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DETECTING NONLINEAR BEHAVIOUR USING THE VOLTERRA SERIES TO ASSESS DAMAGE IN BEAM-LIKE STRUCTURES

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Using the example of a cracked cantilever beam, this paper illustrates a means of identifying damage in structures using the so-called higher order Frequency Response Function (FRFs) which are based on the Volterra series. It is well known that, when a beam subject to a dynamic excitation vibrates, a transverse "breathing"crack present in the beam can change the state (from open to closed and vice-versa), causing nonlinear dynamic behaviour. A simple model of a cracked cantilever beam vibrating in its first mode is proposed. Across the frequency range which encompasses the first mode of vibration, it is possible to model the response characteristics of a cracked beam using a relatively simple asymmetric bilinear oscillator. As described in this article, it is possible to use these higher order FRFs to characterise the nonlinear behaviour of the cantilever beam and investigate the qualitative relation with the parameters of the fault such as entity and location. In this study, the case of single harmonic excitation has been considered initially. Then, a new characteristic function, again based on the higher order FRFs, is proposed for detecting the crack by exploiting the fact that due to the second-order nonlinear behaviour, two harmonic inputs combine to excite the sum of their frequencies. Comparisons are made between results derived using the simple model described and those obtained from a FE model simulating some experimental tests on the beam.

Key words: breathing crack, nonlinear oscillation, Volterra series, higher order Frequency Response Functions

1. Introduction

In the past, a series of studies illustrated that a crack in a structure such as a beam may cause the structure to exhibit nonlinear behaviour if the crack is open during a part of the response and closed over the remaining intervals. Nonlinear behaviour of this type has been confirmed by experimental testing (Gudmunson, 1983), the published results indicating also that the natural frequencies of a cracked beam cannot be simulated accurately using a model of a crack which is always open. In practice, an alternately opening and closing or 'breathing' crack gives rise to natural frequencies of the beam which fall in the range between those corresponding to the always-open and always-closed (i.e. undamaged) cases.

In recent years, relatively in-depth studies on beams with a breathing crack have been undertaken from both analytical and experimental viewpoints. Friswell and Penny (1992) simulated the nonlinear behaviour of a beam vibrating in its first mode of vibration using a simple one degree-of-freedom model with bilinear stiffness. Shen and Chu simulated the dynamic response of simply supported beams with a breathing crack by using a bilinear equation of motion for each mode of vibration, and then analysing the response spectra in order to detect changes as a potentially useful means of damage assessment (Chu and Shen, 1992; Shen and Chu, 1992). Krawczuk and Ostachowicz simulated the breathing crack using springs with periodically time-varying stiffness (Ostachowicz and Krawczuk, 1990) and analysing the forced vibrations using the harmonic balance technique to solve the nonlinear equations of motion (Krawczuk and Ostachowicz, 1994). The technique of harmonic balance has been used also by Pugno *et al.* (2000) in an attempt to study the case of several breathing cracks.

In (Bovsunovsky and Surace, 2005), the authors analyse, from both an experimental and numerical standpoint, the forced vibrations of beams with a breathing crack, demonstrating that one of the main distinctive features of such a vibration system are the occurrence of super-harmonic resonances, the significant nonlinearity of the vibration responses (displacement, acceleration, strain etc.) at these super-harmonic resonances, and the fact that the presence of a crack can cause the damping to increase. In particular, the super-harmonics, which arise because of the nonlinearity, are much more sensitive to the presence of the crack than the change of natural frequencies and modeshapes (by one or even two orders of magnitude). Furthermore, in (Bovsunovskii *et al.*, 2006) also the sub-harmonic resonances were considered to be potential indicators of damage in structures.

In a series of studies dating back to the 1980's, it has been found that higher order FRFs defined using the Volterra Series provide an extremely sensitive means of quantifying nonlinear behaviour in systems. In general, the Volterra series forms a solid mathematical foundation for describing the dynamic behaviour of a class of nonlinear systems which can be represented by polynomial equations of motion. For application to the bilinear oscillator used to model the cracked beam, it is necessary to approximate the restoring force of the system with a polynomial function in order to define a corresponding Volterra Series and the respective higher order FRFs. Furthermore, since the form of these FRFs is directly related to the coefficients of terms of the polynomial function in the governing equations of motion, it should be possible to determine the correlation between the higher order FRFs and the crack depth and/or position.

