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ON THE NONSTANDARD ANALYSIS AND THE INTERRELATION BETWEEN MECHANICS OF MASS-POINT SYSTEMS AND CONTINUUM MECHANICS

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INTRODUCTION. Methods of the nonstandard analysis, introduced for the first time by A. ROBINSON, [1, 2], and then developped in many publications, cf. [3 - 11], are based on the fact that for every mathematical structure $\mathfrak{M}$ there exists another structure $^*\mathfrak{M}$ which is called an enlargement of $\mathfrak{M}$. By the mathematical structure we mean here a pair $\mathfrak{M} = (X, M)$, where X is an infinite set of elements called individuals(1) and M is a system of relations (of an arbitrary order, i.e., including also relations between relations and between individuals and relations, etc.) for which X is its „underlying'' set. The enlargement $^*\mathfrak{M} = (^*X, ^*M)$ is a model of $\mathfrak{M} = (X, M)$, i.e., every statement about $\mathfrak{M}$ (expressed in a certain formal language) which is meaningfull and true is also meaningfull and true as a statement about $^*\mathfrak{M}$(2). At the same time $^*\mathfrak{M}$ is an extension of $\mathfrak{M}$, i.e., $X \subset {}^*X$ and $M \subset {}^*M$; elements of X and those of M are called standard entities of $^*\mathfrak{M}$. If X is an infinite set then $^*X$ is a proper extension of X, i.e., $^*X$ contains nonstandard elements. Moreover, every infinite set consisting of standard entities only is not contained in the structure $^*\mathfrak{M}$ and is called external in $^*\mathfrak{M}$ (is not an element of $^*M$). Entities belonging to $^*M$ are called relations internal in $^*\mathfrak{M}$. It must be emphasized that the statements which are meaningfull and true for $\mathfrak{M}$ are also meaningfull and true for $^*\mathfrak{M}$ provided that we interpret them exclusively in terms of the totality of internal entities only (individuals and relations of $^*\mathfrak{M}$). Following [6] we recapitulate the key properties of an enlargement $^*\mathfrak{M} = (^*X, ^*M)$ of $\mathfrak{M} = (X, M)$ by the principles stated below:

1. *Permanence Principle.* Every mathematical statement which is meaningfull and true for $\mathfrak{M}$ is also meaningfull and true for $^*\mathfrak{M}$, provided that it is interpreted exclusively in terms of internal entities, i.e., entities of $^*M$.

2. *Extension Principle.* Every mathematical notion which is meaningfull for $\mathfrak{M}$ is also meaningfull for $^*\mathfrak{M}$. It follows that any entity of $\mathfrak{M}$ extends naturally and uniquely to an entity of $^*\mathfrak{M}$. The extended entity is called standard in $^*\mathfrak{M}$.

3. *Enlargement Principle.* Every standard set $^*S$ of $^*\mathfrak{M}$(3) which is infinite, and only in this case, contains nonstandard elements, i.e., $^*S \setminus S \neq \emptyset$, where S is a set of all standard elements of $^*S$.

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(1) We assume that elements of X are not sets, i.e., if $x \in X$ then $x \neq \emptyset$ and the assertion $t \in x$ is always false, cf. [11], p. 11.

(2) We have assumed that a single formal language describes both structures $\mathfrak{M}$ and $^*\mathfrak{M}$.

(3) Sets are treated as a special kinds of relations; If $r \in M$ then the corresponding standard entity of $^*M$ will be denoted by $^*r$. Thus $^*S$ is an extension of a set S in M.

4. *Externity Principle.* Every infinite set S which consists of only standard elements does not belong to $*M$ (is said to be external in $*\mathfrak{M}$).

The enlargement $*\mathfrak{M}$ of a given mathematical structure $\mathfrak{M}$ is not defined uniquely. However, from a point of view of applications, all we need is that such enlargement exists and has the relevant properties outlined above. Putting $X = R$ and assuming that M is the set of all relations for which the real number system R is the underlying set[(4)], we shall refer the enlargement $*\mathfrak{M} = (*X, *M)$ to as a nonstandard model of analysis. We have $R \nsubseteq *R$ where R is a set of all standard real numbers in $*\mathfrak{M}$. Moreover, $*R$ constitutes a non-Archimedean ordered field, i.e., it contains positive numbers which are greater then any standard number (infinite positive real numbers). The reciprocals of infinite positive real numbers are infinitesimal numbers; they are positive and smaller then any positive standard real number. The set of all infinitesimal numbers is denoted by $\mu(0)$ and is said to be the monad of zero. By the monad of an arbitrary standard number r, $r \in R$, we mean the set $\mu(r) := \{a | a \in *R, a-r \in \mu(0)\}$. Every finite number of $*R$ (i.e., the number which is not unfinite) can be uniquely represented by a sum $r = {}^\circ r+\varepsilon$, where ${}^\circ r$ is a standard number and ε is an infinitesimal number, ${}^\circ r \in R$, $\varepsilon \in \mu(0)$. The number ${}^\circ r$ is called the standard part of a finite number r. Analogously, in every Euclidean space $*R^n$ we define the set R^n, $R^n \subset *R^n$, of standard points, and for every point $x \in *R^n$ we define its monad $\mu(x)$ putting $\mu(x) := \{y\} | \varrho(x, y) \in \mu(0)\}$, $\varrho: *R^n \times *R^n \to *R$ being the distance function in $*R$. Points of $*R$ with all finite coordinates are said to be finite. Every finite point x has a unique representation $x = {}^\circ x+\delta$, with ${}^\circ x$ as a standard point and δ as an infinitesimal vector (all components of δ are infinitesimal numbers). For further informations the reader may consult ref. [1, 2, 8, 11].

In this paper we are to show that, using the methods of the nonstandard analysis, the fundamental relations of continuum mechanics (for an elastic response) can be derived directly from the Newtonian mass-point mechanics (cf. also [14]). To do this we shall include the basic relation of Newtonian mechanics into a certain structure $\mathfrak{M} = (X, M)$ and then reinterpret them within an enlargement $*\mathfrak{M} = (*X, *M)$ of $\mathfrak{M}$. This procedure was detailed in [12] and in a simplified form will be outlined in Sec. 1. Then we shall prove that there exists a class of „nonstandard" mass-point systems which have „standard" properties of some continuous systems. The presented approach has two main advantages. Firstly, it treats the continuum mechanics as a special case of the Newtonian mass-point mechanics. Secondly, it yields an interpretation of the basic concepts of continuum mechanics (such as a mass density, body force, stress tensor, strain energy function, etc.) in terms of the concepts of mass-point mechanics. In the first case the non-standard approach to continuum mechanics is conservative because any standard result that has been obtained by nonstandard methods can be also obtained without using these methods, [2]. However, the methods of the nonstandard analysis are more desirable from a purely analytical point of view, mainly by the avoidance of passages to a limit at different stages, [5]. They are also more desirable from an heuristic point of view, namely the obtained standard

(4) The set M of „all" relations based on R contains only relations of a definite type, i.e., we exclude from M certain abnormal relations such as sets containing simultaneously individuals and sets of individuals, etc, cf. [1, 2, 9].

relations of continuum mechanics describe certain properties of some „nonstandard" mass point systems and are not limit cases of the relations of mass-point mechanics. As we have mentioned above, the nonstandard passage from Newtonian mass-point mechanics to continuum mechanics also yields an interrelation between the known continuum concepts and those of the mass-point mechanics. Such interrelation can be formulated only in nonstandard terms. It must be also emphasized that the nonstandard formulation of the Newtonian mass-point mechanics yields more extensive class of mathematical models of the real bodies then the classical formulation. The nonstandard terms used in a description of different phenomena within mechanics have, as a rule, well determined physical meaning. For example, the infinitesimal interpartide distances or the infinitesimal masses of points can be treated as distances and masses, respectively, which can not be neglected but are too small to be measured in a class of problems under consideration, [12]. At the same time the standard parts of finite numbers can be treated as suitable approximations due to the character of the mathematical models of physical problems we deal with.

