpha\varphi|_{(a)}\cdot t^\alpha_{(i)}\cdot l.$$

In order to simplify notations the values of a certain function f in a point „a" will be denoted by the sign „v", i.e. $f|_{(a)} = \check{f}$. Thus the above relations can be rewritten to the form

$$u^\alpha_{(i)} = \check{u}^\alpha + \partial_\beta\check{u}^\alpha\cdot t^\beta_{(i)} l, \quad \varphi_{(i)} = \check{\varphi} + \partial_\alpha\check{\varphi}\, t^\alpha_{(i)} l.$$

Quantities $\check{u}^\alpha$ and $\check{\varphi}$ ought not to be misinterpreted as displacements of the „a" node; the latters are denoted by u^α_a, φ_a. By means of appropriate projections of $u^\alpha_{(i)}$ and u^α_a on the directions of $t_{(i)}$ and $\tilde{t}_{(i)}$ vectors the displacements of the ends of the bar $a-S_i$, referred to the local base $t_{(i)}, \tilde{t}_{(i)}$, can be calculated as follows

$$u_{ai} = t^\alpha_{(i)} u^\beta_a \delta_{\alpha\beta}, \quad u_{ia} = t^\alpha_{(i)} u^\beta_{(i)} \delta_{\alpha\beta},$$
$$w_{ai} = \tilde{t}^\alpha_{(i)} u^\beta_a \delta_{\alpha\beta}, \quad w_{ia} = \tilde{t}^\alpha_{(i)} u^\beta_{(i)} \delta_{\alpha\beta}.$$

Then the slope deflection $\psi_{(i)}$ and the extension $\gamma_{(i)}$ of the bar $a-S_i$, defined by

$$\psi_{(i)} = (w_{ia} - w_{ai})/l, \quad \gamma_{(i)} = (u_{ia} - u_{ai})/l,$$

can be easily rearranged to the form

$$\psi_{(i)} = \delta u^{\alpha}_{a} \tilde{t}^{\beta}_{(i)} \delta_{\alpha\beta} + \tilde{t}^{\alpha}_{(i)} t^{\beta}_{(i)} \check{\gamma}_{\beta\alpha} + \check{\varphi},$$

$$\gamma_{(i)} = \delta u^{\alpha} t^{\beta}_{(i)} \delta_{\alpha\beta} + t^{\alpha}_{(i)} t^{\beta}_{(i)} \check{\gamma}_{\alpha\beta},$$

and, similarly,

$$\Delta\varphi_{(i)} \equiv \varphi_a - \varphi_i = \delta\varphi - \check{\varkappa}_{\alpha} t^{\alpha}_{(i)} \cdot l,$$

where

$$\delta u^{\alpha} = (\check{u}^{\alpha} - u^{\alpha}_{a})/l, \quad \delta\varphi = \varphi_a - \check{\varphi},$$

and, the components of the state of strain referred to the point „a" read

$$\check{\gamma}_{\alpha\beta} = \partial_{\alpha}\check{u}_{\beta} - e_{\alpha\beta}\check{\varphi}, \quad \check{\varkappa}_{\alpha} = \partial_{\alpha}\check{\varphi}.$$

If one inserts the quantities $\psi_{(i)}$, $\gamma_{(i)}$ and $\Delta\varphi_{(i)}$ into slope-deflection equations, the internal forces $M_{(i)}$, $T_{(i)}$, $N_{(i)}$ (referred to the middles of bars $a-S_i$, cf. Fig. 3.1), expressed in terms of strain components $\check{\gamma}_{\alpha\beta}$ and $\check{\varkappa}_{\alpha}$, and with the aid of $\delta\varphi$, δu^{α}

$$\check{M}_{(i)} = \frac{r-s}{2}\frac{EJ}{l}[\delta\varphi - t^{\alpha}_{(i)}\check{\varkappa}_{\alpha}\cdot l],$$

$$\check{N}_{(i)} = 2(r+s)\frac{EJ}{l^2}[t^{\alpha}_{(i)}\delta u^{\beta}\delta_{\alpha\beta} + t^{\alpha}_{(i)} t^{\beta}_{(i)}\check{\gamma}_{\alpha\beta}], \tag{3.2}$$

$$\check{T}_{(i)} = -2(r+s)\cdot\frac{EJ}{l^2}\left[\tilde{t}^{\alpha}_{(i)}\delta u^{\beta}\delta_{\alpha\beta} - \frac{1}{2}\delta\varphi + \tilde{t}^{\beta}_{(i)} t^{\alpha}_{(i)}\check{\gamma}_{\alpha\beta} - \frac{1}{2} t^{\alpha}_{(i)}\check{\varkappa}_{\alpha} l\right]$$

are obtained. The quantities δu^{α}, $\delta\varphi$ can be expressed in terms of strain components and the loads $\overset{*}{F}{}^{\alpha}$, $\overset{*}{M}{}^{*}$, subjected to the node „a". To this end, consider the equations of equilibrium of the node „a", see Fig. 5

$$\sum_{i=I}^{III}(\check{N}_{(i)} t^{\alpha}_{(i)} - \check{T}_{(i)}\tilde{t}^{\alpha}_{(i)}) + \overset{*}{F}{}^{\alpha} = 0,$$

$$\sum_{i=I}^{III}\check{M}_{(i)} - \frac{l}{2}\cdot\sum_{i=I}^{III}\check{T}_{(i)} + \overset{*}{M} = 0.$$

By substituting the formulae (3.2) into above equations and by making use of (3.1), we arrive at the diagonal set of algebraic equations, the solutions of which read

$$\delta u^1 = \frac{1}{2(1+\eta)}\cdot[-l\cdot\check{\varkappa}_2 + (1-\eta)\cdot(\check{\gamma}_{12}+\check{\gamma}_{21})] - \frac{\overset{*}{F}{}^1 l^2}{3(1+\eta)(r+s)EJ},$$

$$\delta u^2 = \frac{1}{2(1+\eta)}\cdot[l\check{\varkappa}_1 + (1-\eta)\cdot(\check{\gamma}_{11}-\check{\gamma}_{22})] - \frac{\overset{*}{F}{}^2 l^2}{3(1+\eta)\cdot(r+s)EJ}, \tag{3.3}$$

$$\delta\varphi = \frac{r+s}{2s}\cdot(\check{\gamma}_{12}-\check{\gamma}_{21}) + \frac{\overset{*}{M} l}{3sEJ}.$$