Of relevance, particularly as concerns practical implementation, it was found in early studies by the authors of this article that certain key features of the nonlinear behaviour of structures could be identified relatively easily by exciting the system with a simple sinusoidal forcing function and measuring so-called higher order Transfer Functions (TFs) again based on the Volterra Series (Storer, 1991). Since even relatively slight damage in structural systems can cause sufficient nonlinear dynamic behaviour to be detected, it may be feasible to use experimentally determined higher order TFs as a means of detecting the presence of damage in structures at a relatively early stage. On this basis, in (Ruotolo *et al.*, 1996) a step-sine test on a cantilever beam with a closing crack was simulated numerically and the amplitude response of higher order TFs was computed and proposed as a measurable characteristic function to be used to identify damage in a structure in practice.

The previous work has also focused on how to represent a cantilevered beam with a breathing crack analytically. In (Crespo *et al.*, 1996; Ruotolo, 1997; Ruotolo *et al.*,1999; Ruotolo *et al.*, 1999) a single degree-of-freedom model of the beam is presented and the bilinear restoring force is approximated with a fourth-order polynomial, enabling the principal diagonals of the higher order FRFs to be expressed directly. (The same principle was re-proposed by Chatterjee, 2010 and Chatterjee, 2010 in more recent work on this subject in which the bilinear restoring force is approximated by a second-order polynomial.)

The aim of the present paper is to describe a simple single degree-of-freedom model of the nonlinear cracked beam (that to date has only been presented by these authors in conference proceedings) which is extended to study the case of two harmonic excitations in order to analyse also the effect of their combination due to the nonlinear behaviour. The results derived analytically on the simple model are also compared with those obtained by simulating an experimental test using a nonlinear finite-element model of the beam. The results serve to demonstrate that the breathing crack can be detected defining characteristic functions related to the higher order FRFs based on the Volterra Series.

2. A simple nonlinear model of a cracked beam

2.1. The bilinear oscillator

In order to model the dynamic behaviour in bending of a simple cantilever beam with a transverse crack it is assumed that the crack has infinitely small width. An oscillating force applied perpendicularly to the beam with respect to its axis will cause the beam to bend and it can be assumed that a crack on the upper side of the beam will remain closed when the beam bends upwards but open when the beam bends downwards, as shown in Fig. 1. In general, when the crack opens, the stiffness reduces as compared to when the crack is closed.

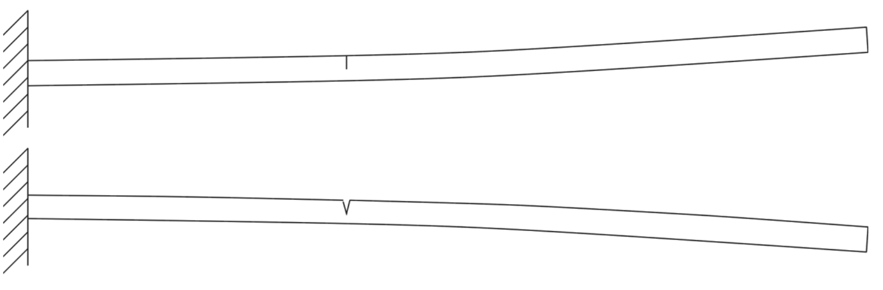

Fig. 1. Beam with a 'breathing' crack

In order to represent the first mode of vibration of a beam, it is appropriate to consider a single degree-of-freedom system with constant mass m and stiffness k when the crack is closed and $k - \delta k$ when the crack is open, i.e. a bilinear restoring force that is a function of the variable $y(t)$ representing the free end displacement of the beam (Fig. 2). To determine the dynamic response of this system, it is possible to integrate the equations of motion in the time domain by switching between the two formulations of stiffness functions depending on whether $y(t) > 0$ or $y(t) < 0$ (Fig. 3) (Friswell and Penny, 1992).

