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SHORT RESEARCH COMMUNICATION

A NOTE ON NON-ASSOCIATED DRUCKER-PRAGER PLASTIC FLOW IN TERMS OF FRACTIONAL CALCULUS

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In this paper, we consider a special case of the general fractional plastic flow rule, namely the one which is equivalent to the classical non-associated Drucker-Prager (D-P) plasticity model. Fractional plastic flow is obtained from the classical flow rule by generalisation of the classical gradient of a plastic potential with a fractional gradient operator. It is important that, contrary to the classical models, non-associativity of fractional flow appears without introduction of the additional potential. The classical associative D-P plasticity is obtained as a special case. The discussion on objectivity of the fractional gradient is also presented also.

Keywords: fractional calculus, plastic flow, non-normality

1. Fractional plastic flow – general setup

In the paper by Sumelka (2014) the concept of generalisation of the classical plastic/viscoplastic flow rule utilising fractional calculus was presented. The fundamental role in this new formulation plays the definition of directions of plastic strain given as a *fractional gradient* of a plastic potential. The concept can be stressed as follows.

On the assumption of small strain, we accept classical additive decomposition of the total strain rate, namely (Lubliner, 1990)

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p \tag{1.1}$$

where $\dot{\boldsymbol{\varepsilon}}$ stands for the total strain rate and $\dot{\boldsymbol{\varepsilon}}^e$, $\dot{\boldsymbol{\varepsilon}}^p$ denote elastic and plastic/viscoplastic parts, respectively.

The elastic strain components can be achieved from Hooke's law

$$\boldsymbol{\sigma}^e = \mathcal{L}^e : \boldsymbol{\varepsilon}^e \tag{1.2}$$

where $\boldsymbol{\sigma}^e$ denotes the Cauchy stress tensor and $\mathcal{L}^e$ denotes the fourth rank tensor of elastic stiffness.

Now, we postulate that the rate of plastic strain can be written as

$$\dot{\boldsymbol{\varepsilon}}^p = \Lambda \mathbf{p}, \tag{1.3}$$

where Λ is a scalar multiplier (calculated through the rules common for rate-independent (plasticity) or rate-dependent (viscoplasticity) concepts) and $\mathbf{p}$ represents the second order tensor which defines the direction of flow (Euclidean norm of $\mathbf{p}$ is one).

Fractional plastic flow is obtained, if one postulates that the direction of flow is stated in terms of the fractional gradient

$$\mathbf{p} = \underset{\boldsymbol{\sigma}}{D}^{\alpha} f \left\| \underset{\boldsymbol{\sigma}}{D}^{\alpha} f \right\|^{-1} \tag{1.4}$$

where f is a plastic potential, D^α denotes partial fractional differentiation, and α denotes the order of the derivative. It is clear that for $\alpha = 1$, classical (associative) solution is recovered.

Remark 1. ***On the fractional differential operator.*** *There are many definitions of fractional differential operators (Samko et al., 1993; Podlubny, 1999; Kilbas et al., 2006; Leszczyński, 2011). In this sense, Eq. (1.4) can be redefined in terms of them. We claim that for a specific material (concrete, steel, rubber, etc.) there should exist the optimal choice of the specific definition mentioned.*

2. Application of Caputo's operator

Throughout this paper we utilise both sided Caputo's derivative for the explicit definition of Eq. (1.4). We call such a derivative the Riesz-Caputo (RC) derivative (cf. Agrawal, 2007; Frederico and Tores, 2010).

So, for a function f over the interval $t \in (a,b)$, we have

$$D^{\alpha} f(t) = {}^{RC}_{\ a}D^{\alpha}_{b}\, f(t) = \frac{1}{2}\Big({}^{C}_{a}D^{\alpha}_{t}\, f(t) + (-1)^n\, {}^{C}_{t}D^{\alpha}_{b}\, f(t) \Big) \tag{2.1}$$

where a, t, b are so called terminals, ${}^{C}_{a}D^{\alpha}_{t}$ and ${}^{C}_{t}D^{\alpha}_{b}$ denote the left and right sided Caputo's derivatives, respectively, and $n = [\alpha] + 1$. In our case (Eq. (1.4)), the interval $t \in (a,b)$ can change dependently on partial fractional differentiation.