By inserting the above equations into (3.2), the internal forces $\check{M}_{(k)}$, $\check{T}_{(k)}$, and $\check{N}_{(k)}$ as functions of strain measures

$$\check{M}_{(k)} = \frac{r-s}{4s}\cdot\frac{EJ}{l}\left[-ls\sqrt{3}\,\varepsilon_{k2}\check{\varkappa}_1 + ls(3\delta_{k2}-1)\check{\varkappa}_2 + (s+r)(\check{\gamma}_{12}-\check{\gamma}_{21})\right] + \frac{r-s}{6\cdot s}\cdot\overset{*}{M},$$

$$\check{T}_{(k)} = \frac{-(r+s)EJ}{l^2}\left\{-\frac{\sqrt{3}}{2}\cdot\frac{\bar{\eta}}{\bar{\eta}+1}\cdot l\cdot\varepsilon_{k2}\cdot\check{\varkappa}_1 + \frac{\bar{\eta}}{2(1+\bar{\eta})}\cdot l\cdot(3\delta_{k2}-1)\check{\varkappa}_2 - \right.$$

$$-\frac{\sqrt{3}\cdot\bar{\eta}}{1+\bar{\eta}}\varepsilon_{k2}\cdot(\check{\gamma}_{11}-\check{\gamma}_{22}) + \left[-\frac{r+s}{2s} + \frac{1}{1+\bar{\eta}} - (2+3\delta_{k2})\frac{\bar{\eta}}{1+\bar{\eta}}\right]\check{\gamma}_{12} +$$

$$\left. + \left[\frac{s+r}{2s} - \frac{1}{1+\bar{\eta}} - 3\delta_{k2}\frac{\bar{\eta}}{1+\bar{\eta}}\right]\check{\gamma}_{21}\right\} + \frac{(3\delta_{k2}-1)}{3\cdot(1+\bar{\eta})}\cdot\overset{*}{F}{}^1 +$$

$$+\frac{\sqrt{3}}{3}\frac{1}{1+\bar{\eta}}\varepsilon_{k2}\overset{*}{F}{}^2 + \frac{s+r}{3\cdot s}\frac{\overset{*}{M}}{l}, \tag{3.4}$$

$$\check{N}_{(k)} = \frac{(s+r)\bar{\eta}}{(1+\bar{\eta})}\frac{EJ}{l^2}\left\{-\frac{1}{2}(3\delta_{k2}-1)l\check{\varkappa}_1 - l\cdot\frac{\sqrt{3}}{2}\varepsilon_{k2}\check{\varkappa}_2 + [\bar{\eta}+(2-3\delta_{k2})]\check{\gamma}_{11} + \right.$$

$$\left. +(\bar{\eta}+3\delta_{k2})\check{\gamma}_{22} + \sqrt{3}\,\varepsilon_{k2}(\check{\gamma}_{12}+\check{\gamma}_{21})\right\} - \frac{\sqrt{3}\bar{\eta}}{3(1+\bar{\eta})}\varepsilon_{k2}\overset{*}{F}{}^1 + \frac{\bar{\eta}}{3(1+\bar{\eta})}(3\delta_{k2}-1)\overset{*}{F}{}^2.$$

are finally found. Strain energy of the rods $a-S_i$, i = I, II, III, belonging to the segment can be calculated as follows

$$E_{(I)} = \sum_{i=I}^{III} E_i, \qquad E_i = E_i^M + E_i^T + E_i^N, \text{ where}$$

$$E_i^M = \int_0^{l/2}\frac{(\check{M}_{(i)} - \bar{x}\cdot\check{T}_{(i)})^2 d\bar{x}}{2\cdot E\bar{J}(\bar{x})} + \int_0^{l/2}\frac{(\check{M}_{(i)} + \bar{x}\check{T}_{(i)})^2 d\bar{x}}{2E\bar{J}(\bar{x})}, \tag{3.5}$$

$$E_i^N = 2\int_0^{l/2}\frac{\check{N}_{(i)}^2 d\bar{x}}{2E\bar{A}(\bar{x})}, \qquad E_i^T = 2\cdot\int_0^{l/2}\frac{\varkappa\check{T}_{(i)}^2}{2G\bar{A}(\bar{x})}d\bar{x}, \qquad G = \frac{E}{2(1+\nu)}.$$

The potential $\sigma_I \equiv \sigma_{(v)} = E_{(I)}/P$, $P = 1.5\cdot\sqrt{3}l^2$.

Carrying out the integration we obtain

$$\sigma_I = \sigma_I^0 + \sigma_I^*, \quad \sigma_I^0 = \frac{1}{2}\check{A}^{\alpha\beta\gamma\delta}\check{\gamma}_{\alpha\beta}\check{\gamma}_{\gamma\delta} + \check{B}^{\alpha\beta\gamma}\check{\gamma}_{\alpha\beta}\check{\varkappa}_\gamma + \frac{1}{2}\check{C}^{\alpha\beta}\check{\varkappa}_\alpha\check{\varkappa}_\beta, \tag{3.6}$$
$$\overset{*}{\sigma}_I = \check{\overset{*}{p}}{}^{\alpha\beta}\check{\gamma}_{\alpha\beta} + \check{\overset{*}{m}}{}^\alpha\check{\varkappa}_\alpha.$$