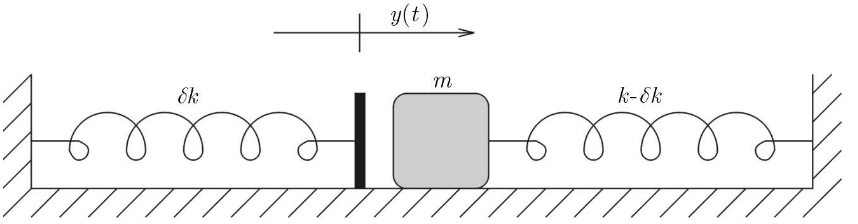

Fig. 2. Single degree of freedom bilinear oscillator

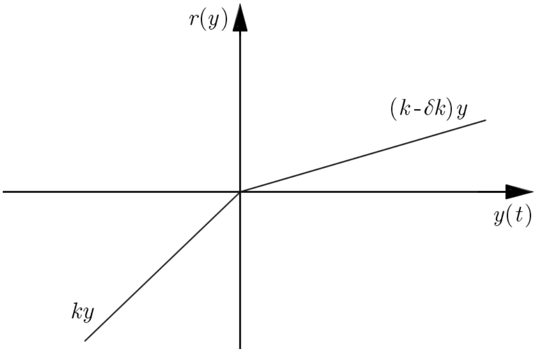

Fig. 3. Bilinear restoring force

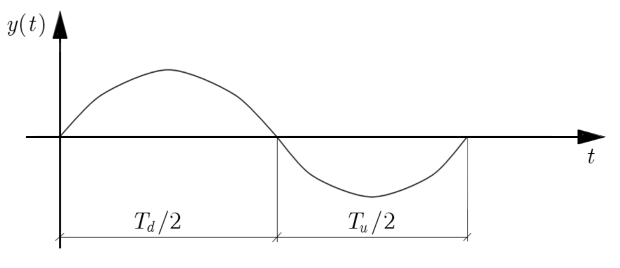

Fig. 4. Response cycle for the bilinear oscillator

For the dynamic motion of this simple system to periodic excitation, it is appropriate to consider that one response cycle is composed of two partial cycles corresponding to the two stiffness formulations respectively. Using $T_d/2$ to denote the duration of the partial cycle when the crack is open, and $T_u/2$ corresponding to when the crack is closed, the total period of the response cycle is: $T_e = (T_d + T_u)/2$ (Fig. 4). Since $T_e = 2\pi/\omega_e$, the previous equation can be written

$$\omega_e = 2\frac{\omega_d \omega_u}{\omega_d + \omega_u}$$

such that oscillations occur at a frequency which does not correspond to the natural frequencies both of the cracked and of the uncracked beam.

2.2. A single degree of freedom model for harmonic inputs

At this stage, it is necessary to quantify the alteration in the stiffness of the beam due to the presence of the crack. The equation of motion of the beam can be written

$$\frac{\partial^2}{\partial z^2}\Big(EI\frac{d^2v(z,t)}{dz^2}\Big) + \rho A\frac{\partial^2 v(z,t)}{\partial t^2} = f(z,t) \tag{2.1}$$

Assuming that

$$f(z,t) = 0 \qquad\qquad v(z,t) = v(z)\mathrm{e}^{\mathrm{j}\Omega t} \tag{2.2}$$

the response behaviour can be determined by introducing the boundary conditions at the fixed end, where the vertical displacement and rotation are zero, and at the free end where the moment and shear are zero, it is possible to calculate the natural frequency Ω_n of the beam and the corresponding mode-shapes $v_n(z)$ which, for convenience, can be expressed in the non-dimensional form

$$\int\limits_0^L \rho A v_n^2(z)\,dz = 1 \tag{2.3}$$

Considering the forcing term

$$f(z,t) = x(z)\mathrm{e}^{\mathrm{j}\omega t} \tag{2.4}$$

in which $x(z)$ represents the spatial distribution of the force applied to the beam and indicating by index I the first mode of vibration, if $\omega \leqslant \Omega_I$, the dynamic response can be determined using the modal decomposition

$$v(z,t) = v_I(z)y_I(t) \tag{2.5}$$

the dynamic response can be considered to be the product of the term $v_I(z)$ which expresses the spatial evolution and the term $y_I(t)$ which expresses the evolution in time.

Substituting (2.5) in (2.1), multiplying both sides by $v_1(z)$ and integrating along the length of the beam

$$m\ddot{y}_I(t) + ky_I(t) = X\mathrm{e}^{\mathrm{j}\omega t} \tag{2.6}$$

which represents the forced oscillations of a SDOF system in which,

$$
\begin{aligned}
m &= \int\limits_0^L \rho A v_I^2(z)\,dz = 1 \qquad\qquad k = \int\limits_0^L EI v_I''^2(z)\,dz \\
X &= \int\limits_0^L x(z) v_I(z)\,dz
\end{aligned}
\tag{2.7}
$$

Using this simple model, it is possible to calculate the response $y_I(t)$ to an excitation with spatial distribution $x(z)$ and frequency ω; given $y_I(t)$, it is possible to determine the dynamic response of the whole beam using (2.5).