Remark 2. ***Approximation of the left and right sided Caputo's derivatives.*** *Analytical solutions utilising fractional differentiation are very limited. Due to this reason, many numerical approximations were recently proposed (Diethelm et al., 2005; Odibat, 2006). In this paper, we follow the concept discussed by Leszczyński (2011). It causes that Eq. (2.1) reduces to an appropriate sum of classical derivatives of plastic potential function from the specific point of interest and its surrounding. The size of the surrounding is controlled by the interval over which the derivative is calculated. In this sense, the fractional derivative is non-local.*

Thus, the direction of fractional flow depends not only on the information in a point (contrary to the classical derivative) but also depends on the information from the surrounding.

3. Drucker-Prager plastic flow in terms of fractional calculus

Classical non-associated linear D-P model is governed by the yield criterion

$$f(\boldsymbol{\sigma}) = \sqrt{J_2} - A - BI_1 = 0 \tag{3.1}$$

flow potential

$$g(\boldsymbol{\sigma}) = \sqrt{J_2} - CI_1 \tag{3.2}$$

and, in consequence, unnormalised flow directions

$$\frac{\partial g}{\partial \boldsymbol{\sigma}} = \frac{1}{2\sqrt{J_2}}\mathbf{s} - C\mathbf{I} \tag{3.3}$$

where A, B, C are material constants, J_2 is the second invariant of the deviatoric part of the Cauchy stress, and I_1 is the first invariant of the Cauchy stress, $\mathbf{s}$ is the stress deviator, and $\mathbf{I}$ denotes the unit tensor.

Now, using the fractional flow concept cf. Eq. (1.4), it is enough to assume flow the potential equivalent with the yield function. The directions of flow are then functions of the order of the fractional derivative α, and length of the interval over which partial the fractional derivative is calculated. In this particular example, the unnormalised fractional flow directions, in the principal directions of stress tensor, are as follows

$$\begin{aligned}
{}_{a_1}D^{\alpha}_{\sigma_1 b_1} f &= \frac{1}{2}\frac{{h_1}^{1-\alpha}}{\Gamma(3-\alpha)}\Big\{[1-\alpha 2^{1-\alpha}][f_1^{(1)}(\sigma_1-2h_1,\sigma_2,\sigma_3)+f_1^{(1)}(\sigma_1+2h_1,\sigma_2,\sigma_3)] \\
&\quad + 2f_1^{(1)}(\sigma_1,\sigma_2,\sigma_3)+[2^{1-\alpha}-2][f_1^{(1)}(\sigma_1-h_1,\sigma_2,\sigma_3)+f_1^{(1)}(\sigma_1+h_1,\sigma_2,\sigma_3)]\Big\} \\
{}_{a_2}D^{\alpha}_{\sigma_2 b_2} f &= \frac{1}{2}\frac{{h_2}^{1-\alpha}}{\Gamma(3-\alpha)}\Big\{[1-\alpha 2^{1-\alpha}][f_2^{(1)}(\sigma_1,\sigma_2-2h_2,\sigma_3)+f_2^{(1)}(\sigma_1,\sigma_2+2h_2,\sigma_3)] \\
&\quad + 2f_2^{(1)}(\sigma_1,\sigma_2,\sigma_3)+[2^{1-\alpha}-2][f_2^{(1)}(\sigma_1,\sigma_2-h_2,\sigma_3)+f_2^{(1)}(\sigma_1,\sigma_2+h_2,\sigma_3)]\Big\} \\
{}_{a_3}D^{\alpha}_{\sigma_3 b_3} f &= \frac{1}{2}\frac{{h_3}^{1-\alpha}}{\Gamma(3-\alpha)}\Big\{[1-\alpha 2^{1-\alpha}][f_3^{(1)}(\sigma_1,\sigma_2,\sigma_3-2h_3)+f_3^{(1)}(\sigma_1,\sigma_2,\sigma_3+2h_3)] \\
&\quad + 2f_3^{(1)}(\sigma_1,\sigma_2,\sigma_3)+[2^{1-\alpha}-2][f_3^{(1)}(\sigma_1,\sigma_2,\sigma_3-h_3)+f_3^{(1)}(\sigma_1,\sigma_2,\sigma_3+h_3)]\Big\}
\end{aligned} \tag{3.4}$$