1. Nonstandard model of Newtonian mechanics.

To develop Newtonian mechanics of mass-point systems within certain mathematical structure $\mathfrak{M} = (X, M)$, we shall assume that $R \subset X$ and $\mathcal{M} \subset X$, $\mathcal{M}$ being certain infinite but countable set of elements called points. Since we are to deal with finite systems of points, we shall assume that there is known an arbitrary but fixed sequence

$$\varphi = \{D_n\}_{n\in N^+}; \qquad \bigcup_{n=1}^{\infty} D_n = \mathcal{M}, \qquad \overline{\overline{D}}_n = n.$$

Every D, $D \subset \mathcal{M}$, such that $D = D_n$ for some positive integer n, is called a point system. By $C(D)$ we shall define the set of all injections $\varkappa: D \ni P \to \varkappa(P) \in R^3$. A continuous mapping $\bar{I} \ni t \to \varkappa_t \in C(D)$, I being an open interval of R, such that $\dot{\varkappa}_t(P)$, $\ddot{\varkappa}_t(P)$ exist for every $t \in I$, $P \in D$, will be called a motion of D. Let $D \circ D := \{(P, Q) | P, Q \in D, P \neq Q\}$. By $\mathcal{N}$ we shall define the set of all quadruples $(D, (m_P)_{P\in D}, (f_P)_{P\in D}, (\sigma_{PQ})_{(P,Q)\in D\circ D})$ where $D \in \{D_1, D_2, D_3, \ldots\}$, $m_P \in R^+$ and

$$f_P: R^3 \times R^3 \to R^3, \qquad \sigma_{PQ}: R^+ \to R; \qquad \sigma_{PQ} = \sigma_{QP},$$

are sufficiently regular functions. An arbitrary element of $\mathcal{N}$ will be called Newtonian mass-point system; D is a point system, m_P is a mass assigned to a point P, $f_P(\varkappa_t(P), \dot{\varkappa}_t(P))$ $t \in I$, is an external force acting at P in an arbitrary motion of D and $\sigma_{PQ}(\varrho(\varkappa_t(P), \varkappa_t(Q)))$, $t \in I$ ($\varrho: R^3 \times R^3 \to R$ is a distance function) will be treated as a value of an interaction force between points $P, Q \in D$ in this motion. As a basic statement of Newtonian mechanics we shall assume that for every Newtonian mass-point system $(D, (m_P)_{P\in D}, (f_P)_{P\in D}, (\sigma_{PQ})_{(P,Q)D\circ D})$, a motion of its point system D has to satisfy the relation

$$m_P \ddot{\varkappa}_t(P) = f_P(\varkappa_t(P), \dot{\varkappa}_t(P)) + \sum_{Q\in D\setminus\{P\}} f_{PQ}(\varkappa_t(P), \varkappa_t(Q)), \qquad (1.1)$$

$$P \in D, t \in I,$$

where we have denoted

$$f_{PQ}\big(\varkappa_t(P), \varkappa_t(Q)\big) = \sigma_{PQ}\big(\varrho(\varkappa_t(P), \varkappa_t(Q))\big)\frac{\varkappa_t(P)-\varkappa_t(Q)}{\varrho(\varkappa_t(P), \varkappa_t(Q))} \tag{1.2}$$

We have tacitly assumed here that $\mathscr{N}$ is a set of all unconstrained Newtonian mass-point systems (cf. also [12]). Substituting RHS of Eqs. (1.2) into Eqs. (1.1) we arrive at the well known Newtonian equations. Every motion of a point system D satisfying Newton equation (i.e., Eqs. (1.1) with the denotations (1.2)) will be referred to as motion of a Newtonian mass-point system $(D, (m_P)_{P\in D}, (f_P)_{P\in D}, (\sigma_{PQ})_{(P,Q)\in D\circ D})$.

Passing to an enlargement $^*\mathfrak{M} = (^*X, ^*\mathfrak{M})$ of $\mathfrak{M} = (X, M)$, we obtain $^*R \subset {}^*X$, $^*\mathscr{M} \subset {}^*X$. A sequence φ is now uniquely extended to a standard sequence $^*\varphi = \{D_n\}_{n\in {}^*N^+}$ with $\bigcup\limits_{n=1}^{\infty} D_n = {}^*\mathscr{M}$, $\overline{\overline{D}}_n = n$, where n runs over all positive integers $^*N^+$ (finite and infinite). The set $C(D)$ (here and in what follows $D = D_n$ for some $n \in {}^*N^+$) analogously as before, is the set of all internal injections $\varkappa: D \ni P \to \varkappa(P) \in {}^*R^3$, which will be called configurations of D. Symbol I stands now for an arbitrary internal interval of $^*R$. An arbitrary internal continuous mapping $\bar{I} \ni t \to \varkappa_t \in C(D)$, such that $\dot{\varkappa}_t(P)$, $\ddot{\varkappa}_t(P)$ exist for every $t \in I$, $P \in D$, is said to be a motion of D. The set $\mathscr{N}$ extends uniquely to a standard set $^*\mathscr{N}$ of all quadruples $s = (D, (m_P)_{P\in D}, (f_P)_{P\in D}, (\sigma_{PQ})_{(P,Q)\in D\circ D})$, where $f_P: {}^*R^3 \times {}^*R^3 \to {}^*R^3$ and $\sigma_{PQ}: {}^*R^+ \to {}^*R$, $\sigma_{PQ} = \sigma_{QP}$, are sufficiently regular internal functions. An arbitrary element s of $^*\mathscr{N}$ will be called a Newtonian mass-point system with D as a point system (without any specification; mind, that $D = D_n$ for some $n \in {}^*N^+$), m_P as a mass of P, $f_P(\varkappa_t(P), \dot{\varkappa}_t(P))$ as an external force acting on P and $\sigma_{PQ}\big(\varrho(\varkappa_t(P), \varkappa_t(Q))\big)$ as a value of an interaction between P, Q in an arbitrary motion of D (by the definition every motion is an internal mapping). By $\mathscr{N}$ we shall denote the set of all quadruples $(D, (m_P)_{P\in D}, (f_P)_{P\in D}, (\sigma_{PQ})_{(P,Q)\in D\circ D})$ consisting exclusively of standard elements (here $D = D_n$ for some standard n, $n \in N^+$); elements of $\mathscr{N}$ will be called standard mass-point systems[5]. It is obvious that $\mathscr{N} \nsubseteq {}^*\mathscr{N}$, i.e., there exist nonstandard mass-point systems (cf. also the Enlargement Principle). Such systems have no counterparts in the known formulation of mechanics. Thus, in the nonstandard model of Newtonian mechanics, we deal with more extensive class of mass-point systems (i.e., more extensive class of mathematical models of certain physical phenomena) then that in the classical (standard) model of Newtonian mechanics. The basic statement of Newtonian mechanics (which can be formulated within a certain formal language, cf. [2], p. 60), formulated above, is also true in $^*\mathfrak{M} = (^*X, ^*M)$. It means that for every $s = \{D, (m_P)_{P\in D}, (f_P)_{P\in D}, (\sigma_{PQ})_{(P,Q)\in D\circ D}\}$, motion of D has to satisfy Eqs. (1.1), (1.2). Thus the form of Newton's equations of motion remains unchanged after passage to a non-standard model of Newtonian mechanics. At the same time these equations now describe more extensive class of mathematical models of physical phenomena then the „standard" equations. Generally speaking, within nonstandard model of Newtonian mechanics we can deal with point systems D which are infinite from the „standard" point of view (i.e., $\overline{\overline{D}} = n$ where $n \in {}^*N \setminus N$ is a fixed but

