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OPTIMUM DESIGN OF THIN-WALLED I-BEAM SUBJECTED TO STRESS CONSTRAINT

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This paper deals with the problem of optimization of a thin-walled open section I-beam loaded in a complex way, subjected to the bending, torsion and constrained torsion. A general case of bending moments about two centroidal axes, the torsion and the bimoment acting simultaneously, is derived and then some particular loading cases are considered. The problem is reduced to the determination of minimum mass, i.e. minimum cross-sectional area of structural thin-walled beam elements of the chosen shape for the given complex loads, material and geometrical characteristics. The optimization parameters have been determined by Lagrange's multipliers method. The area of the cross-section has been selected as the objective function. The stress constraint is introduced and used as the constraint function. The obtained results are used for numerical calculation.

Key words: optimization, thin-walled beams, optimal dimensions

1. Introduction

Investigations of the behaviour of thin-walled members with open cross-sections have been carried out extensively since the early works of Timoshenko and Gere (1961), who were among the first to publish a number of books on the strength of materials, the theory of elasticity and the theory of stability, and who also developed the theory of bending of beams and plates. Vlasov (1959) contributed largely to the theory of thin-walled structures by developing the theory of thin-walled open section beams. Kollbruner and Hajdin (1970) expanded the field of thin-walled structures by a range of works. Also, Murray (1984) and Rhodes and Spence (1984) should be mentioned for introducing the theoretical aspects of the behaviour of thin-walled structures. The problem of solution of various optimization tasks has been discussed in a number of works and monographs.

Due to their low weight, thin-walled open section beams are widely applied in many structures. Many modern metal structures (motor and railroad vehicles, naval structures, turbine blades) are manufactured using thin-walled elements (shells, plates, thin-walled beams) which are subjected to complex loads. In most structures, it is possible to find elements in which, depending on loading cases and the way they are introduced, the effect of constrained torsion is present and its consequences are particularly evident in the case of thin-walled profiles.

Among the authors who developed theoretical fundamentals of the optimization method, Fox (1971), Brousse (1975), Prager (1974) and Rozvany (1992) should be given the most prominent place. Many studies have been conducted on optimization problems, treating the cases where geometric configurations of structures are specified and only the dimensions of structural members and the areas of their cross-sections are determined in order to obtain the minimum structural weight or cost (Rong and Yi, 2010; Mijailović, 2010). Tian and Lu (2004) presented a combined theoretical and experimental study on the minimum weight and the associated optimal geometric dimensions of an open-channel steel section.

Many authors, including Farkas (1984), applied mathematical problems to the conditional extreme of the function with more variables onto the cross-sectional area of the structure and defined the optimum cross-section from the aspect of load and consumption of the material. Then, there is a series of works where the optimization problem of various cross-sections, such as triangular cross-section (Selmic *et al.*, 2006), I-section (Andjelic, 2007), channel-section (Andjelic and Milosevic-Mitic, 2006) or Z-section beams (Andjelic *et al.*, 2009) is solved by applying the Lagrange multiplier method.

One of the thin-walled profiles commonly used in steel structures is the I-cross section.The aim of this study is to expand these works and present one approach to the optimization of a thin-walled I-section beam.

2. Formulation of the optimization problem

In the analysis of thin-walled beams, the specific geometric nature of the beam consisting of an assembly of thin sheets will be exploited to simplify the problem formulation.

It is assumed that the load can be applied to the I-beam in an arbitrary way.

In the process of structure dimensioning, apart from defining the requested dimensions necessary to permit the particular part of the structure to support the applied loads, it is often of significance to determine the optimal values of the dimensions. The I-cross section, being a very often used thin-walled profile in steel structures, is considered here as the object of optimization.

The cross-section of the considered beam (Fig. 1) with principal centroidal axes X_i $(i = 1, 2)$ has an axis of symmetry. It is assumed that its flanges have equal widths $b_1 = b_3$, and thicknesses $t_1 = t_3$, and that its web has the width b_2 and thickness t_2. The ratios of thicknesses and widths of the flanges and web are treated as non-constant quantities.