In Eqs. (3.4), the approximations of Caputo's derivatives discussed by Leszczyński (2011) are applied on the assumption that the length of the interval over which the partial fractional derivative is calculated is equal $4h_i$ (the point of interest lays in the middle of this interval cf. Fig. 1). Other denote: $f_i^{(1)} = \partial f/\partial\sigma_i$ – the classical i-th partial derivative of f, Γ – the Gamma function, and σ_i – i-th principal value of the stress tensor.

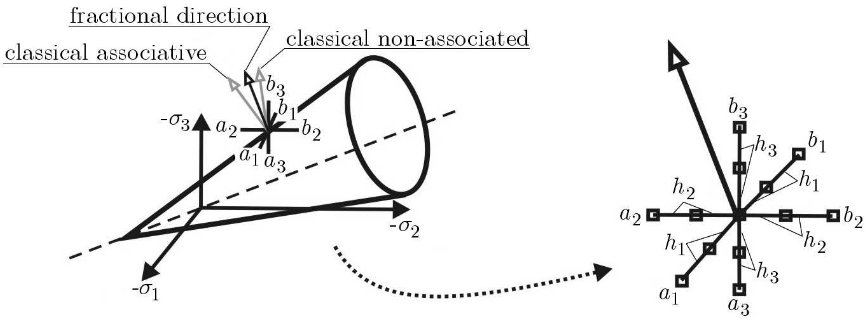

Fig. 1. The concept of fractional plastic flow for the Drucker-Prager model

It is clear that one can chose α, h_1, h_2, h_3 such that the flow directions described by Eqs. (3.3) and (3.4) are equal (simply by solving the set of uncoupled nonlinear equations obtained form comparison of components of the classical and fractional flows directions.). In this sense, utilising the concept of fractional flow, one can obtain a non-associated D-P model without the necessity of making an additional potential assumption.

Remark 3. ***On possible fractional flow directions.*** *Please notice that in this particular example the number of material parameters is greater comparing with the classical model. However, only few additional material parameters give a vast range of possible flow directions, where the one equivalent with the classical non-associated D-P model is a particular case. Thus, we can control the flow directions not only in the meridional plane, but also in the deviatoric (Π) plane.*

Remark 4. ***Objectivity of fractional flow.*** *It is important to mention that the fractional gradient of an isotropic scalar value function of a tensorial argument does not lead to an isotropic tensor function (contrary to the classical gradient operator). In this sense, from the first sight, the objectivity requirement is violated, what is of course not acceptable in continuum mechanics. However, if we enforce appropriate transformation rules for the* *intervals* (h_i) *over which the fractional gradient is calculated (cf. Fig. 1) the objectivity requirement if fulfilled. Thus, the formula for intervals* h_{ij} *in an arbitrary coordinate system can be easily deduced from fulfilling classical transformation rules for the second rank tensors, namely*

$$\widetilde{p}_{ij}(\widetilde{h}_{ij},\widetilde{\sigma}_{ij}) = R_{ik}R_{jl}p_{kl}(h_{ij},\sigma_{ij}) \tag{3.5}$$

where $\widetilde{(\cdot)}$ *denotes the new coordinate system, and* $\mathbf{R}$ *denotes rigid rotation.*

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