Parameters k and m are associated with the undamaged beam, and to determine the characteristics of the cracked beam it is possible to apply the following equation (Ruotolo, 1997)

$$
\delta\nu_I = -\frac{[EI v_I''(z_0)]^2}{k_T}
$$

where $\delta\nu_I$ is the variation caused by the crack of the eigenvalue of the first mode of the undamaged beam, z_0 is the position of the crack and $1/k_T$ is the flexibility of a spring equivalent to the crack whose value is calculated using basic fracture mechanics notions. Using the relation between the variation in the eigenvalue and the change in the stiffness (Ruotolo, 1997)

$$
\frac{\delta\nu_i}{\nu_i} = \frac{\delta k_i}{k_i}
$$

which is valid assuming the structure does not experience a change in mass, it is possible to determine the variation in stiffness

$$
\delta k = k\frac{\delta\nu_I}{\nu_I} \tag{2.8}
$$

relative to the first mode of vibration of the structure. Since $\nu_1 = k/m$ and from (2.3) $m = 1$, it is possible to write that

$$
\delta k = \delta\nu_I = -\frac{[EI v_I''(z_0)]^2}{k_T} \tag{2.9}
$$

Using (2.9) and (2.8), it is possible to determine the stiffness of the cracked beam.

The equation of motion of the beam with the 'breathing' crack can be written as

$$m\ddot{y}_I(t) + c\dot{y}_I(t) + r[y_I(t)] = X\mathrm{e}^{\mathrm{j}\omega t} \tag{2.10}$$

in which the elastic restoring force $r[y_I(t)]$ can be expressed as

$$r(y_I) = \begin{cases} ky_I & y_I < 0 \\ (k-\delta k)y_I & y_I \geqslant 0 \end{cases}$$

introducing the damping term $c\dot{y}_I(t)$ with $c = 2\zeta m\sqrt{k/m}$, in which ζ is the modal damping coefficient.

In certain situation, it may be appropriate to approximate the nonlinear term in the interval $\Delta = [-\Delta_0, \Delta_0]$ with a polynomial series of the order n i.e.

$$r(y_I) \simeq \sum_{i=0}^{n} k_i y_I^i \tag{2.11}$$

It is possible to re-write equation (2.10) in a polynomial form (upto 4th order) using the orthogonal polynomials of Forsythe (Forsythe, 1957)

$$r(y_I) \simeq k_0 + k_1 y_I + k_2 y_I^2 + k_3 y_I^3 + k_4 y_I^4 \tag{2.12}$$

with

$$\begin{array}{lll} k_0 = -\dfrac{15}{256}\delta k \Delta_0 & k_1 = k - \dfrac{\delta k}{2} & k_2 = -\dfrac{105}{128}\dfrac{\delta k}{\Delta_0} \\ k_3 = 0 & k_4 = \dfrac{105}{256}\dfrac{\delta k}{\Delta_0^3} & \end{array} \tag{2.13}$$

In practice, the term k_0 in series (2.12) can be neglected when no static force is applied to the structure.

The amplitude of the response, in which equation (2.11) should approximate the elastic restoring force, varies with the excitation frequency ω. Thus it is necessary that the limits of the interval Δ depend on ω

$$\Delta_0(\omega) = \Big|\frac{X}{k + \mathrm{j}c\omega - m\omega^2}\Big| \tag{2.14}$$

Substituting (2.12)-(2.14) in equation (2.10), it is possible to formulate the following equation of motion

$$\begin{aligned} m\ddot{y}_I(t) + c\dot{y}_I(t) + \Big(k - \frac{\delta k}{2}\Big)y_I(t) - \frac{105}{128}\frac{\delta k}{\Delta_0(\omega)}y_I^2(t) \\ + \frac{105}{256}\frac{\delta k}{\Delta_0^3(\omega)}y_I^4(t) = X\mathrm{e}^{\mathrm{j}\omega t} \end{aligned} \tag{2.15}$$

Furthermore, supposing the beam is excited with two harmonic inputs with amplitude X_1 and X_2 and frequencies ω_1 and ω_2 whose sum is smaller than the first natural frequency, the equation of motion will be

$$\begin{aligned} & m\ddot{y}_I(t) + c\dot{y}_I(t) + \Big(k - \frac{\delta k}{2}\Big) y_I(t) - \frac{105}{128}\frac{\delta k}{\varDelta_0(\omega_1,\omega_2)} y_I^2(t) \\ & + \frac{105}{256}\frac{\delta k}{\varDelta_0^3(\omega_1,\omega_2)} y_I^4(t) = X_1 \mathrm{e}^{\mathrm{j}\omega_1 t} + X_2 \mathrm{e}^{\mathrm{j}\omega_2 t} \end{aligned} \tag{2.16}$$