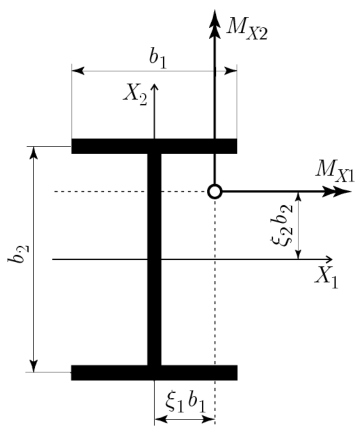

Fig. 1. Cross-section

It is also assumed that the loads are applied in two longitudinal planes, parallel to the principal axes X_i $(i = 1, 2)$ of the cross-section at the distances $\xi_i b_i$ $(i = 1, 2)$ (Fig. 1).

If applied in such a way, the loads will cause the bending moments M_{Xi} $(i = 1, 2)$, acting in the above mentioned two planes parallel to the longitudinal axis of the beam, and consequently the effects of the constrained torsion will occur in form of the bimoment B, making the stresses depend on the boundary conditions (Kollbruner and Hajdin, 1970; Ružić, 1995).

The aim of the paper is to determine the minimal mass of the beam or, in other words, to find the minimal cross-sectional area

$$A = A_{min} \tag{2.1}$$

for the given loads and material and geometrical properties of the considered beam, while satisfying the constraints.

The process of selecting the best solution from various possible solutions must be based on a prescribed criterion known as the objective function. In the considered problem, the cross-sectional area will be treated as the objective function, and it is evident from Fig. 1 that

$$A = \sum_{i=1}^{3} b_i t_i = \sum_{i=1}^{3} \mu_i b_i^2 \tag{2.2}$$

The coefficients μ_i, Eq. (2.3), represent the thickness-length ratios of the cross-sectional walls (Fig. 1)

$$\mu_i = \frac{t_i}{b_i} \qquad i = 1,2,3 \tag{2.3}$$

where t_i and b_i are thickness and widths of the flanges and web.

Because $b_1 = b_3$, equation (2.2) is possible to be presented in the form

$$A = A(b_1, b_2) = 2b_1t_1 + b_2t_2 \tag{2.4}$$

2.1. Constraints

The formulation is restricted to stress analysis of thin-walled beams with open sections.

When the bending moments act in planes parallel to the longitudinal axis (Fig. 1) at the distances $\xi_i b_i$ $(i = 1,2)$, the bimoment will occur as a consequence, and it can be expressed as the function of the bending moments and the eccentricities of their planes $\xi_i b_i$ $(i = 1,2)$ in the following way (Kollbruner and Hajdin, 1970; Ružić, 1995)

$$B = \sum_{i=1}^{2} \xi_i b_i M_{Xi} \tag{2.5}$$

The normal stresses are the consequences of the bending moments M_{Xi} $(i = 1,2)$, and of the bimoment B that occurs if the constrained torsion exists, and they will be denoted by σ_{Xi} $(i = 1,2)$ and σ_ω, respectively.

The maximal normal stresses (Kollbruner and Hajdin, 1970; Ružić, 1995) are defined in the form

$$\sigma_{max} = \sum_{i=1}^{2} \sigma_{Xi\,max} + \sigma_{\omega\,max} \tag{2.6}$$

where

$$\sigma_{Xi\,max} = \frac{M_{Xi}}{W_{Xi}} \qquad i = 1,2 \qquad\qquad \sigma_{\omega\,max} = \frac{B}{W_\omega} \tag{2.7}$$

and W_{Xi} $(i = 1,2)$ are the section moduli and W_ω is the sectorial section modulus for the considered cross-section. Incorporating (2.7) into expression (2.6), the maximal normal stress will become

$$\sigma_{max} = \frac{M_{X1}}{W_{X1}} + \frac{M_{X2}}{W_{X2}} + \frac{B}{W_\omega} \tag{2.8}$$

If the moment of torsion acts simultaneously with the bending moment, the expression for the shear stress is introduced in addition to the one for normal stress

$$\tau_{max} = \frac{M_{t\,max}}{W_t} \tag{2.9}$$

where $M_{t\,max}$ is the maximum value of the concentrated torque, and W_t is the torsion modulus.

Then the equivalent stress is calculated according to

$$\sigma_e = \sqrt{\sigma_{max}^2 + \alpha\tau_{max}^2} \tag{2.10}$$

where the coefficient α indicates whether it is the Hypothesis of maximum shear stress $(\alpha = 4)$ or the Hypothesis of the maximum specific deformation work expended in the change of shape $(\alpha = 3)$.