[5] Mind that $\mathscr{N}$ is an external relation (cf. the Externity Principle).

nonstandard natural number[6]. To each point we can assign an infinitesimal (infinite) mass. Distances and values of interactions between points can be infinitesimal or infinite. Thus the question arises how to interpret, from the purely physical point of view, the nonstandard quantities (nonstandard real numbers) in problems of mechanics. The answer to this question depends on the physical character of the problem under consideration. Roughly speaking, the quantities of the different order in magnitude (i.e., not belonging to the same Archimedean system[7]) will be treated as describing the features of phenomena which can not be simultaneously measured and compared (from a quantitative point of view) in an experiment. In what follows we shall see that an existence of quantities of a different order in magnitude (an existence of non-Archimedean systems in the nonstandard analysis) makes it possible to investigate continuum mechanics as a special case of mass-point mechanics.

2. Kinematics of nonstandard point systems.

Let $s = \big(D, (m_P)_{P\in D}, (f_P)_{P\in D}, (\sigma_{PQ})_{(P,Q)\in D\circ D}\big)$ be a certain (fixed in what follows) nonstandard Newtonian mass-point system in which $D = D_n$ for some infinite n, $n \in {}^*N\setminus N$. Every set $D_n, n \in {}^*N\setminus N$, is a nonstandard point system. By $C_0(D)$ we shall denote the subset of $C(D)$, defined by $C_0(D) := \{\varkappa | \varkappa \in C(D) \varkappa(P)$ is a finite point in ${}^*R^3$ for every $P \in D\}$. Following [2], for an arbitrary subset K of ${}^*R^3$ we shall define the set (possible empty) ${}^\circ K$, putting ${}^\circ K := \{x | x \in R^3 \wedge [\mu(x) \cap K] \neq \emptyset\}$. The set ${}^\circ K$ will be called the standard representation of a set K, provided that all points of K are finite (cf. Introduction). It can be proved that if K is an internal set in ${}^*R^3$ then ${}^\circ K$ is closed in R^3 (cf. [2], p. 101). Let $\varkappa_R: D \to {}^*R^3$ be the known configuration of D such that $\varkappa_R \in C_0(D)$ and ${}^\circ\varkappa_R(D) = \bar{\Omega}$, where Ω is a certain regular region in R^3 (here $\bar{\Omega} = {}^\circ({}^*\Omega)$ and ${}^*\Omega$ is a standard regular region in ${}^*R^3$, cf. [2], p. 102[8]). The set of all such configurations will be denoted by $C_S(D)$. The triples $\Theta_P = (\Theta_P^\alpha) \equiv \varkappa_R(P)$, $\Theta = (\Theta^\alpha) \equiv {}^\circ\varkappa_R(P), P \in D$. $\alpha = 1, 2, 3$, will be referred to as Q-material and S-material coordinates of P, respectively. It can be easily observed that the Q-material coordinates Θ_P are related to a discrete structure of an internal set $\varkappa_R(D)$ in ${}^*R^3$ and play the role of certain micro-coordinates of D. At the same time S-material coordinates Θ (standard coordinates) can be interpreted as macro-coordinates; mind that all points of $\varkappa_R(D)$ belonging to one monad have the same S-material coordinates. Thus the internal set $\varkappa_R(D)$ in ${}^*R^3$, where $\varkappa_R \in C_S(D)$, having the standard reprezentation $\bar{\Omega} = {}^\circ\varkappa_R(D)$ (Ω is a regular region in R^3), can be interpreted from two different points of view. Firstly, it is a discrete set in ${}^*R^3$, i.e., for every $\varkappa_R(P)$, $P \in D$, there exists a ball $B\big(\varkappa_R(P), r\big)$ with a center $\varkappa_R(P)$ and a radius $r \in {}^*R^+$, such that $B\big(\varkappa_R(P), r\big) \cap [\varkappa_R(D) \setminus \{\varkappa_R(P)\}] = \emptyset$. Secondly, to $\varkappa_R(D)$ we can uniquely assign a re-

(6) Mind, that from the point of view of the nonstandard analysis all point systems under consideration are finite.

(7) The numbers α, $\beta \in {}^*R$, $0 < \alpha < \beta$, are assumed to belong to the same Archimedean system if and only if there exists such standard natural number n, $n \in N$, that $n\alpha \geqslant \beta$.

(8) A region Ω in R^3 extends uniquely to a standard region ${}^*\Omega$ in ${}^*R^3$, cf. also the footnote in the Introduction.

gular standard region $^*\Omega$ in $^*R^3$, such that $\bar{\Omega} = {}^\circ\varkappa_R(D)$ is a standard representation of $\varkappa_R(D)$ in R^3. It means that the nonstandard discrete set $\varkappa_R(D)$ in $^*R^3$ has the features of a certain standard region $^*\Omega$ and a nonstandard point systems D in every configuration $\varkappa \in C_S(D)$ has certain properties of a standard but ,,continuous'' system[(9)].

Let $\bar{D}$ be a nonstandard point system ($D = D_n$) for an infinite positive integer n) and $\varkappa$ be its arbitrary configuration such that $\varkappa \in C_S(D)$ (i.e. $\varkappa(D)$ has a standard representation in a form of a closure of a certain regular standard region). Let $\bar{\Omega} \equiv {}^\circ\varkappa(D)$ stands for a standard representation of $\varkappa(D)$ and let us define

$$\partial_S^* \Omega := \{x | \mu(x) \cap {}^*\partial\Omega \neq \emptyset\},$$
$$\mathrm{int}_S {}^*\Omega := \{x | \mu(x) \subset {}^*\Omega\}.$$

The foregoing sets are said to be S-boundary and S-interior of $^*\Omega$, respectively, cf. [2] p. 107 - 108. Now putting $\mathrm{Bound}\,\varkappa(D) \equiv \varkappa(D) \cap \partial_S {}^*\Omega$, $\mathrm{Int}\varkappa(D) \equiv \varkappa(D) \cap \mathrm{int}_S {}^*\Omega$, we shall refer $\mathrm{Bound}\,\varkappa(D)$ and $\mathrm{Int}\varkappa(D)$ to as a boundary and an interior, respectively, of a discrete set $\varkappa(D)$ in $^*R^3$. It means that to every configuration $\varkappa$, $\varkappa \in C_S(D)$, of a nonstandard point system D, we can uniquely assign a set of boundary points and a set of interior points. Analogously, denoting by S an arbitrary smooth surface in $\bar{\Omega} = {}^\circ\varkappa(D)$ and putting $L_S :=$ $= \{x | \mu(x) \cap {}^*S \neq \emptyset\}$, $L_S \subset {}^*R^3$, we shall refer the set $\varkappa(D) \cap L_S$ to as a discrete material surface in $\varkappa(D)$. Thus we conclude that for every $\varkappa \in C_S(D)$ there exists one-to-one correspondence between certain discrete subsets of a discrete set $\varkappa(D)$ in $^*R^3$ and certain smooth manifolds of a closure of a regular region Ω in R^3. This correspondence is not only formal but also gives interpretation of a material smooth surface or a boundary of a continuous body in more physical terms of configurations of mass-point systems.