Tensors $\check{\mathbf{A}}$, $\check{\mathbf{B}}$, $\check{\mathbf{C}}$, $\check{\overset{*}{\mathbf{p}}}$, $\check{\overset{*}{\mathbf{m}}}$ take the form

$$\check{A}^{\alpha\beta\gamma\delta} = \check{\lambda}\delta^{\alpha\beta}\delta^{\gamma\delta} + (\check{\mu}+\check{\alpha})\delta^{\alpha\gamma}\delta^{\beta\delta} + (\check{\mu}-\check{\alpha})\cdot\delta^{\alpha\delta}\delta^{\beta\gamma},$$
$$\check{B}^{111} = -\check{B}^{122} = -B^{221} = -B^{212} = B, \quad \text{the others} \quad \check{B}^{\alpha\beta\gamma} = 0, \tag{3.7}$$
$$\check{C}^{\alpha\beta} = \check{C}\delta^{\alpha\beta}, \quad \check{\overset{*}{p}}{}^{\alpha\beta} = 0, \quad \check{\overset{*}{m}}{}^\alpha = 0 \quad \forall(\overset{*}{F}{}^\alpha, \overset{*}{M}),$$

where moduli $\check{\lambda}$, $\check{\mu}$, $\check{\alpha}$, $\check{B}$ and $\check{C}$ are defined as follows

$$\check{\lambda} = \frac{2\sqrt{3}\cdot\eta(\bar{\eta}-1)}{(\bar{\eta}+1)}\frac{EJ}{l^3}, \quad \check{\mu} = \frac{4\sqrt{3}\cdot\eta}{1+\bar{\eta}}\cdot\frac{EJ}{l^3}, \tag{3.8}$$
$$\check{\alpha} = \frac{2\sqrt{3}\cdot\eta}{\bar{\eta}+3\eta}\frac{EJ}{l^3}, \quad \check{B} = \frac{2\sqrt{3}\cdot\eta}{\bar{\eta}+1}\cdot\frac{EJ}{l^2}, \quad \check{C} = \frac{\sqrt{3}\cdot(3\eta+\bar{\eta}+1)}{3(1+\bar{\eta})}\cdot\frac{EJ}{l}.$$

The quantities A and J are fixed acc. to (2.5). The parameters η and $\bar{\eta}$ are defined by (2.9) provided the bars are prismatic. Moreover, if the grid members are slender one can substitute $\eta = \bar{\eta}$ into (3.8), cf. (2.10), to obtain effective moduli independent of l[2]

$$\check{\lambda}/E = \frac{\sqrt{3}(\eta-1)}{6\sqrt{\eta}(\eta+1)}, \quad \check{\mu}/E = \frac{\sqrt{3}}{3\sqrt{\eta}\cdot(1+\eta)}, \quad \check{\alpha}/E = \frac{\sqrt{3}}{24\eta\sqrt{\eta}} \tag{3.9}$$

and „micropolar" constants

$$\check{B}/E = \frac{\sqrt{3}}{6\sqrt{\eta}(1+\eta)}l, \quad \check{C}/E = \frac{\sqrt{3}\cdot(4\eta+1)}{36\eta\sqrt{\eta}\cdot(1+\eta)}\cdot l^2, \tag{3.10}$$

proportional to l^1 and l^2 respectively.

Thus the elastic properties of the hexagonal plate in the plane-stress state are described by the tensors $\check{\mathbf{A}}$, $\check{\mathbf{B}}$, and $\check{C}$. The tensors $\check{\mathbf{A}}$ and $\check{\mathbf{C}}$ are isotropic because the geometry of the lattice (observing, say, a rotation of it around the fixed main node) is invariant under the rotation at the angles $2/3\pi n$, $n = 1, 2, \ldots$. Tensors of the second and the fourth orders, which are invariant under such transformations, have isotropic forms, invariant under arbitrary change of the coordinate system. Thus the components of the tensors $\check{\mathbf{A}}$ and $\check{\mathbf{C}}$ do not vary provided the main nodes are defined as intermediate and vice versa. One can say that these tensors do not depend of the choice of main nodes.

The $\check{\mathbf{B}}$ tensor is characterised by different properties. It can be shown that the components of $\check{\mathbf{B}}$, referred to the cartesian coordinate system $x^{\alpha'}$ rotated at an angle ψ (cf. Fig. 5), can be written as follows

$$\check{B}^{\alpha'\beta'\gamma'} = \check{B}\cdot\chi^{\alpha'\beta'\gamma'}, \tag{3.11}$$

where components of the tensor χ read

$$\chi^{1'1'1'} = -\chi^{1'2'2'} = -\chi^{2'2'1'} = -\chi^{2'1'2'} = \cos 3\psi, \tag{3.12}$$
$$\chi^{2'2'2'} = -\chi^{1'1'2'} = -\chi^{1'2'1'} = -\chi^{2'1'1'} = \sin 3\psi.$$

Components of the tensor $\check{\mathbf{B}}$ depend on the choice of the coordinate system as well as on the choice of main nodes; namely, an interchanging of main and intermediate nonodes imply changes of signs of all components $\check{B}^{\alpha\beta\gamma}$. The tensor $\check{\mathbf{B}}$ couples constitutive equations. Its existence results from the lack of centrosymmetry of the lattice, i.e. from the noncentrosymmetry of the vicinity of the each lattice node. Thus the continuum description of the honeycomb plate requires to apply the uncentrosymmetrical models, cf. [11].

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[2] Horvay's results [3, 4] yield the same definitions of the effective moduli $\check{\lambda}$ and $\check{\mu}$.

The tensors $\overset{*}{\check{\mathbf{p}}}$ and $\overset{*}{\check{\mathbf{m}}}$ are identically equal to zero for the fixed (cf. Fig. 1) coordinate system x^α. Thus the mentioned tensors vanish in an arbitrary coordinate system.

The following factors have inclined the author to recall the Klemm-Woźniak, [6], derivation of constitutive equations:

a) some of the components of the tensor $\check{\mathbf{A}}$ obtained in [6] are incorrect, so that an isotropy of this tensor as well as its relation to Horvay's results could not be revealed

b) considerations have been generalised by taking into account the transverse shear deformations of the lattice (not necessarily prismatic) rods

c) tensors $\overset{*}{\check{\mathbf{p}}}$ and $\overset{*}{\check{\mathbf{m}}}$ vanish. This fact has not been shown in [6].