If the allowable stress is denoted by σ_0, the constraint function can be written as

$$\varphi = \sigma_e - \sigma_0 \leqslant 0 \tag{2.11}$$

Therefrom, the constraint function is obtained

$$\varphi(\sigma,\tau) = \sigma_{max}^2 + \alpha\tau_{max}^2 - \sigma_0^2 \leqslant 0 \tag{2.12}$$

The normal and shear stresses are taken into account in the considerations that follow, and that is the reason why the constraints treated in the paper are the stress constraints.

After incorporating W_{X1}, W_{X2}, W_ω and W_t into expression (2.12), the constraint function becomes

$$\begin{aligned}\varphi(b_1,b_2) &= 144M_{X1}^2 + 36M_{X2}^2C^2\Big(\frac{b_2}{b_1}\Big)^2 + 144B^2C^2\frac{1}{b_1^2} + 144M_{X1}M_{X2}C\frac{b_2}{b_1}\\ &\quad + 288M_{X1}BC\frac{1}{b_1} + 4\frac{C^2}{C_1^2}\frac{\alpha^2}{\mu_1^4} - 4\sigma_0^2\mu_1^2b_1^4b_2^2C^2\end{aligned} \tag{2.13}$$

where

$$C = 6 + \frac{\mu_2}{\mu_1}\Big(\frac{b_2}{b_1}\Big)^2 = 6 + \frac{t_2b_2}{t_1b_1} \tag{2.14}$$

Expression (2.13) represents the constraint function corresponding to the given stress constraints.

3. Results and discussion

3.1. Lagrange multiplier method

The Lagrange multiplier method (Fox, 1971; Mijailović, 2010; Onwubiko, 2000; Zoller, 1972) is a powerful tool for solving these types of problems and represents a classical approach to constraint optimization. It is a method for finding the extremum of a function of several variables, when the solution must satisfy a set of constraints. The Lagrange multiplier, labelled as λ, measures the change of the objective function with respect to the constraint.

Applying the Lagrange multiplier method to the vector that depends on two parameters b_i $(i = 1,2)$

$$\frac{\partial}{\partial b_i}[A(b_1,b_2) + \lambda\varphi(b_1,b_2)] = 0 \qquad i = 1,2 \tag{3.1}$$

a system of equations is obtained, and after the elimination of the multiplier λ, it becomes

$$\frac{\partial A(b_1,b_2)}{\partial b_1}\frac{\partial\varphi(b_1,b_2)}{\partial b_2} = \frac{\partial A(b_1,b_2)}{\partial b_2}\frac{\partial\varphi(b_1,b_2)}{\partial b_1} \tag{3.2}$$

3.2. Analytic solution

Let the ratio

$$z = \frac{b_2}{b_1} \tag{3.3}$$

be the optimal ratio of the parts of the considered cross-section and let

$$\psi = \frac{t_2}{t_1} \tag{3.4}$$

be the ratio of the flange and web thicknesses.

After incorporating expression (2.5) for the bimoment in equation (2.13), equation (3.2) can be reduced to an equation of the ninth order (3.5), whose solutions yield the optimal values of ratio (3.3)

$$\sum_{k=0}^{9} c_k z^k = 0 \tag{3.5}$$

The coefficients c_k in (3.5) are given in the Appendix.

3.2.1. Some particular cases

The obtained results are used for the calculation that follows. In the case when only bending moments act, some particular cases can be considered.

In the case when only normal stresses occur in the cross-section, it is possible to write the constraint function in the form

$$\varphi = \sigma_{max} - \sigma_0 \leqslant 0 \tag{3.6}$$

It must be underlined that in this case, when shear stresses are disregarded, the constraint function is considerably simplified

$$\varphi(\sigma) = \sigma_{max} = \sigma_{X1\,max} + \sigma_{X2\,max} + \sigma_{\omega\,max} \leqslant \sigma_0 \tag{3.7}$$

For the allowable stress σ_0, according to equations (3.3) and (3.4), the constraint function can be reduced and written as

$$\varphi = \varphi(b_1, b_2) = 6M_{X1}\frac{1}{t_1 b_1 b_2(6+\psi z)} + 3M_{X2}\frac{1}{t_1 b_1^2} + 6B\frac{1}{t_1 b_1^2 b_2} - \sigma_0 \leqslant 0 \tag{3.8}$$

After incorporating expression (2.5) for the bimoment in equation (3.8), equation (3.2) can be reduced to an equation of the fourth order (3.9), whose solutions yield the optimal values of ratio (3.3)

$$\sum_{k=0}^{k=4} c_k z^k = 0 \tag{3.9}$$

The coefficients c_k in (3.9) are given in the Appendix.