Now let $I = (\tau_0, \tau_1)$ be an open interval in R and let $^*\bar{I} \ni t \to \varkappa_t \in C_S(D)$ be a certain motion of a nonstandard point system D. Let us define the function $\bar{\Omega} \times \bar{I} \ni (\Theta, t) \to$ $\to p(\Theta, t) \in R^3$ setting $p(\Theta, t) = {}^\circ\varkappa_t(P)$ with $\Theta = {}^\circ\varkappa_R(P)$, for every $P \in D$, $t \in \bar{I}$. Let $p : \bar{\Omega} \times \bar{I} \to R^3$ be a function, such that $p(\cdot\,, t)$ is smooth in Ω and invertible in $\bar{\Omega}$ for every $t \in I$ (i.e., $\det\nabla p(\Theta, t) > 0$, $\Theta \in \Omega$), having continuous first and second time derivaties, and satisfying conditions: $p(\Theta, t) = {}^\circ\varkappa_t(P)$, $\dot{p}(\Theta, t) = {}^\circ\dot{\varkappa}_t(P)$, $\ddot{p}(\Theta, t) = {}^\circ\ddot{\varkappa}_t(P)$, $\Theta =$ ${}^\circ\varkappa_R(P)$, for every $t \in {}^*\bar{I}$, $P \in D$. Function $p(\cdot)$ will be referred to as the deformation function (related to the reference configuration $\varkappa_R \in C_S(D)$) for a motion $^*\bar{I} \ni t \to \varkappa_t \in$ $\in C_S(D)$. Motions of D for which there exist deformation functions (related to a certain reference configuration $\varkappa_R : D \to {}^*R^3$) will be called S-regular[(10)]. Putting $q(\Theta_P, t) \equiv$ $\equiv \varkappa_t(P)$, $P \in D$, $t \in {}^*\bar{I}$, we can define the function $q : \varkappa_R(D) \times {}^*\bar{I} \to {}^*R^3$, representing the motion of D by use of the ,,microcoordinates'' $\Theta_P \in \varkappa_R(D)$, $P \in D$. It can be seen that the deformation function for this motion (if it exists) is nothing else but a standard part of the function q, i.e., $p(\cdot) = {}^\circ q(\cdot)$ (c.f. [2], p. 115, for the definition of a standard part of a function).

In the sequel we are to show under which conditions a motion a nonstandard point system D (provided that D belongs to a certain nonstandard Newtonian mass-point system) can be S-regular.

(9) The problem of different interpretations of discrete sets of points in $^*R^3$ has been detailed in [13].

(10) A terminology used here slightly differs from that used in [13].

3. Mass-distribution in certain nonstandard Newtonian mass-point systems

Let $\varkappa_R \in C_S(D)$ be fixed reference configuration of a point-system $D(D = D_n$ for some infinite n, $n \in {}^*N \setminus N)$ belonging to a certain Newtonian mass-point system $s = (D, (m_P)_{P\in D}, (f_P)_{P\in D}, (\sigma_{PQ})_{(P,Q)\in D\circ D})$. We have ${}^\circ\varkappa_R(D) = \bar{\Omega}$, Ω being a regular region in R^3 (c.f. Sec. 2). Let Δ be an arbitrary subset of ${}^*R^3$. To every Δ we shall assign (provided that $\varkappa_R$ is fixed) the subset $D_R(\Delta)$ of D, putting

$$D_R(\Delta) := \{P | P \in D \wedge \varkappa_R(P) \in \Delta\}. \tag{3.1}$$

Thus $D_R(\Delta)$ is a set of points of D which in the reference configuration $\varkappa_R$ occupy the places in ${}^*R^3$ belonging to Δ.

Now let Θ be an arbitrary point in S-interior of ${}^*\Omega$, $\Theta \in \mathrm{int}_S{}^*\Omega$, and let r_1 stands for an arbitrary but fixed positive standard number. Setting $r_m = r_1/m$ for $m = 1, 2, 3, \ldots$ (m runs over the sequence of all positive integers, finite and unfinite) and denoting by $B(\Theta, r_m)$ the ball in ${}^*R^3$ with a center Θ and a radius r, we shall construct the sequence

$$\varrho_m(\Theta) = \frac{1}{\mathrm{vol}\, B(\Theta, r_m)} \sum_{P \in D_R(B(\Theta, r_m))} m_P, \qquad m = 1, 2, 3, \ldots, \tag{3.2}$$

where $\mathrm{vol}\, B(\Theta, r_m) = 4r_m^3\pi/3$ is a volume of $B(\Theta, r_m)$. We see that $\varrho_m(\Theta)$ is a mean mass density (in a ball with a center $\Theta \in \mathrm{int}_S{}^*\Omega$ and a radius r_m) of a mass-point system under consideration in its reference configuration. Sequences (3.2) are obviously not convergent[11].

In what follows we shall apply the known concept of an F-limit of an infinite sequence $\{a_n\}$, $n \in {}^*N$ of points a_n in a certain metric space $({}^*T, \varrho)$ (cf. 2, p. 109). The space $({}^*T, \varrho)$ is an extension of a metric space (T, ϱ), where ϱ is a distance function in T and hence a distance function in ${}^*T$. In the sequel ${}^*T$ will always stand for a Euclidean space ${}^*R^k$, k being a fixed positive standard integer. We say that point a, $a \in {}^*T$, is a F-limit of $\{a_n\}$, $a \in F\lim a_n$, if and only if for every $\varepsilon \in R^+$ there exists $n_0 \in N^+$ such that $\varrho(a, a_n) < \varepsilon$ for all finite $n, n > n_0$. If $a \in F\lim a_n$ is a finite point in ${}^*T$ (i.e., if there exist a standard point x in ${}^*T$ such that $\varrho(a, x) \in \mu(0)$[12]) then a standard point ${}^\circ a$ will be called S-limit of a sequence $\{a_t\}$. Mind that if $a = F\lim a_n$ then for every $b \in \mu(a)$ (b is an arbitrary point in ${}^*T$ such that $\varrho(a, b)$ is infinitesimal positive number) we also have $b \in F\lim a_n$. It follows that F-limit of a sequence $\{a_n\}$ (if it exists) is not determined uniquely (but S-limit is defined uniquely).