4. The second version of constitutive equations (variant II)

New procedure, based on the second (II) method (see Sec. 2.2), of defining the strain energy density σ, is proposed here. A starting point is a division of the grid plate into repeated segments of the II type, their centres being in main nodes. Consider the circular vicinity $(r \leqslant l\sqrt{3})$ of the main node „i", Fig. 6. Six main nodes Λ_k, $\Lambda = \mathrm{I}, \mathrm{II}$, $k = 1, 2, 3$ lie on the circumference $r = l\sqrt{3}$. The functions u^α, φ are assumed to be linear in the circle $r \leqslant l\sqrt{3}$. Displacements of main nodes adjoining the node „i" can be expressed by means

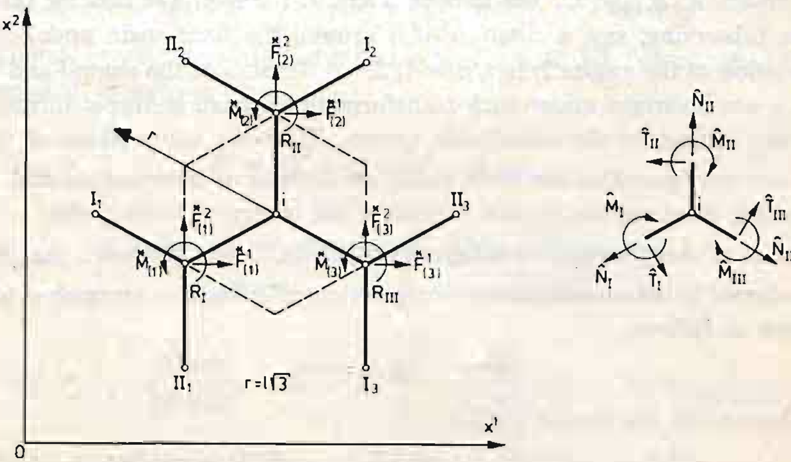

Fig. 6

of the values of functions u^α, φ and their first derivatives in this point. Displacements of the intermediate nodes R_k, $k = \mathrm{I}, \mathrm{II}, \mathrm{III}$, can be found with the aid of the conditions of their equilibrium, analogously to the derivation outlined in Sec. 3. Tiresome rearrangements prove that also the latter displacements can be expressed in terms of the functions u^α, φ, $\partial_\beta u^\alpha$ and $\partial_\beta \varphi$ referred to the point „i". Then on substituting these expressions into slope-deflection Eqs. (2.6) the internal forces $\hat{M}_{(k)}$, $\hat{N}_{(k)}$ and $\hat{T}_{(k)}$ in the middles of the rods $i - R_k$ (see Fig. 6)

$$\hat{M}_{(k)} = \frac{r-s}{4s}\frac{EJ}{l}\left[-\sqrt{3}\,rl\varepsilon_{k2}\hat{\varkappa}_1 + rl(1-3\delta_{k2})\hat{\varkappa}_2 - (r+s)(\hat{\gamma}_{21}-\hat{\gamma}_{12})\right] + \frac{r-s}{6s}\overset{*}{M}_{(k)}, \quad (4.1)$$

$$\hat{T}_{(k)} = \frac{-(r+s)EJ}{l^2}\left\{l\left(\frac{1}{1+\bar{\eta}}-\frac{r}{s}\right)\cdot\left[\frac{1}{2}(3\delta_{k2}-1)\hat{\varkappa}_2+\frac{\sqrt{3}}{2}\varepsilon_{k2}\hat{\varkappa}_1\right]+\right.$$
$$+\left[\frac{r+s}{2s}-\frac{1}{1+\bar{\eta}}+(3\delta_{k2}-2)\frac{\bar{\eta}}{1+\bar{\eta}}\right]\hat{\gamma}_{12}+\left[\frac{-(r+s)}{2s}+\right.$$
$$\left.\left.+\frac{1}{(1+\bar{\eta})}+3\delta_{k2}\frac{\bar{\eta}}{1+\bar{\eta}}\right]\hat{\gamma}_{21}-\frac{\sqrt{3}\cdot\bar{\eta}}{1+\bar{\eta}}\varepsilon_{k2}(\hat{\gamma}_{11}-\hat{\gamma}_{22})\right\}+$$
$$-\frac{(r+s)}{s3l}M_{(k)}-\frac{1}{3(1+\bar{\eta})}(3\delta_{k2}-1)\overset{*}{F}{}^1_{(k)}+\frac{\sqrt{3}}{3(1+\bar{\eta})}\varepsilon_{k2}\overset{*}{F}{}^2_{(k)}, \tag{4.1}$$

[cont.]

$$N_{(k)} = \frac{(r+s)\bar{\eta}}{1+\bar{\eta}}\frac{EJ}{l^2}\left\{\frac{1}{2}(1-3\delta_{k2})l\hat{\varkappa}_1+\frac{\sqrt{3}}{2}\varepsilon_{k2}l\hat{\varkappa}_2+\right.$$
$$\left.+[\bar{\eta}+(2-3\delta_{k2})]\hat{\gamma}_{11}+(\bar{\eta}+3\delta_{k2})\hat{\gamma}_{22}-\sqrt{3}\,\varepsilon_{k2}(\hat{\gamma}_{12}+\hat{\gamma}_{21})\right\}+$$
$$\frac{\bar{\eta}}{3(1+\bar{\eta})}(3\delta_{k2}-1)\overset{*}{F}{}^2_{(k)}+\frac{\sqrt{3}}{3}\frac{\bar{\eta}}{1+\bar{\eta}}\varepsilon_{k2}\overset{*}{F}{}^1_{(k)}$$

are obtained. For details the reader is referred to [12]. The components of the state of strain computed in the point „i" are denoted by $\hat{\gamma}_{\alpha\beta}$ and $\hat{\varkappa}_\alpha$.