It is obvious that the coefficients c_k $(k = 1, 2, \dots, 6)$ depend on the ratio of the bending moments M_{X2}/M_{X1} and on the eccentricities ξ_1 and ξ_2 of their planes.

The results that follow were obtained by an analytical approach.

3.2.2. *Optimal values* $z = b_2/b_1$

From the general case, when the bending moments about both axes occur simultaneously with the bimoment, some particular cases can be considered, depending on the ratio M_{X2}/M_{X1}.

The optimal ratios $z = b_2/b_1$ defined by (3.3) and obtained from equation (3.9) are calculated for $M_{X2}/M_{X1} = 0$, $\psi = 0.5$, 0.75, 1 and ξ_1, $\xi_2 = 0$, 0.2, 0.4, 0.6, 0.8, 1.0, or in other way, for $0 \leqslant \xi_1 \leqslant 1$, $0 \leqslant \xi_2 \leqslant 1$.

The optimal values of z for $M_{X2}/M_{X1} = 0$ are shown in Table 1 as functions of ξ_1 and ψ. From equation (3.9), according to the coefficients presented in Appendix, it is obvious that the quantity z does not depend on the eccentricity ξ_2.

Table 1. Optimal z for $M_{X2}/M_{X1} = 0$

ψ	ξ_1					
	0	0.2	0.4	0.6	0.8	1.0
0.5	12	2.83	2.46	2.32	2.24	2.19
0.75	8	1.89	1.64	1.54	1.49	1.46
1	6	1.42	1.23	1.16	1.12	1.09

It is evident from Fig. 2 that the quantity z is decreasing when the eccentricity ξ_1 increases. Also, it can be inferred that the values of z are decreasing when the ratio (3.4) $\psi = t_2/t_1$ increases.

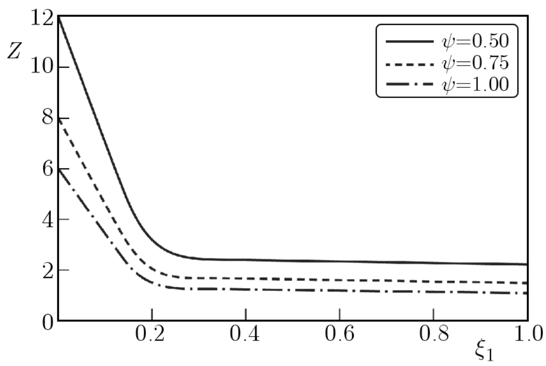

Fig. 2. Optimal z for $M_{X2}/M_{X1} = 0$

4. The loading cases

Some particular cases can be considered depending on the loading case. The loading cases were considered when concentrated bending moments were applied at the free end for three positions of the load plane with respect to the shearing plane:

(a) **Loading case 1:** A beam loaded with a concentrated bending moment at the free end.

In the present section, the cantilever I-beam is fixed at one end and subjected to the concentrated bending moment $M_{X1} = 100\,\mathrm{Nm}$, $M_{X2} = 0$ $(M_{X2}/M_{X1} = 0)$.

Two loading cases (two ways of introducing the concentrated bending moment) for relations (3.4) $\psi = 0.5$, 0.75, 1 and for the eccentricities $0 \leqslant \xi_1 \leqslant 1$, $0 \leqslant \xi_2 \leqslant 1$ (Fig. 1) are considered:

– Loading case 1.1: $\xi_1 = \xi_2 = 0$
– Loading case 1.2: $\xi_1 = 0.5$, $\xi_2 = 0$

The results for ratios (3.3) $z = b_2/b_1$ obtained from equation (3.9) are given in Table 1.

(b) **Loading case 2:** A beam loaded by a concentrated force at the free end.

The considered cantilever I-beam of the length $l = 1500\,\mathrm{mm}$, loaded by the concentrated force $F^* = 1000\,\mathrm{N}$ passing through the shear centre plane, are presented. In the case of the I-beam, the shear center plane coincides with the web.

The optimal values z_{opt} are calculated as above explained. The results for ratios (3.3) $z = b_2/b_1$ obtained from equation (3.9) are the same as the results for load 1.1, and they are presented in Table 1.