In this case $\varDelta_0$ is given by the amplitude of the dynamic response due to both excitations at frequencies ω_1 and ω_2. By approximating the response of the nonlinear oscillator to frequency ω_i with the response of the corresponding linear oscillator

$$\varDelta_{0i}(\omega_i) = \frac{X_i}{k + \mathrm{j}c\omega_i - m\omega_i^2} \tag{2.17}$$

it follows that the amplitude of the response of the linear oscillator to both excitations with frequency ω_1 and ω_2 will be

$$\varDelta_0(\omega_1,\omega_2) = \Big|\varDelta_{01} + \varDelta_{02}\Big| = \Big|\frac{X_1}{k + \mathrm{j}c\omega_1 - m\omega_1^2} + \frac{X_2}{k + \mathrm{j}c\omega_2 - m\omega_2^2}\Big| \tag{2.18}$$

3. The Volterra series and higher order Frequency Response Functions

3.1. Defining higher order FRFs from the Volterra series

Recently, investigations into the behaviour of nonlinear structures have used the concepts of higher order Frequency Response Functions (FRFs) defined from the Volterra series (Gifford and Tomlinson, 1989), a mathematical basis for the analysis of differential equations with polynomial-type nonlinearities (Schetzen, 1980). The Volterra series extends the familiar concept of the convolution integrals for linear systems to a series of multi-dimensional convolution integrals necessary for polynomial-type nonlinearities. For any linear system, with an input $x(t)$ and output $y(t)$, the convolution integral is written

$$y(t) = \int h(\tau) x(t-\tau)\, d\tau \tag{3.1}$$

Correspondingly, for a nonlinear Volterra system, the Volterra series is written

$$\begin{aligned} y(t) = \int h_1(\tau_1)x(t-\tau_1)\,d\tau_1 + \iint h_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)\,d\tau_1\,d\tau_2 \\ + \int \cdots \int h_n(\tau_1,\ldots,\tau_n)\prod_{i=1}^{n} x(t-\tau_i)',d\tau_1\ldots d\tau_n \end{aligned} \tag{3.2}$$

In the analysis of linear systems, the relationship between the Impulse Response Function, $h(\tau)$, and the FRF $H(\omega)$ is well known, and since the first term in the series has exactly the same form as the linear convolution integral, the first order FRF can be realised with the conventional one-dimensional Fourier Transform

$$H_1(\omega_1) = \int_{-\infty}^{\infty} h_1(\tau_1)\mathrm{e}^{-\mathrm{j}\omega_1\tau_1}\,d\tau_1 \tag{3.3}$$

A similar relationship can be defined for all the kernels, $h_n(\tau_1,\ldots,\tau_i)$, in the Volterra series by using Multi-dimensional Fourier Transforms

$$H_n(\omega_1,\ldots,\omega_n) = \int \cdots \int h_n(\tau_1,\ldots,\tau_n)\mathrm{e}^{-\mathrm{j}(\omega_1\tau_1+\ldots+\omega_n\tau_n)}\,d\tau_1\ldots d\tau_n \tag{3.4}$$

The series of multi-dimensional higher order FRFs defined in this way provides a very general representation of this class of polynomial nonlinear systems and can be used to explain how systems excited by, for example, a broadband random signal tend to distribute energy between frequencies in a way that reflects the type of nonlinearity present in the system.

3.2. Higher order FRFs for harmonic inputs

For a nonlinear system which can be represented by the Volterra series, the response to a single harmonic input $x(t) = X\mathrm{e}^{\mathrm{j}\omega t}$ can be written

$$y(t) = H_1(\omega)X\mathrm{e}^{\mathrm{j}\omega t} + H_2(\omega,\omega)X^2\mathrm{e}^{\mathrm{j}2\omega t} + \ldots + H_n(\omega,\ldots,\omega)X^n\mathrm{e}^{\mathrm{j}n\omega t} \tag{3.5}$$