Now assume that there exists the standard continuous function $\varrho_R: {}^*\Omega \to {}^*R^+$ (obtained as a unique extension of the continuous function $\varrho_R: \Omega \to R^+$), such that

$$\varrho_R(\Theta) \in F\lim \varrho_m(\Theta); \; \Theta \in \mathrm{int}_S{}^*\Omega. \tag{3.3}$$

It follows that $\varrho_R(\Theta) = S\lim \varrho_m(\Theta)$, $\Theta \in \Omega$. We have assumed here that every infinite sequence $\{\varrho_m(\Theta)\}$, $\Theta \in \mathrm{int}_S{}^*\Omega$, has such finite F-limit $\varrho_R(\Theta)$, that $\varrho_R(\cdot)$ is a continuous

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(11) The concept of a limit in an enlargement ${}^*\mathfrak{M}$ of a certain structure $\mathfrak{M}$ is analogous to that of a limit in the structure $\mathfrak{M}$ (cf. the Extension Principle in Introduction). For example, the real number $r \in {}^*R$ is, by definition, a limit point of a sequence $\{r_m\}$, $m \in {}^*N$, in ${}^*R$, if for every $\varepsilon \in {}^*R^+$ and for every $\nu \in {}^*N$ there exists the natural number n, $n > \nu$, such that $|r - r_n| < \varepsilon$.

(12) Finite points in ${}^*T$ are also called near-standard points, cf. [2], p. 93.

function defined on Ω (mind, that $\Omega \subset \mathrm{int}_S {}^*\Omega$, where Ω is a set of all standard points in $\mathrm{int}_S {}^*\Omega$). The existence of a function $\varrho_R(\cdot)$ depends only on mass distribution $(m_P)_{P\in D}$ and on the choice of the reference configuration $\varkappa_R$ of D, $\varkappa_R \in C_S(D)$. The standard function $\varrho_R: {}^*\Omega \to {}^*R^+$ (if it exists) will be called S-density of mass in a reference configuration $\varkappa_R$ of a mass-point system. In what follows we shall assume that for the system $(D, (m_P)_{P\in D}, (f_P)_{P\in D}, (\sigma_{PQ})_{(P,Q)\in D\circ D})$ there exists the reference configuration $\varkappa_R \in C_S(D)$ with the S-density of mass ϱ_R. It means that the mass-point system under consideration has certain property of a material continuum which will be referred to as S-regular mass-distribution in a configuration $\varkappa_R$. We can observe that the masses m_P, for every $P \in D$, have to be infinitesimal.

The interrelation between the ,,discrete" mass distribution $\varkappa_R(D) \ni \Theta_P \to m(\Theta_P) \in {}^*R^+$, where $m(\Theta_P) \equiv m_P$, and the ,,continuous" standard mass distribution $\varrho_R: {}^*\Omega \to {}^*R^+$, can be written down explicity due to the following theorem on F-limits (cf. [2], p. 110). Namely, if $\{a_n\}$, $n \in {}^*N$, is an internal sequence of points $a_n \in {}^*T$ having F-limit, then there exists an infinite natural number λ, $\lambda \in {}^*N \setminus N$, such that $F\lim a_n = a_\nu$ for every infinite ν and $\nu < \lambda$ (mind, that F-limits are not uniquely defined).

Since every infinite sequence (3.2) is internal and is assumed to have S-limit[13], we obtain

$$\varrho_R(\Theta) = {}^0\left(\frac{1}{\mathrm{vol}\, B(\Theta, r_\nu)} \sum_{P\in D_R(B(\Theta, r_\nu))} m_P\right), \quad \nu < \lambda_0, \ \nu \in {}^*N \setminus N, \tag{3.4}$$

for every $\Theta \in \Omega \subset {}^*\Omega$. The RHS of Eq. (3.4) represents the standard part of an arbitrary standard number in a bracket (i.e., for an arbitrary infinite positive integer ν, such that $\nu < \lambda_0$). Using Q-material coordinates Θ_P, $\Theta_P = \varkappa_R(D)$, and setting $m_P(\Theta) \equiv m_P$, we obtain an alternative form of Eq. (3.4), given by

$$\varrho_R(\Theta) = {}^0\left(\frac{1}{\mathrm{vol}\, B(\Theta, r_\nu)} \sum_{\Theta_P\in B(\Theta, r_\nu)\cap\varkappa_R(D)} m(\Theta_P)\right), \quad \nu < \lambda_0, \ \nu \in {}^*N \setminus N, \tag{3.5}$$

for every $\Theta \in \Omega \subset {}^*\Omega$. Eqs. (3.4) or (3.5) yield the direct interrelation between the ,,discrete" mass distribution in a nonstandard mass-point system and a standard ,,continuous" mass distribution. The physical sense of Eqs. (3.4) or (3.5) is evident; the values of ,,continuous" mass density at every standard point $\Theta \in \Omega$ of ${}^*\Omega$ are obtained (if they exist) as standard parts of mean mass densities in a ball with a center in a point Θ, provided that the radius r_ν of this ball is infinitesimal but, roughly speaking, not too small (i.e., $r_\nu > r_{\lambda_0}$ for some infinite λ_0 and $\nu \in {}^*N \setminus N$).

4. Distributions of external and internal forces in certain non-standard Newtonian mass-point systems.

Now let ${}^*\bar{I} \in t \to \varkappa_t \in C_S(D)$ be an arbitrary S-regular motion of the nonstandard point system and let us construct the sequences

[13] We confine ourselves to mass-point systems with S-regular mass-distribution in a reference configuration $\varkappa_R$.

$$b_m(\Theta, t) \equiv \frac{1}{\operatorname{vol} B(\Theta, r_m)} \sum_{P \in D_R(B(\Theta, r_m))} f_P(\varkappa_t(P), \dot{\varkappa}_t(P)), \tag{4.1}$$

$$d_m(\Theta, t) \equiv \frac{1}{\operatorname{vol} B(\Theta, r_m)} \sum_{\substack{P \in D_R(B(\Theta, r_m)) \\ Q \in D \setminus \{P\}}} \sum f_{PQ}((\varkappa_t(P), \varkappa_t(Q)),$$

for every $\Theta \in \operatorname{int}_S {}^*\Omega$, $t \in {}^*\bar{I}$. It can be easily seen that $b_m(\Theta, t)$, $d_m(\Theta, t)$ are mean densities of external and internal forces (in a ball with a center Θ and a radius $r_m = r_1/m$, $m \in {}^*N^+$) for a certain S-regular motion of a mass-point system under consideration. As a rule, the sequences (4.1) are not convergent. However, it may happen that the sequences $\{b_m(\Theta, t)\}$, $\{d_m(\Theta, t)\}$ have S-limits for every $\Theta \in \operatorname{int}_S {}^*\Omega$, $t \in {}^*\bar{I}$. In what follows we shall confine ourselves only to such non-standard mass-point systems $s = (D, (m_P)_{P \in D}, (f_P)_{P \in D}, (\sigma_{PQ})_{(P,Q) \in D \circ D})$, that for every S-regular motion of D there exist the standard continuous functions $b_R(\Theta, t)$, $d_R(\Theta, t)$, $\Theta \in {}^*\Omega$, $t \in {}^*\bar{I}$ (i.e., the extensions of continuous functions $b_R: \Omega \times \bar{I} \to R^3$, $d_R: \Omega \times \bar{I} \to R^3$, respectively), such that

$$\begin{aligned} b_R(\Theta, t) &= \operatorname{Slim} b_m(\Theta, t), \\ d_R(\Theta, t) &= \operatorname{Slim} d_m(\Theta, t); \quad \Theta \in \operatorname{int}_S {}^*\Omega, \ t \in {}^*\bar{I}. \end{aligned} \tag{4.2}$$

From the foregoing assumption it follows that $b_R(\Theta, t) = \operatorname{Slim} b_m(\Theta, t)$, $d_R(\Theta, t) = \operatorname{Slim} d_m(\Theta, t)$ for every standard $(\Theta, t) \in \Omega \times \bar{I} \subset {}^*\Omega \times {}^*\bar{I}$. The standard functions $b_R: {}^*\Omega \times {}^*\bar{I} \to {}^*R^3$, $d_R: {}^*\Omega \times {}^*\bar{I} \to {}^*R^3$ will be called S-body force and S-density of interaction, respectively, related to a reference configuration $\varkappa_R$, $\varkappa_R \in C_S(D)$.