5. Numerical example and analysis of results

As a numerical example, the considered cantilever beam with the length $l = 1500\,\mathrm{mm}$, fixed at one end, is subjected to the bending moments $M_{X1} = 100\,\mathrm{Nm}$, $M_{X2} = 0$.

The initial cross-sectional geometrical characteristics are calculated taking into account the initial dimensions of the I-section beam. It is assumed that the considered section has the initial cross-sectional geometrical characteristics: $b_1 = 51.75\,\mathrm{mm}$, $b_2 = 92\,\mathrm{mm}$, $t_1 = 8\,\mathrm{mm}$, $t_2 = 6.5\,\mathrm{mm}$. It represents the *initial model* with the *initial area* of the cross-section. For the given loads and the defined geometry of the profile, the initial stresses are calculated.

Starting from the initial relation zinitial and for the initial wall thicknesses t_1 and t_2 the optimal relation zoptimal is calculated defining the *optimal area* of the cross-section.

5.1. Determination of the minimum cross-sectional area

The problem is considered in two ways:

(1) The optimal dimensions of the cross-sections $b_{1\,opt}$ and $b_{2\,opt}$ are obtained by equalizing the *initial* and the *optimal area* ($A_{init} = A_{opt}$) and by using the calculated optimal relation z. In that case, the normal stress, lower than the initial one, is obtained ($\sigma_{opt} < \sigma_{init}$). It represents *optimal model no. 1* (Table 2).

(2) From the condition requiring that the stresses must be lower than the allowable one, i.e. the *initial stress*, the optimal values $b_{1\,opt}$ and $b_{2\,opt}$ are obtained using the calculated optimal relation z and comparing the stress defined by the optimal geometrical characteristics of the *initial stress*. It represents *optimal model no. 2*. Starting from the optimal cross-sectional dimensions ($b_{1\,opt}$ and $b_{2\,opt}$, the optimal – minimum cross-sectional area A_{min} is calculated for each loading case, and the results including the saved mass of the material are given in Table 2.

It is noticeable from Table 2 that for all the loading cases the level of stresses is decreased in *optimal model no. 1* with the area of the cross-section having the same value as in the *initial model*. The saved mass of material is increased with respect to the initial stress limits in *optimal model no. 2*, where the area is smaller than the initial one. The calculation showed that the maximum saved material is obtained in loading case 1.1, and the minimum in loading case 1.2. The result for loading case 2.1 for the saved mass is identical with the results for load 1.1, and they are presented in Table 2.

This allows the conclusion that if the distance of the loading plane from the shearing plane is increased, the optimization of the cross-section is less necessary.

Table 2. Normal stresses and saved mass: $t_1 = 8$ mm and $t_2 = 6.5$ mm, $z_{init} = 1.78$

Loading case	z_{opt}	σ_{init} [MPa]	$\sigma_{opt\ no.1}$ [MPa]	$\sigma_{opt\ no.2}$ [MPa]	$A_{init} = A_{opt\ no.1}$ [mm^2]	$A_{init} = A_{opt\ no.2}$ [mm^2]	Saved mass no. 2 [%]
1.1	7.38	2.01	1.59	2.01		1260	11.64
1.2	1.46	9.43	9.37	9.43	1426	1423	0.217
2.1	7.38	30.2	23.8	30.2		1260	11.64

6. Application of the finite element method

The presented loading cases are treated also by the Finite Element Method (FEM) (Zloković *et al.*, 2004). The model consists of 360 2D plate finite elements. The flanges are divided into 90 elements each, and the web into 180 elements. The FEM was applied to check the results obtained in the above section.

Loading case 1. The beam loaded with a concentrated bending moment at the free end of the beam.

- Loading case 1.1: $\xi_1 = \xi_2 = 0$

  The introduction of the load is modelled in three ways:

  (a) The concentrated bending moment $M^* = 100$ Nm is introduced in the nodal point situated at the connection of the upper flange and the web (Fig. 3a).

  In case (a), the maximal stress concentration occurs at the place of load introduction. At the distance of $1.45b_2$ from the load introduction place, the stresses correspond to the analytically obtained values.

  (b) Two concentrated bending moments $M^* = 50$ Nm each, having the total value $M^* = 100$ Nm, are introduced in the nodal points situated at the connections of the horizontal flanges and the web (Fig. 3b).

  The same results are obtained for the elements in the upper and lower flanges.

  In case (b), the maximal stress concentration occurs in the elements of load introduction, but it is 50% lower than in case (a).