In this case, only the leading diagonals of the higher order FRFs need to be considered in order to represent the response spectra, i.e. where $\omega_1 = \omega_2 = \ldots = \omega_n$. These functions, denoted $H_n(\omega,\ldots,\omega)$, become one-dimensional in frequency and can be determined using the one-dimensional Fourier Transform. Thus using $Y(\omega)$ to denote the fundamental spectral line of the output $y(t)$, and $Y(n\omega)$ as the n-th order harmonic at n times the input frequency, the diagonal of the n-th order FRF can be defined as

$$H_n(\omega,\ldots,\omega) = \frac{2^{n-1}Y(n\omega)}{[X(\omega)]^n} \tag{3.6}$$

Furthermore, the multi-dimensional second-order FRF can be used to quantify the interaction between any two harmonic components of the input signal. For the input $x(t) = X_1 e^{\omega_1 t} + X_2 e^{\omega_2 t}$, the displacement can be written

$$\begin{aligned} y(t) = H_1(\omega_1)X_1 e^{j\omega_1 t} + H_2(\omega_2)X_2 e^{j\omega_2 t} + H_2(\omega_1,\omega_1)X_1^2 e^{j2\omega_1 t} \\ + H_2(\omega_2,\omega_2)X_2^2 e^{j2\omega_1 t} + 2H_2(\omega_1,\omega_2)X_1X_2 e^{j(\omega_1 t+\omega_1 t)} + \dots \end{aligned} \tag{3.7}$$

and the second order FRF can be defined as

$$H_2(\omega_1+\omega_2) = \frac{Y(\omega_1+\omega_2)}{X_1(\omega_1)X_2(\omega_2)} \tag{3.8}$$

4. Identifying higher order FRFs for the cracked beam via harmonic probing

4.1. Expressing the higher order FRFs analytically

Using the technique of harmonic probing that consists in introducing formulation (3.5) into equation of motion (2.15), and equating the coefficients, it is possible to obtain closed-form expressions for each of the functions $H_n(\omega,\dots,\omega)$ in terms of different parameters of the system

$$\begin{aligned} &H_1(\omega) = \frac{1}{k_1 + jc\omega - m\omega^2} \qquad H_2(\omega,\omega) = -k_2H_1^2(\omega)H_1(2\omega) \\ &H_3(\omega,\omega,\omega) = -2H_1(\omega)H_2(\omega,\omega)k_2H_1(3\omega) \\ &H_4(\omega,\dots,\omega) = -[k_2(H_2^2(\omega,\omega) + 2H_1(\omega)H_3(\omega,\omega,\omega)) + k_4H_1^4(\omega)]H_1(4\omega) \end{aligned} \tag{4.1}$$

It is important to observe that, since the different stiffness coefficients k_i are related to the depth and position of the crack, the form of higher order FRFs can be used to characterise the damage present in a structure.

The expression for $H_2(\omega_1,\omega_2)$ can be obtained by subsituting $y(t)$, $\dot{y}(t)$ and $\ddot{y}(t)$ into equation (2.16) and equating the coefficients of $X_1X_2\exp[j(\omega_1 t + \omega_1 t)]$

$$H_2(\omega_1,\omega_2) = -k_2H_1(\omega_1)H_1(\omega_2)H_1(\omega_1+\omega_2) \tag{4.2}$$

$H_2(\omega_1,\omega_2)$ for values of $\omega_1+\omega_2 = \Omega_I$ is proposed here for the first time as an indicator of damage, exploiting the fact that the nonlinear behaviour exhibited due to the presence of the crack causes two harmonics to combine and excite the sum of their respective frequencies.

4.2. Application and results

The first two fourth order FRFs illustrated in Figs. 5 and 6 were calculated for a beam with the following characteristics: Young's Modulus $1.8 \cdot 10^{11}\ \mathrm{N/m^2}$, density $7850\ \mathrm{kg/m^3}$, length $0.7\ \mathrm{m}$ and square cross-section $0.02\ \mathrm{m} \times 0.02\ \mathrm{m}$, and different damage configurations (L denotes the distance of the crack from the fixed end of the beam) shown in Table 1.