Compare Eqs. (3.4) and (4.1). Neglecting differences in signs, which depend on the numbering of the nodes, we have

$$\check{M}_{(k)}(\check{\gamma}_{\alpha\beta},\check{\varkappa}_\alpha) = \hat{M}_{(k)}(\hat{\gamma}_{\alpha\beta},\hat{\varkappa}_\alpha), \qquad \check{T}_{(k)}(\check{\gamma}_{\alpha\beta},\check{\varkappa}_\alpha) = \hat{T}_{(k)}(\hat{\gamma}_{\alpha\beta},\hat{\varkappa}_\alpha),$$

provided there is inserted $\check{\gamma}_{\alpha\beta} = \hat{\gamma}_{\alpha\beta}$, $\check{\varkappa}_\alpha = \hat{\varkappa}_\alpha = 0$; and

$$\check{N}_{(k)}(\check{\gamma}_{\alpha\beta},\check{\varkappa}_\alpha) = \hat{N}_{(k)}(\hat{\gamma}_{\alpha\beta},\hat{\varkappa}_\alpha),$$

provided one substitutes $\check{\gamma}_{\alpha\beta} = \hat{\gamma}_{\alpha\beta}$ and $\check{\varkappa}_\alpha = \hat{\varkappa}_\alpha$. Therefore, Eqs. (3.4) and (4.1) have different right hand sides, if $\varkappa_\alpha$ exist. This fact implies, what will be shown further, that the second version analysed herein leads to the different tensors of elastic moduli from those obtained via Woźniak-Klemm's method.

Proceeding similarly as in Sec. 3 an energy $E_{(II)}$ accumulated in the rods $i-R_k$, belonging to the segment of the type II (cf. Fig. 4), can be evaluated. The energy density $\sigma_{II} = \sigma_{(\wedge)}$ is defined as a quotient E_{II}/P, $P = 1.5\sqrt{3}l^2$. After appropriate rearrangements we finally obtain

$$\sigma_{II} = \mathring{\sigma}_{II}+\sigma^*_{II}, \qquad \sigma^*_{II} = \hat{\overset{*}{p}}{}^{\alpha\beta}\hat{\gamma}_{\alpha\beta}+\hat{\overset{*}{m}}{}^{\alpha}\hat{\varkappa}_\alpha,$$
$$\mathring{\sigma}_{II} = \frac{1}{2}\hat{A}^{\alpha\beta\gamma\delta}\hat{\gamma}_{\alpha\beta}\hat{\gamma}_{\gamma\delta}+\hat{B}^{\alpha\beta\gamma}\hat{\gamma}_{\alpha\beta}\hat{\varkappa}_\gamma+\frac{1}{2}\hat{C}^{\alpha\beta}\hat{\varkappa}_\alpha\hat{\varkappa}_\beta. \tag{4.2}$$

Tensors $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ have the forms

$$\hat{A}^{\alpha\beta\gamma\delta} = \check{A}^{\alpha\beta\gamma\delta}, \qquad \hat{B}^{\alpha\beta\gamma} = \hat{B}\chi^{\alpha\beta\gamma}, \qquad \hat{C}^{\alpha\beta} = \hat{C}\delta^{\alpha\beta}, \tag{4.3}$$

where

$$\hat{B} = \frac{2\sqrt{3}\eta(3\eta-\bar{\eta})}{(1+\bar{\eta})(3\eta+\bar{\eta})}\frac{EJ}{l^2}, \qquad \hat{C} = \frac{\sqrt{3}\,[(3\eta-\bar{\eta})^2+(3\eta+\bar{\eta})]}{3(1+\bar{\eta})(\bar{\eta}+3\eta)}\frac{EJ}{l}. \tag{4.4}$$

Quantities A, J and $\bar{\eta}$ are fixed according to Eqs. (2.5) and (2.9). The tensors $\grave{\mathbf{A}}$ and χ are defined in Sec. 3. In the case of the grid constructed from slender rods ($\bar{\eta} \approx \eta$), we have

$$\hat{B} = \frac{1}{2}\check{B}, \quad \hat{C} = \frac{1+\eta}{1+4\eta}\check{C} \approx \frac{1}{4}\check{C}, \tag{4.5}$$

where $\check{B}$ and $\check{C}$ are defined by Eqs. $(3.8)_{4,5}$.

The components of tensors $\hat{\mathbf{p}}^*$ and $\hat{m}^*$ depend, in a complicated way, on the external loads $\overset{*}{F}{}^{\alpha}_{(k)}$, $\overset{:}{M}_{(k)}$, $k = \mathrm{I, II, III}$, subjected to intermediate nodes. For the sake of brevity, these formulae (obtained in [12]) will not be reported here. However, it is worth mentioning that $\hat{p}^{*\alpha\beta} \neq 0$ and $\hat{m}^{*\alpha} \neq 0$, provided the loads in the intermediate joints exist.

5. Estimations of elastic moduli (resulting from the positive determination of the strain energy)

Obtained in the preceding sections the sets of elastic moduli $(\lambda, \mu, \alpha, \hat{B}, \hat{C})$ and $(\lambda, \mu, \alpha, \hat{B}, \hat{C})$ satisfy the conditions which yield from the positive definition of the quadratic forms $\mathring{\sigma}_I \equiv \mathring{\sigma}_{(\vee)}$, $\mathring{\sigma}_{II} \equiv \mathring{\sigma}_{(\wedge)}$ defined by

$$\mathring{\sigma}_{(\tau)} = \frac{1}{2}\overset{(\tau)}{A}{}^{\alpha\beta\gamma\delta}\overset{(\tau)}{\gamma}_{\alpha\beta}\overset{(\tau)}{\gamma}_{\gamma\delta} + \overset{(\tau)}{B}{}^{\alpha\beta\gamma}\overset{(\tau)}{\gamma}_{\alpha\beta}\overset{(\tau)}{\varkappa}_{\gamma} + \frac{1}{2}\overset{(\tau)}{C}{}^{\alpha\beta}\overset{(\tau)}{\varkappa}_{\alpha}\overset{(\tau)}{\varkappa}_{\beta}, \qquad \tau = \vee, \wedge. \tag{5.1}$$