  The stresses corresponding to the analytically obtained values are again at the distance of $1.45b_2$ from the load introduction place.

  (c) The concentrated bending moment $M^* = 100$ Nm is represented by the couple produced by two parallel vertical concentrated forces $F^* = 3000$ N introduced in the nodal points situated in the centroid and on the centroidal axis at the distance of 33.3 mm from the end of the beam (Fig. 3c).

  In case (c), the stress concentration is minimal, compared to cases (a) and (b), and the highest value appears at the load introduction place.

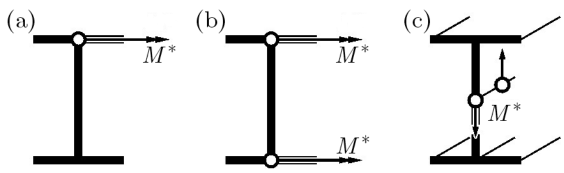

Fig. 3. Introduction of loads in loading case 1.1

- Loading case 1.2: $\xi_1 = 0.5$, $\xi_2 = 0$

  The load introduction is modeled in three ways:

(a) The concentrated bending moment $M^* = 100\,\mathrm{Nm}$ is introduced in the model at the nodal point situated at the end of the upper flange (Fig. 4a).

In case (a), the location of maximal stress concentration is at the load introduction place. The stresses corresponding to the analytically obtained values are at the distance of $1.08b_2$ from the load introduction place.

(b) The concentrated bending moment $M^* = 100\,\mathrm{Nm}$ is represented by the couple produced by two parallel vertical concentrated forces $F^* = 3000\,\mathrm{N}$ introduced in the nodal points situated at the end of the upper flange and at the distance of 33.3 mm from the end of the beam (Fig. 4b).

The stresses corresponding to the analytically obtained values are at the distance of $1.08b_2$ from the load introduction place.

(c) The cocentrated bending moment $M^* = 100\,\mathrm{Nm}$ is introduced in the same way as in case (a), but the end of the cantilever beam is stiffened by the vertical rectangular plate (Fig. 4c).

The stresses corresponding to the analytically obtained values are again at the distance of $1.08b_2$ from the load introduction place.

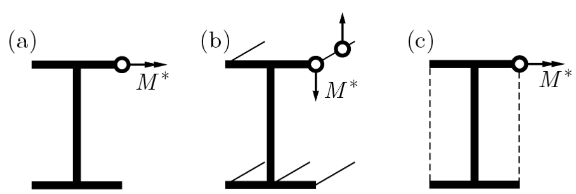

Fig. 4. Introduction of loads in loading case 1.2

Loading case 2. Concentrated forces along the web

The load introduction is modelled using 3D finite elements. Two vertical concentrated forces $F^* = 500\,\mathrm{N}$ each, having the total value $F^* = 1000\,\mathrm{N}$, are introduced in the model at the nodal points situated on the centroidal axis on both sides of the web (Fig. 5).

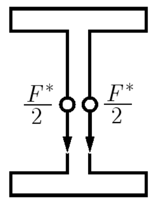

Fig. 5. Introduction of loads in loading case 2

6.1. Discussion

The results of the normal stress obtained by the FEM (Table 3) for loading case 2 seem almost identical and correspond to the analytically obtained values (Table 2).

The results are presented in Table 3 for for the previously defined models: initial, optimal model no. 1 and optimal model no. 2.

On the basis of the proposed optimization procedure, it is possible to calculate in a very simple way the optimal ratios between the parts of the considered thin-walled profiles.

For all loading cases, it is possible to find the decreased level of the stresses in the optimal model no. 1 as well as the saved mass of material with respect to the initial stress limits.

The maximum normal stresses depend on the manner of load introduction (the stress concentration occurs at the place where the loads are introduced).

Table 3. Normal stresses

Model	FEM results σ [MPa]	Analytical results σ [MPa]
Initial model	29.6	30.2
Optimal model no. 1	22.8	23.8
Optimal model no. 2	28.2	30.2

It must be underlined that the results obtained by the FEM show and prove the existence of the Saint-Venant principle. As it is known, the influence of the stress concentration disappears at the distance between one and two cross-sectional dimensions.

7. Conclusion

The paper presents one approach to the optimization of thin-walled I-section beams, loaded in a complex way, using the Lagrange multiplier method.