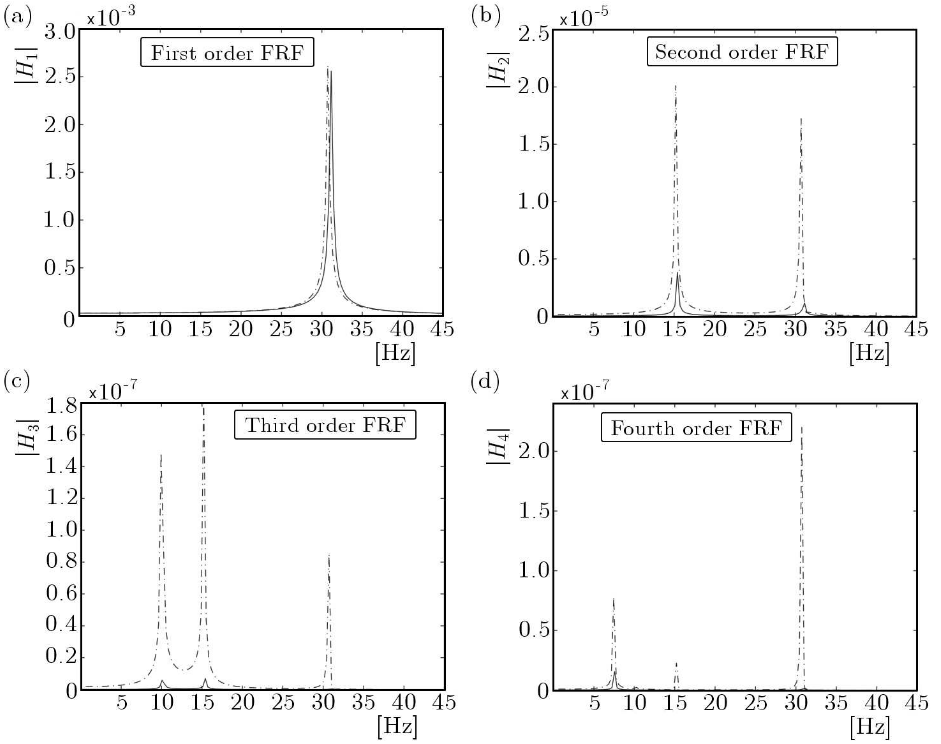

Fig. 5. Higher order FRFs for damage conditions 1 (—) and 2 ($-\cdot-$)

Table 1. Damage Configurations

Configuration	Depth [mm]	L_1 [m]
1	4	0.15
2	8	0.15
3	8	0.25

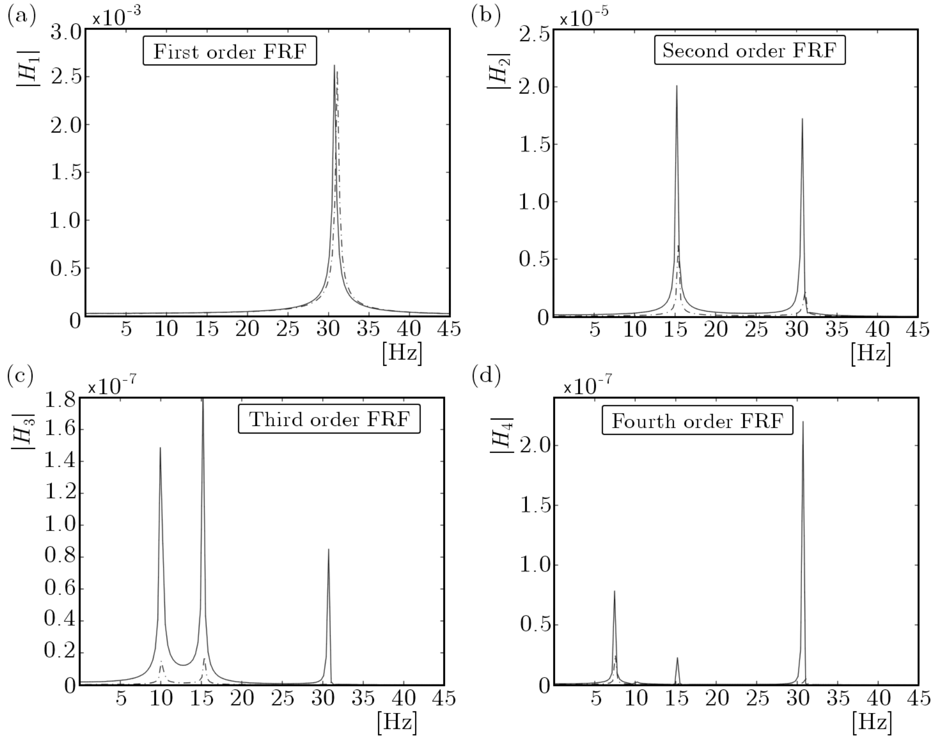

Fig. 6. Higher order FRFs for damage conditions 2 (—) and 3 ($- \cdot -$)

In particular, Fig. 5 shows damage cases 1 and 2, where the crack is in the same position but has two different depths, while Fig. 6 presents damage cases 2 and 3 where the crack has the same depth but two different positions. As can be observed from the results, the amplitude of the different higher order FRFs alters significantly depending on how the depth or location of the crack varies.