Since the infinite sequences $b_m(\Theta, t)$, $d_m(\Theta, t)$ are internal, then by virtue of a theorem on F-limits (cf. Sec. 3) we obtain

$$b_R(\Theta, t) = {}^0\!\left(\frac{1}{\operatorname{vol} B(\Theta, r_\nu)} \sum_{P \in D_R(B(\Theta, r_\nu))} f_P(\varkappa_t(P), \dot{\varkappa}_t(P))\right), \quad \nu < \lambda_1, \tag{4.3}$$

$$d_R(\Theta, t) = {}^0\!\left(\frac{1}{\operatorname{vol} B(\Theta, r_\nu)} \sum_{\substack{P \in D_R(B(\Theta, r_\nu)) \\ Q \in D \setminus \{P\}}} \sum f_{PQ}(\varkappa_t(P), \varkappa_t(Q))\right), \quad \nu < \lambda_2,$$

for every standard $(\Theta, t) \in \Omega \times \bar{I} \subset {}^*\Omega \times {}^*\bar{I}$, $\nu \in {}^*N \setminus N$.

Thus we conclude, that the Newtonian mass-point system under consideration, in an arbitrary S-regular motion of its point system D, has certain features of a material continuum. These features are expressed by the existence of uniquely defined continuous fields $b_R: \Omega \times \bar{I} \to R^3$, $d_R: \Omega \times \bar{I} \to R^3$, characterizing the distribution of external and internal forces. At the same time Eqs. (4.3) yield an interrelation between the system of forces in a „discrete" mass-point system and a certain „continuous" distribution of forces (S-body force and S-density of interaction). The physical interpretation of the RHS of Eqs. (4.3) is rather clear; we deal here with certain mean densities of forces in an infinitesimal ball $B(\Theta, r_\nu)$ which, roughly speaking, is not sufficiently small (has an infinitesimal radius r_ν but greater then r_λ, $\lambda \equiv \max(\lambda_1, \lambda_2)$).

5. Passage to standard laws of motion.

From now on we shall assume that the Newtonian nonstandard mass-point system $s = \left(D, (m_P)_{P\in D}, (f_P)_{P\in D}, (\sigma_{PQ})_{(P,Q)\in D\circ D}\right)$ under consideration satisfies all assumptions introduced in Secs. 3.4. Thus we assume that there exists the reference configuration $\varkappa_R\colon D \to {}^*R^3$, such that $\overline{\Omega} \equiv {}^\circ\varkappa_R(D)$ is a closure of a certain regular region Ω in R^3 and such that the function $\varrho_R\colon\Omega \to R^+$, defined by Eq. (3.4), exists and is continuous in Ω. Moreover, we assume that for every S-regular motion of D there exist functions $\boldsymbol{b}_R\colon\Omega\times \times \bar{I} \to R^3$, $\boldsymbol{d}_R\colon\Omega\times\bar{I}\to R^3$, defined by Eqs. (4.3), which are continuous in $\overline{\Omega}\times I$. A Newtonian mass-point system satisfying the forementioned conditions will be called regular. Now the question arises which necessary conditions are imposed on S-regular motion of D (if it exists) by Newton's equations of motion (1.1), (1.2) for a regular Newtonian mass-point system.

To obtain these conditions let us observe that for every $\Theta \in \Omega$, $t\in I$, $m\in {}^*N^+$, from Eqs. (1.1) it follows that

$$(5.1)\qquad \frac{1}{\operatorname{vol}B(\Theta, r_m)} \sum_{P\in D_R(B(\Theta, r_m))} m_P\ddot{\varkappa}_t(P) - \frac{1}{\operatorname{vol}B(\Theta, r_m)} \sum_{P\in D_R(B(\Theta, r_m))} f_P\big(\varkappa_t(P), \dot{\varkappa}_t(P)\big) - \frac{1}{\operatorname{vol}B(\Theta, r_m)} \sum_{P\in D_R(B(\Theta, r_m))}\sum_{Q\in D\setminus\{P\}} f_{PQ}\big(\varkappa_t(P), \varkappa_t(Q)\big) = 0,$$

where ${}^*\bar{I}\ni t \to \varkappa_t \in C(D)$ is a motion of the point system D. Let Eqs. (5.1) be satisfied by a certain S-regular motion. It means that

$$(5.2)\qquad \begin{aligned} \varkappa_t(P) &= \boldsymbol{p}(\Theta, t)+\boldsymbol{u}_t(P),\\ \dot{\varkappa}_t(P) &= \dot{\boldsymbol{p}}(\Theta, t)+\dot{\boldsymbol{u}}_t(P),\\ \ddot{\varkappa}_t(P) &= \ddot{\boldsymbol{p}}(\Theta, t)+\ddot{\boldsymbol{u}}_t(P);\quad \Theta \equiv {}^\circ\varkappa_R(P), \end{aligned}$$

hold for every $P\in D$, $t\in{}^*\bar{I}$, where $\boldsymbol{u}_t(P), \dot{\boldsymbol{u}}_t(P), \ddot{\boldsymbol{u}}_t(P)$ are certain infinitesimal vectors in ${}^*R^3$. Substituting the RHS of Eq. (5.2) into Eq. (5.1) and putting $m = \nu$, where $\nu < \lambda$, $\lambda = \max(\lambda_0, \lambda_1, \lambda_2)$ and $\nu\in{}^*N\setminus N$, cf. Eqs. (3.5), (4.3), we shall arrive at the relation

$$(5.3)\qquad \varrho_R(\Theta)\ddot{\boldsymbol{p}}(\Theta, t) = \boldsymbol{b}_R(\Theta, t)+\boldsymbol{d}_R(\Theta, t),$$

which has to hold for every $\Theta\in\Omega$, $t\in\bar{I}$. Passing from Eqs. (5.1), $(5.2)_3$ to Eqs. (5.3) we have taken into account formulas (3.5), (4.3) and a relation

$$(5.4)\qquad {}^0\!\left(\frac{1}{\operatorname{vol}B(\Theta, r_\nu)} \sum_{P\in D_R(B(\Theta, r_\nu))} m_P\ddot{\boldsymbol{u}}_t(P)\right) = 0, \qquad \Theta\in\Omega.$$

In order to prove that Eqs. (5.4) holds let us observe that the RHS of the foregoing formula can be interpreted as S-limits of internal sequences

$$(5.5)\qquad \frac{1}{\operatorname{vol}B(\Theta, r_m)} \sum_{P\in D\;(B(\Theta, r_m))} m_P\ddot{\boldsymbol{u}}_t(P), \qquad m\in{}^*N^+.$$

But the existence of F-limit of an infinite sequence $\{a_n\}$, $n \in {}^*N$, of points in a certain metric space $^*T$ depends only on terms a_n for $n \in N$. Because all these terms for sequence (5.5) are infinitesimal (it follows from the fact that all such terms of sequence (3.2) are finite) then S-limit of this sequence is equal to zero and Eq. (5.4) hold for every $\Theta \in \bar{\Omega}$.