This fact follows from the derivation of $\mathring{\sigma}_{(\tau)}$: e.g., when $\tau = \mathrm{I}$, the RHS of the Eq. $(3.5)_1$, which defines an energy $E_{(I)}$ accumulated in the rods belonging to a segment, is expressed by means of integrals with positive integrand functions; thus the energy $E_{(\tau)}$ is positive definite for all arbitrary values of components $\overset{(\tau)}{\gamma}_{\alpha\beta}$ and $\overset{(\tau)}{\varkappa}_{\alpha}$. Nevertheless, the explicit form of energy estimations, which impose certain restrictions on the values of effective elastic moduli, is worth considering.

Let us transform the function $\mathring{\sigma}$ (an index τ is neglected now), to the convenient form for the further analysis

$$\mathring{\sigma} = \frac{1}{2}\Xi^{\alpha\beta}\eta_\alpha\eta_\beta, \qquad \alpha, \beta = 1, 2, \ldots, 6, \tag{5.2}$$

where $\eta_1 = \gamma_{11}$, $\eta_2 = \gamma_{22}$, $\eta_3 = \gamma_{12}$, $\eta_4 = \gamma_{21}$, $\eta_5 = \varkappa_1$, $\eta_6 = \varkappa_2$. A coordinate system is fixed as in Fig. 1. The matrix Ξ can be written in the form

$\Xi =$

$2\mu+\lambda$	λ			B	
λ	$2\mu+\lambda$			$-$B	
		$\mu+\alpha$	$\mu-\alpha$		$-$B
		$\mu-\alpha$	$\mu-\alpha$		$-$B
B	$-$B			C	
		$-$B	$-$B		C

By applying Sylvester theorem the following necessary and sufficient conditions for the matrix Ξ to be positive definite

$$\mu > 0, \quad \alpha > 0, \quad \mu+\lambda > 0, \quad C > 0, \quad B^2 < C\mu \tag{5.3}$$

are obtained. Positive definition of the quadratic form (5.2) does not depend of the choice of a coordinate system. Therefore, the inequalities (5.3) are sufficient for $\mathring{\sigma}$ to be positive determined. Note yet that the sign B (which depends on the choice of main nodes) does not affect in (5.3). The inequality $(5.3)_5$ shows that the moduli B and C are not arbitrary; this estimation can be treated as an upper bound for B or a lower one for C.

6. Effective Young moduli and Poisson's ratios

The tensor **A** (symmetrised in respect to both pairs of indices) can be written in the form

$$A^{(\alpha\beta)(\gamma\delta)} = \frac{E_1}{1+\nu_1}\cdot\left[\frac{\nu_1}{1-\nu_1}\delta^{\alpha\beta}\delta^{\gamma\delta}+\frac{1}{2}(\delta^{\alpha\gamma}\delta^{\beta\delta}+\delta^{\alpha\delta}\delta^{\beta\gamma})\right] \tag{6.1}$$

similar to that known from a classical theory of a plane-stress state.

Moduli E_1 and ν_1, being effective Young and Poisson constants, can be expressed by means of Horvay's [3] formulae

$$E_1 = \frac{4\mu(\mu+\lambda)}{2\mu+\lambda} = \frac{4}{\sqrt{3\eta}(\eta+3)}\cdot E,$$
$$\nu_1 = \frac{\lambda}{2\mu+\lambda} = \frac{\eta-1}{\eta+3}. \tag{6.2}$$

Energy inequalities (5.3) imply estimations

$$E_1 > 0, \quad -1 < \nu_1 < 1, \tag{6.3}$$

weaker, than those known from a classical three-dimensional theory of elasticity: $E > 0$, $-1 < \nu < 1/2$. Effective Young and Poisson's moduli can be defined in different way, taking as a starting point the reverse form of the constitutive equations (2.13)

$$\gamma_{\alpha\beta} = \overset{-1}{A}_{\alpha\beta\gamma\delta}p^{\gamma\delta}+\overset{-1}{B}_{\alpha\beta\gamma}m^{\gamma}+\gamma^*_{\alpha\beta},$$
$$\varkappa_{\alpha} = \overset{-1}{B}_{\alpha\beta\gamma}p^{\beta\gamma}+\overset{-1}{C}_{\alpha\beta}m^{\beta}+\varkappa^*_{\alpha} \tag{6.4}$$

Displaying the symmetrized part of the tensor $\overset{-1}{\mathbf{A}}$ in the form

$$\overset{-1}{A}_{(\alpha\beta)(\gamma\delta)} = -\frac{\nu_2}{E_2}\delta_{\alpha\beta}\delta_{\gamma\delta}+\frac{1+\nu_2}{2E_2}(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}),$$

we obtain

$$E_2 = \frac{4(\lambda+\mu)(\mu-B^2/C)}{2\mu+\lambda-B^2/C}, \quad \nu_2 = \frac{\lambda+B^2/C}{2\mu+\lambda-B^2/C}. \tag{6.5}$$

It is not diffucult to prove that constants E_α, ν_α, $\alpha = 1,2$, satisfy inequalities

$$E_2 < E_1, \quad \nu_2 > \nu_1 \tag{6.6}$$

and

$$E_2 > 0, \quad -1 < \nu_2 < 1, \tag{6.7}$$

the latter of which are identical with (6.3). Note that moduli E_2 and ν_2 do not depend of α constant. In the case of $B = 0$, we have $E_1 = E_2, \nu_1 = \nu_2$, of course.