Accepting the cross-sectional area as the objective function and the stress constraints as the constrained functions, it is possible to calculate in a simple way the optimal ratios of the webs and the flanges of the considered thin-walled profiles.

In addition to the general case, when the I-beam is loaded in a complex way, subjected to bending, torsion and constrained torsion, some particular loading cases are considered. As a result of the calculation, the modified constrained functions are derived as polynomials of the ninth order in a general case, and as polynomials of the fourth order in some particular loading cases.

Particular attention is directed to the calculation of the saved mass using the proposed analytical approach. It is also possible to calculate the saved mass of the used material for different loading cases.

The aim of the paper is the optimization of thin-walled elements subjected to complex loads, and it can be concluded that the paper gives general results allowing for the derivation of the expressions recommendable for technical applications.

Appendix

The coefficients c_k in (3.5):

$$c_0 = -768[1+12\xi_1(1+3\xi_1)]$$

$$c_1 = 128\psi\{1-9\psi^2+12\xi_1[2-9\psi^2+9\xi_1(1-3\psi^2)]\} - 2304(1+6\xi_1+4\xi_2+24\xi_1\xi_2)\frac{M_{X2}}{M_{X1}}$$

$$\begin{aligned}c_2 &= 192\psi^2\{\psi^2(1-3\psi^2)+4\xi_1[(1+6\psi^2-9\psi^4)+3\xi_1(5+9\psi^2-9\psi^4)]\}\\ &\quad -13824\xi_2(1+2\xi_2)\frac{M_{X2}^2}{M_{X1}^2}+384\psi[3(2-3\psi^2)+54\xi_1(1-\psi^2)+4\xi_2(2-9\psi^2)\\ &\quad +72\xi_1\xi_2(1-3\psi^2)]\frac{M_{X2}}{M_{X1}}\end{aligned}$$

$$\begin{aligned}c_3 &= 32\psi^3\{3\psi^4(1-\psi^2)+4\xi_1[1+18\psi^2+18\psi^4-9\psi^6+\xi_1(17+135\psi^2+81\psi^4-27\psi^6)]\}\\ &\quad +6912\psi[1+3\xi_2(1-\psi^2)+2\xi_2^2(1-3\psi^2)]\frac{M_{X2}^2}{M_{X1}^2}+64\psi^2[13+54\psi^2-27\psi^4\\ &\quad +18\xi_1(11+27\psi^2-9\psi^4)+12\xi_2(2+6\psi^2-9\psi^4)+72\xi_1\xi_2(5+9\psi^2-9\psi^4)]\frac{M_{X2}}{M_{X1}}\end{aligned}$$

$$
\begin{aligned}
c_4 &= 16\psi^4\{\psi^6 + 4\xi_1[3\psi^2(1+6\psi^2+2\psi^4) + \xi_1(2+51\psi^2+135\psi^4+27\psi^6)]\}\\
&\quad +1152\psi^2[3(1+3\psi^2) + \xi_2(11+27\psi^2-9\psi^4) + 2\xi_2^2(5+9\psi-9\psi^4)]\frac{M_{X2}^2}{M_{X1}^2}\\
&\quad +32\psi^3[2+39\psi^2+54\psi^4-9\psi^6+2\xi_1(35+297\psi^2+27\psi^4-27\psi^6)\\
&\quad +4\xi_2(1+18\psi^2+18\psi^4-9\psi^6)+8\xi_1\xi_2(17+135\psi^2+81\psi^4-27\psi^6)]\frac{M_{X2}}{M_{X1}}\\
&\quad +432\alpha^2\frac{1}{\mu_1^4}\frac{M_t^2}{M_{X1}^2}
\end{aligned}
$$

$$
\begin{aligned}
c_5 &= 96\xi_1\psi^7[\psi^2(1+2\psi^2) + \xi_1(2+17\psi^2+15\psi^4)] + 64\psi^3[9(1+9\psi^2+9\psi^4)\\
&\quad +\xi_2(35+297\psi^2+243\psi^4-27\psi^6) + 2\xi_2^2(17+35\psi^2+81\psi^4-27\psi^6)]\frac{M_{X2}^2}{M_{X1}^2}\\
&\quad +16\psi^4[3\psi^2(2+13\psi^2+6\psi^4)+2\xi_1(4+105\psi^2+297\psi^4+81\psi^6)\\
&\quad +12\xi_2\psi^2(1+6\psi^2+2\psi^4)+8\xi_1\xi_2(2+51\psi^2+135\psi^4+27\psi^6)]\frac{M_{X2}}{M_{X1}} + 432\psi\alpha^2\frac{1}{\mu_1^4}\frac{M_t^2}{M_{X1}^2}
\end{aligned}
$$