Eqs. (5.3) constitute the interrelation among the deformation function $p:\Omega \times \bar{I} \to R^3$, S-density of mass $\varrho_R:\Omega \to R^+$, S-density of interaction $d_R:\Omega \times \bar{I} \to R^3$ and S-body force $b_R:\Omega \times \bar{I} \to R^3$. Thus Eqs. (5.3) can be called standard laws of motion and their form coincides with that of laws of motion for a certain material continuum, occupying in the reference configuration a regular region Ω in R^3. Because the interactions have been assumed non-local, we do not deal here with any contact forces (which are introduced and detailed in [15]). It must be emphasized that Eqs. (5.3) have to hold only if the motion of a nonstandard point system D, satisfying Eqs. (1.1), (1.2), is S-regular. At the same time Eq. (5.3) (in which $\ddot{p}(\Theta, t) = {}^0\ddot{\varkappa}_R(P)$, $\Theta \equiv {}^0\varkappa_R(P)$, cf. $(5.2)_3$(together with Eqs. (3.4), (4.3) represent the necessary condition imposed on the S-regular motion of a regular Newtonian mass-point system (provided that such motion exists).

6. Passage to standard constitutive relations.

Now let us substitute the RHS of Eqs. $(5.2)_{1,2}$ into Eqs. $(4.3)_1$. Setting

$$\Delta f_P(p(\Theta, t), \dot{p}(\Theta, t); \dot{u}_t(P), \dot{u}_t(P)) \equiv$$
$$\equiv f_P(p(\Theta, t)+\dot{u}_t(P), \dot{p}(\Theta, t)+\dot{u}_t(P))-f_P(p(\Theta, t), \dot{p}(\Theta, t)),$$

let us assume that the relation

$$(6.1)\qquad {}^0\left(\frac{1}{\operatorname{vol} B(\Theta, r_\nu)} \sum_{P \in D_R(B(\Theta, r_\nu))} \Delta f_P(p(\Theta, t), \dot{p}(\Theta, t), u_t(P), \dot{u}_t(P))\right) \equiv 0$$

holds for every infinitesimal $u_t(P)$, $\dot{u}_t(P)$. Let us also define the function $\beta_R:\Omega \times R^3 \times R^3 \to R^3$ by means of

$$(6.2)\qquad \beta_R(\Theta, p(\Theta, t), \dot{p}(\Theta, t)) \equiv {}^0\left(\frac{1}{\operatorname{vol} B(\Theta, r_\nu)} \sum_{P \in D_R(B(\Theta, r_\nu))} f_P(p(\Theta, t), \dot{p}(\Theta, t))\right).$$

In Eqs. (6.1), (6.2), as usual, we have $\nu \in {}^*N\setminus N$ and $\nu < \lambda$ for a certain infinite positive integer λ. Thus we conclude that if the conditions of the form (6.1) are satisfied for every $\Theta \in \Omega$ then we can characterize the S-body forces by the formulas

$$(6.3)\qquad b_R(\Theta, t) = \beta_R(\Theta, p(\Theta, t), \dot{p}(\Theta, t)), \quad \Theta \in \Omega,\ t \in \bar{I},$$

with the RHS of Eqs. (6.3) defined by Eqs. (6.2). Eqs. (6.2), (6.3) yield the interrelation between the „continuous" S-body force and the „discrete" distribution of external forces in the regular Newtonian mass-point system under consideration. This interrelation is valid under the conditions that the value of S-body force in any S-regular motion of a nonstandard point system D (cf. Sec. 2) depends only on the deformation function for this motion. It can be shown that such situation will take place if the external fields in $^*R^3$, determining the form of functions $f_P:{}^*R^3 \times {}^*R^3 \to {}^*R^3$, are standard.

Now let us detail the possible interrelation between the S-density of interaction $d_R(\Theta, t)$ and the deformation function $p(\cdot)$ of an arbitrary S-regular motion of a nonstandard point system D. To this aid we shall use Eq. $(4.3)_2$ with the functions $f_{PQ}: {}^*R^3 \times R^3 \to {}^*R^3$ defined by Eq. (1.2). For every S-regular motion ${}^*\bar{I} \ni t \to \varkappa_t \in C_S(D)$ with the deformation function $\Omega \times \bar{I} \ni (\Theta, t) \to p(\Theta, t) \in R^3$ (where $p(\Theta, t) = {}^0\varkappa_t(P)$ with $\Theta = {}^0\varkappa_R(P)$ cf. Sec. 2) we have

$$\varkappa_t(P) = p(\Theta_P, t) + w_t(P); \qquad \Theta_P = \varkappa_R(P),\ P \in D,\ t \in {}^*\bar{I}, \tag{6.4}$$

where now $p: {}^*\Omega \times {}^*\bar{I} \to {}^*R^3$ stands for an extension of the deformation function (which can be called a standard deformation function) and $w_t(P)$ are infinitesimal vectors in ${}^*R^3$. Instead of S-material coordinates $\Theta = {}^0\varkappa_R(P)$ (macro-coordinates), which have been used before (cf. Eq. (5.2)), we apply now Q-material coordinates $\Theta_P = \varkappa_R(P)$ (micro-coordinates). If $w_t(P) = 0$, $P \in D$, $t \in {}^*\bar{I}$, then Eq. (6.4) will represent a special S-regular motion of D in which material points are „frozen" in a certain standard „material continuum"; motion of this „material continuum" is described by a standard deformation function $p: {}^*\Omega \times {}^*\bar{I} \ni (\Theta, t) \to p(\Theta, t) \in {}^*R^3$ (i.e., by an extension of a deformation function for the motion of D).