Moduli E_2 and ν_2 depend on the choice of the version (I or II) of constitutive relations; this dependence is weak in the case of slender lattice rods (cf. Figs. 7, 8) since then, according to (4.5) one obtains $\hat{B}^2/\hat{C} \approx \check{B}^2/\check{C}$. The patterns of variation of effective moduli

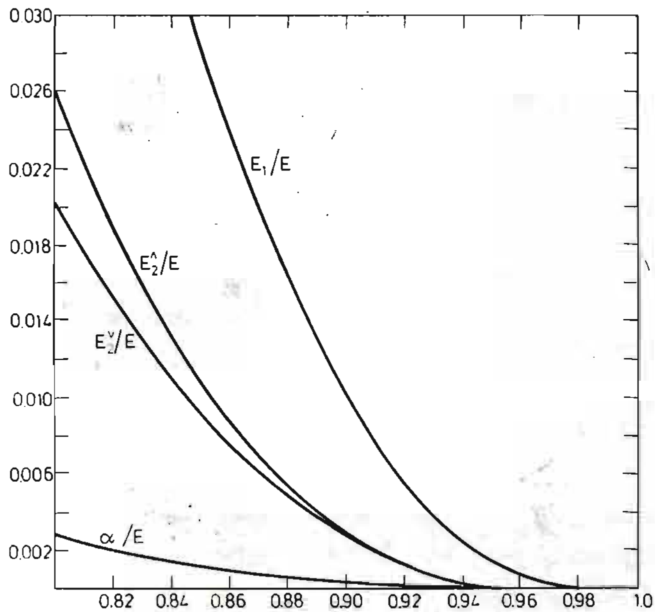

Fig. 7

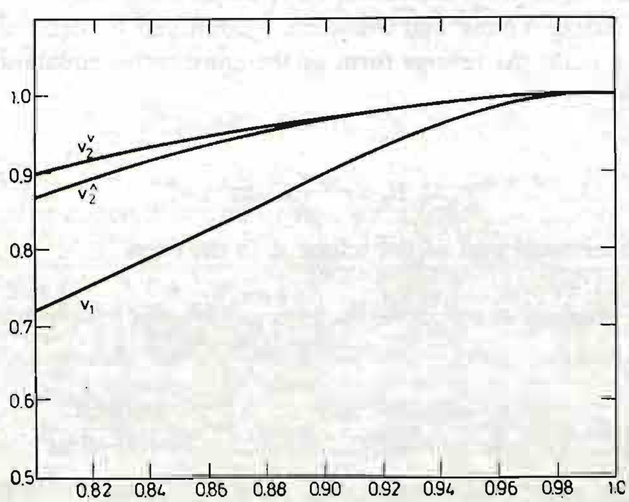

Fig. 8

$E_1, \nu_1, \check{E}_2, \check{\nu}_2, \hat{E}_2, \hat{\nu}_2$ and α depending on the ratio ϱ are shown in Figs. 7, 8. The diagrams were made under the the assumption $\eta = \bar{\eta}$. It is readily seen that

$$\lim_{\varrho\to 0} E_\alpha(\varrho) = 0, \quad \lim_{\varrho\to 1} \nu_\alpha(\varrho) = 1, \quad \alpha = 1, 2.$$

An analysis of variation of moduli B and C will be presented in a separate paper.

7. Governing equations in terms of displacements. Boundary value problems

Consider a lattice-type honeycomb plate, Fig. 9, whose mid-surface is referred to cartesian coordinate system x^α. Assume the family of main nodes according to Fig. 9. A part Γ_1 of the boundary is loaded by forces and couples: $\bar{p}^\alpha$ and $\bar{m}$. On Γ_2 — displacements $\bar{u}^\alpha$ and $\bar{\varphi}$ are known. The loads subjected to internal main nodes are approximated by functions p^α, Y^3. The loads in intermediate nodes are characterized by tensors $\mathbf{p}^*$ and $\mathbf{m}^*$.

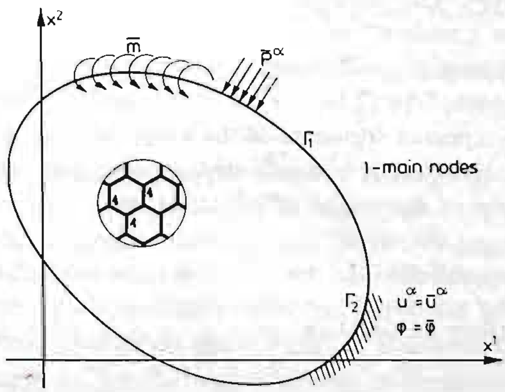

Fig. 9

Substituting constitutive Eqs. (2.13) into equations of equilibrium (2.14) (where $\sigma_{(i)}$ has the form (3.6), provided $i = \mathrm{I}$ or (4.2), provided $i = \mathrm{II}$), and taking into account strain-displacement relations (2.12), the governing set of equilibrium equations in terms of displacements

$$\begin{gathered}
[(2\mu+\lambda)\partial_1^2+(\mu+\alpha)\partial_2^2]u_1+[(\lambda+\mu-\alpha)\partial_1\partial_2]u^2+[B(\partial_1^2-\partial_2^2)+2\alpha\partial_2]\varphi+'p^1 = 0,\\
[(\lambda+\mu-\alpha)\partial_1\partial_2]u^1+[(2\mu+\lambda)\partial_2^2+(\mu+\alpha)\partial_1^2]+[-2B\partial_1\partial_2-2\alpha\partial_1]\varphi+'p^2 = 0, \quad (7.1)\\
[B(\partial_1^2-\partial_2^2)-2\alpha\partial_2]u^1+[-2B\partial_1\partial_2+2\alpha\partial_1]u^2+[C(\partial_1^2+\partial_2^2)-4\alpha]\varphi+'\Psi^3 = 0,
\end{gathered}$$

where

$$'p^\alpha = p^\alpha+\partial_\beta \overset{*}{p}{}^{\beta\alpha}, \qquad 'Y^3 = Y^3+\partial_\alpha \overset{*}{m}+e_{\alpha\beta}\overset{*}{p}{}^{\alpha\beta} \tag{7.2}$$

are obtained. The mixed boundary value problems are formulated due to Woźniak [2]: find the functions u^α and φ satisfying Eqs. (7.1) and boundary conditions

$$\begin{aligned}
u_\alpha = \bar{u}_\alpha, \quad \varphi = \bar{\varphi} \quad &\text{on} \quad \Gamma_2\\
p^{\alpha\beta}n_\beta = \bar{p}^\alpha, \quad m^\alpha n_\alpha = \bar{m} \quad &\text{on} \quad \Gamma_1,
\end{aligned} \tag{7.3}$$

where n^α denote components of a unit vector normal to the boundary.