$$
\begin{aligned}
c_6 &= 16\xi_1\psi^{10}[\psi^2+\xi_1(6+17\psi^2)] + 32\psi^4[1+27\psi^2+81\psi^4+27\psi^6\\
&\quad +\xi_2(4+105\psi^2+297\psi^4+81\psi^6)+2\xi_2^2(2+51\psi^2+135\psi^4+27\psi^6)]\frac{M_{X2}^2}{M_{X1}^2}\\
&\quad +8\psi^7[\psi^2(6+13\psi^2)+6\xi_1(4+35\psi^2+33\psi^4)+12\xi_2\psi^2(1+2\psi^2)\\
&\quad +24\xi_1\xi_2(2+17\psi^2+15\psi^4)]\frac{M_{X2}}{M_{X1}} + 144\psi^2\alpha^2\frac{1}{\mu_1^4}\frac{M_t^2}{M_{X1}^2}
\end{aligned}
$$

$$
\begin{aligned}
c_7 &= 16\xi_1^2\psi^{13} + 48\psi^7[1+9\psi^2+9\psi^4+\xi_2(4+35\psi^2+33\psi^4)+2\xi_2^2(2+17\psi^2+15\psi^4)]\frac{M_{X2}^2}{M_{X1}^2}\\
&\quad +8\psi^{10}[\psi^2+\xi_1(12+35\psi^2)+2\xi_2\psi^2+4\xi_1\xi_2(6+17\psi^2)]\frac{M_{X2}}{M_{X1}} + 20\psi^3\alpha^2\frac{1}{\mu_1^4}\frac{M_t^2}{M_{X1}^2}
\end{aligned}
$$

$$
\begin{aligned}
c_8 &= 8\psi^{10}[3(1+3\psi^2)+\xi_2(12+35\psi^2)+2\xi_2^2(6+17\psi^2)]\frac{M_{X2}^2}{M_{X1}^2} + 8\psi^{13}[2\xi_1(1+2\xi_2)]\frac{M_{X2}}{M_{X1}}\\
&\quad +\psi^4\alpha^2\frac{1}{\mu_1^4}\frac{M_t^2}{M_{X1}^2}
\end{aligned}
$$

$$
c_9 = 4\psi^{13}(1+4\xi_2+4\xi_2^2)\frac{M_{X2}^2}{M_{X1}^2}
$$

The coefficients c_k in (3.9):

$$
c_0 = -12(1+6\xi_1) \qquad c_1 = 2\Big[\psi(1+24\xi_1) - 36\xi_2\frac{M_{X2}}{M_{X1}}\Big]
$$

$$
c_2 = 2\psi\Big[11\psi\xi_1 + 6(3+4\xi_2)\frac{M_{X2}}{M_{X1}}\Big] \qquad c_3 = 2\psi^2\Big[\psi\xi_1 + (6+11\xi_2)\tfrac{M_{X2}}{M_{X1}}\Big]
$$

$$
c_4 = \psi^3(1+2\xi_2)\frac{M_{X2}}{M_{X1}}
$$

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Projekt optymalnego przekroju cienkościennego dwuteownika przy zadanych więzach naprężeniowych

Streszczenie

W pracy zajęto się zagadnieniem optymalizacji cienkościennej belki o otwartym przekroju dwuteowym poddanej złożonemu stanowi obciążenia, tj. zginaniu i skręcaniu przy narzuconym warunku na naprężenia ścinające. Rozważono ogólny przypadek momentów gnących działających względem osi centralnych przekroju przy jednoczesnym obciążeniu skręcaniem oraz bimomentem, a następnie przedyskutowano przypadki szczególne. Problem optymalizacji zredukowano do zadania minimalizacji masy przekroju belki dla zadanego kształtu, charakterystyk materiałowych oraz rodzaju obciążenia. Parametry optymalizacji wyznaczono metodą mnożników Lagrange'a. Na funkcję celu wybrano pole przekroju dwuteownika. Do opisu brzegu obszaru optymalizacji użyto funkcji więzów stanu naprężenia. Otrzymane wyniki posłużyły za podstawę do przeprowadzenia symulacji numerycznych.

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