In what follows we shall confine ourselves to a certain subclass of a class of all S-regular motions of a point system D under consideration. This subclass contains motions in which the values of a function $w_t(P)$, $P \in D$, $t \in {}^*\bar{I}$, in Eq. (6.4) are not only infinitesimal but also, roughly speaking, „sufficiently small". To be more precise we shall assume that for every pair (P, Q) of interacting material points (i.e., points for which $f_{PQ}(\cdot)$ is not identically equal to zero) in the subclass of motions under consideration we have

$$w_t(P) - w_t(Q) = E(P, Q, t)[\varkappa_t(P) - \varkappa_t(Q)], \tag{6.5}$$

where $E(P, Q, t)$ is a certain 3×3 matrix of infinitesimal numbers. Eq. (6.5) can be also written down in a form

$$w_t(P) - w_t(Q) \in \sigma(\varkappa_t(P) - \varkappa_t(Q)),$$

where by $\sigma(x)$, $x \equiv (x_1, x_2, x_3) \in {}^*R^3$, we denote the set of all triples $y = (y_1, y_2, y_3) \in {}^*R^3$, such that $y_i = E_i^j x_j$, $i, j = 1, 2, 3$, where E_j^i are infinitesimal (cf. also [2], p. 79). By virtue of Eq. (1.2), for a class of S-regular motions of D satisfying Eq. (6.5), we obtain

$$f_{PQ}(\varkappa_t(P), \varkappa_t(Q)) - f_{PQ}(p(\Theta_P, t), p(Q_Q, t)) \in \sigma\big(f_{PQ}(\varkappa_t(P), \varkappa_t(Q))\big), \tag{6.6}$$

for every $P, Q \in D$, $P \neq Q$, $t \in {}^*\bar{I}$. It means that, roughly speaking, the interactions in a motion determined by Eqs. (6.4), (6.5) are „nearly the same" as the interactions in a motion characterized by Eq. (6.4) with $w_t(P) = 0$ for every $P \in D$, $t \in {}^*\bar{I}$. Motions of D satisfying Eq. (6.5) (for every pair of interacting points P, Q and every $t \in {}^*I$) will be called strictly S-regular.

For motions of D which are strictly S-regular it can be shown that the S-density of interactions is uniquely determined by the deformation function. Namely from Eq. (6.6) it follows that

$$^0\left(\frac{1}{\operatorname{vol} B(\Theta, r_\nu)} \sum_{\substack{P \in D_R(B(\Theta, r_\nu)) \\ Q \in D \setminus \{P\}}} f_{PQ}(\varkappa_t(P), \varkappa_t(Q))\right) = \tag{6.7}$$

$$(6.7)\ [\text{cont.}] \qquad = {}^{0}\left(\frac{1}{\operatorname{vol} B(\Theta, r_\nu)} \sum_{\substack{P \in D_R(B(\Theta, r_\nu)) \\ Q \in D \setminus \{P\}}} f_{PQ}\big(p(\Theta_P, t), p(\Theta_Q, t)\big)\right),$$

for every $\Theta \in \Omega$, $\nu \in {}^*N \setminus N$ and $\nu < \lambda$.

Introducing the functionals

$$(6.8) \qquad D_R(\Theta, p(\cdot\,, t)) \equiv {}^{0}\left(\frac{1}{\operatorname{vol} B(\Theta, r_\nu)} \sum_{\substack{P \in D_R(B(\Theta, r_\nu)) \\ Q \in D \setminus \{P\}}} f_{PQ}\big(p(\Theta_P, t), p(\Theta_Q, t)\big)\right),$$

defined for every $\Theta \in \Omega$ on the space of all deformation functions $p: \Omega \times \bar{I} \ni (\Theta, t) \to p(\Theta, t) \in R^3$ (and hence on the space of all standard deformation functions $p: {}^*\Omega \times {}^*\bar{I} \ni (\Theta, t) \to p(\Theta, t) \in {}^*R^3$) and taking into account Eqs. $(4.3)_2$ and (6.7), we arrive at the relation

$$(6.9) \qquad d_R(\Theta, t) = D_R(\Theta, p(\cdot\,, t)); \ \Theta \in \Omega,\ t \in \bar{I}.$$

Eqs. (6.9), (6.8) characterize the interrelation between the „continuous" S-density $d_R(\cdot\,, t)$ of interactions and the „discrete" distribution of interactions in the regular Newtonian mass-point system. This interrelation holds in any strictly S-regular motion of the point system D under consideration[14].

Formulas (6.3), (6.9) can be interpreted as the constitutive relations of a certain non-local elastic „material continuum", motion of which is described by an arbitrary deformation function $p: \Omega \times I \ni (\Theta, t) \to p(\Theta, t) \in R^3$. The properties of this „material continuum" are uniquely determined by the properties of a regular Newtonian mass-point system, provided that we confine ourselves to the strictly S-regular motions of its point system.

7. Conclusions.

Summarizing the obtained results we shall formulate the following assertions:

1. Every S-regular motion of an arbitrary regular Newtonian mass-point system[15] (if it exists) has to satisfy Eq. (5.3) together with Eqs. (3.5), (4.3) and with $\ddot{p}(\Theta, t) = {}^{0}\ddot{\varkappa}_t(P)$, $\Theta \in {}^{0}\varkappa_R(P)$, for every $P \in D$, $t \in I$.

2. If there exists strictly S-regular motion of a certain regular Newtonian mass point system then the deformation function for this motion has to satisfy Eqs. (5.3), (6.3), (6.9) with denotations (3.4), (6.2), (6.8).

3. Every regular Newtonian mass-point system uniquely determines certain non-local elastic „material continuum" with governing relations (5.3), (6.3), (6.9). The continuous fields in these governing relations are expressed in terms of Newtonian mass-point mechanics by Eqs. (3.4), 6.2), (6.8) for every strictly S-regular motion of the Newtonian mass-point system under consideration (if it exists).

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[14] This interrelation also holds in any S-regular motion satisfying Eq. (6.7)

[15] A motion of the Newtonian mass-point system was defined in Sec. 1 as the motion of its point system satisfying Newton's equations (1.1), (1.2).

It must be remembered that every regular Newtonian mass-point system is, by definition, a nonstandard Newtonian mass-point system. The number of points in this system is equal to a certain fixed infinite natural number and the masses of points are infinitesimal. For the class of motions under consideration (for S-regular motions) also the values of external forces acting on the points as well as the values of interactions between the points are infinitesimal. The assertions listed above, which interrelate certain nonstandard „discrete" functions (i.e., defined on $\varkappa_R(D) \subset {}^*R^3$) with standard continuous fields (defined on ${}^0\varkappa_R(D) \subset R^3$) can be expressed exclusively in terms of the nonstandard analysis. On the other hand, resulting relations (5.3), (6.3), (6.9), which can be interpreted as describing certain „material continuum", are standard. Thus the method of the nonstandard analysis applied to Newtonian mass-point mechanics makes it possible to define the class of nonstandard mass-point systems (which were called regular Newtonian mass-point systems) having properties of material continua (for more general approaches cf. [13]). In this paper, starting from Newtonian mechanics, we have derived governing relations of a nonlocal continuum mechanics; passage from Newtonian mechanics to the relations of the elasticity theory will be described in the next paper (cf. also [15]). However, the non-standard methods can be also applied directly to some problems of continuum mechanics, [16, 17].

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Резюме

НЕСТАНДАРТНЫЙ АНАЛИЗ И СВЯЗЬ МЕЖДУ МЕХАНИКОЙ МАТЕРИАЛЬНЫХ ТОЧЕК И МЕХАНИКОЙ КОНТИНУУМ.

В работе доказано, что учитывая методы нестандартного анализа из уравнений механики Нютона системы материальных точек можна вывести непоследственно фундаментальные уравнения механики континуум без применения апроксимации и граничных переходов.

Streszczenie

O N ESTANDARDOWEJ ANALIZIE I ZWIĄZKU MIĘDZY MECHANIKĄ PUNKTÓW MATERIALNYCH A MECHANIKĄ KONTINUUM

W pracy wykazano, że korzystając z metod niestandardowej analizy można wyprowadzić podstawowe równania mechaniki kontinuum bez stosowania aproksymacji i przejść granicznych bezpośrednio z równań mechaniki Newtona układów punktów materialnych.

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