8. Final remarks

Two versions of the lattice-type hexagonal plate theory (in plane-stress state) based on the various ways of defining density of strain energy of the structure have been derived. It is worth distinguishing between similarities and differences of the presented variants by Woźniak-Klemm and by the present author.

i) stress tensors ($\check{\mathbf{p}}$,$\check{\mathbf{m}}$) and ($\hat{\mathbf{p}}$, $\hat{\mathbf{m}}$), and strain measures ($\check{\gamma}$, $\check{\varkappa}$), ($\hat{\gamma}$, $\hat{\varkappa}$) as well as displacements u^{α}, φ are referred to intermediate (version I) or to main nodes (version II). This is not in contradiction with the fact, that in both cases, functions u^{α}, φ approximate displacements of main nodes

ii) in both versions constitutive equations have similar form; specifically, tensors $\check{\mathbf{A}}$ and $\hat{\mathbf{A}}$ are identical. The components of $\mathbf{A}$ are expressed by moduli λ, μ and α which do not depend of the length „l" of the bars, but depend on the slenderness ratio η, only. The qualitative differences occur between the tensors $\mathbf{B}$ and $\mathbf{C}$, dependent explicitly on „l" and „l^2", respectively. The mentioned moduli describe a „microstructure" of the grid plate and determine a scale effect

iii) The physical meaning of equilibrium Eqs. (2.14) is different in both versions. In Klemm-Woźniak's approach, Eqs. (2.14) can be understood as approximate conditions of equilibrium of all of the repeated segments of the I type (cf. Fig. 4); thus the equilibrium of intermediate nodes is satisfied. It is worth emphasising, that the latter conditions have been utilised in the course of derivation of the stress-strain relations. Equilibrium of the segments (I) does not imply the equilibrium of main nodes. Therefore, only the necessary equilibrium conditions are satisfied. In the secónd version, Eqs. (2.14) express equilibrium conditions of segments of (II) type, hence the equilibrium equations of main nodes are fulfilled. The equilibrium equations of intermediate nodes have been satisfied in the course of the derivation of stress-strain relations. Therefore both sufficient and necessary conditions are fulfilled

iv) the essential quantitative difference between two analysed approches results from the fact, that in the II (second) version tensors $\overset{*}{\hat{\mathbf{p}}}$ an $\overset{*}{\hat{\mathbf{m}}}$ do not vanish, whereas in the first one these tensors are equal to zero. Therefore, in II variant, constitutive equations depend on the loads subjected to intermediate nodes of the lattice, whereas the loads in main nodes occur in the RHS of equilibrium equations. In the governing equations (7.1) all of the loads have effect.

In version I diffuculties occur, when loads in main nodes are taken into account, because in the RHS of (2.14) only these loads, which are subjected within the segment (I), can be included. Therefore, perhaps, in the first variant the loads in main nodes cannot be considered.

In the subsequent papers an attempt will be made to evaluate the range of applicability of the considered versions of Woźniak's lattice-type, honeycomb plate theory. It will be shown that valuable results can be obtained using the methods of solid state physics cf. [13].

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Резюме

ДВЕ КОНТИНУАЛЬНЫЕ МОДЕЛИ (ПО ВОЗНЯКУ) ГЕКСАГОНАЛЬНЫХ СЕТЧАТЫХ ПЛАСТИНОК

В работе выводятся две концепции континуальной описи густых, упругих, гексагональных сетчатых пластинок. Обе версии базируются на теории Возняка, в которой поведение сетчатых поверхностных конструкций описывается при помощи модели Коссера с волокнистой структурой. Первая версия является обобщением и развитием трудов Клемма и Возняка посвященных сетчатым пластинкам со структурой сотов меда. Во второй версии приняты иные предположения касающиеся метода определения упругого потенциала пластинки. Полученные модели дают разные „микрополярные" константы (B, C), вызывающие масштабные эффекты.

Исследованы ограничения вытекающие из положительности энергии деформации и показано, что модели B и C связанные неравенством $B^2 < C\mu$, где μ — эффективный модуль Ляме.

В работе выводятся уравнения в смещениях и соответствующие краевые условия.

Streszczenie

DWA KONTYNUALNE MODELE (TYPU WOŹNIAKA) HEKSAGONALNYCH TARCZ SIATKOWYCH

W pracy przedstawiono dwie koncepcje opisu kontynualnego gęstych, sprężystych, heksagonalnych tarcz siatkowych. Obie wersje bazują na teorii Woźniaka — aproksymacji zachowania się dźwigarów siatkowych za pomocą modelu matematycznego dwuwymiarowego ośrodka Cosseratów o włóknistej strukturze. Pierwsza wersja stanowi uogólnienie i rozwinięcie wyników pracy Klemma i Woźniaka dotyczącej siatek o strukturze plastra miodu. W drugiej wersji przyjęto nieco inne założenia dotyczące sposobu definiowania potencjału sprężystego tarczy. Otrzymane wersje prowadzą do innych zestawów stałych „mikropolarnych" (B, C) odpowiadających za efekt skali. Zbadano ograniczenia wynikające z warunku dodatniej określoności energii odkształcenia i wykazano, że stałe B i C powinny spełniać nierówność $B^2 < C\mu$, gdzie μ — zastępczy moduł Lamégo. Wyprowadzono równania „przemieszczeniowe" i sformułowano dopuszczalne warunki brzegowe.

Praca zostala złożona w Redakcji 26 kwietnia 1983 roku