In Fig. 7, the absolute value of $H_2(\omega_1, \omega_2)$, the function proposed here as a new indicator of damage, is plotted for values of $\omega_1 + \omega_2 = \Omega_I$. Also in this case the amplitude of this function depends on the intensity of damage. The advantage of this indicator function lies in the fact that it is possible to select an infinite number of frequency couple combinations $(\omega_1,\ \omega_2)$ so as to excite a resonance of the system (eg. the first resonance Ω_I in this case). In general, the nonlinear behaviour exhibited at a resonance is more evident for a nonlinear stiffness function and therefore potentially more sensitive to the presence of the crack than the diagonal $H_2(\omega, \omega)$ shown in Figs. 5 and 6 which corresponds to resonant behaviour only at $\Omega_I/2$ and Ω_I.

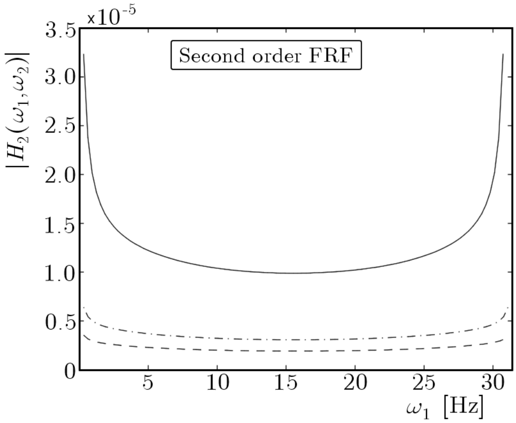

Fig. 7. Second order FRF with two harmonic inputs for damage conditions 1 (– –), 2 (—), and 3 (– · –)

5. Simulation of a stepsine test on the cracked beam

5.1. FE model of the cracked beam

In order to establish the validity of the higher order FRFs obtained in Section 4, considering a simple degree-of-freedom model of the beam, a stepsine test has been simulated using a FEM of the cracked beam presented in the paper by Ruotolo *et al.*, (1996).

For the sake of completeness, the model is presented very shortly in this subsection.

In the simulations it was supposed that the damage affected just the stiffness matrix of the element containing the crack and not the global mass and the damping matrices $\mathbf{M}$ and $\mathbf{C}$. Undamaged sections of the beam were modelled by Euler-type finite elements with two nodes and two degrees-of-freedom (transverse displacement and rotation) at each node. For the section with the crack, the finite element proposed in reference (Qian *et al.*, 1990) has been used.

In order to model accurately the nonlinear behaviour of the beam, it is necessary to determine the precise moment that the beam changes state, i.e. when the crack opens or closes. In the results presented, it has been assumed that the change between fully-open and fully-closed takes place instanteously, giving rise to a bilinear-type stiffness nonlinearity.

When the crack closes and its interfaces are completely in contact with each other, the dynamic response can be determined directly using the global stiffness matrix of the uncracked beam $\mathbf{K}_u$. However, when the crack opens, the stiffness matrix of the cracked element must be introduced in replacement at the appropriate rows and columns of the global stiffness matrix $\mathbf{K}_d$.

In the numerical simulation, the change of state is imposed in terms of the beam curvature at the cracked section: the crack is assumed to be open if the curvature is in the positive sense, otherwise it is closed. Under the action of the excitation force, alternate crack opening and closing causes the equations of motion of the cracked beam to be nonlinear

$$\mathbf{M}\ddot{\boldsymbol{y}}(t) + \mathbf{C}\dot{\boldsymbol{y}}(t) + \mathbf{K}\boldsymbol{y}(t) = \boldsymbol{x}(t) \tag{5.1}$$

where

$$\mathbf{K} = \mathbf{K}_u - \delta \Delta\mathbf{K} \tag{5.2}$$

and by denoting the changes in the global stiffness matrix due to the crack

$$\Delta\mathbf{K} = \mathbf{K}_u - \mathbf{K}_d \tag{5.3}$$

with

$$\delta = \begin{cases} 1 & \text{when the crack is open} \\ 0 & \text{when the crack is closed} \end{cases}$$

For the numerical simulations presented, the nonlinear equations of motion for the cracked beam rewritten in an incremental form have been solved with an implicit time integration scheme and modified Newton iteration according to Bathe and Gracewski, (1981).

5.2. Higher order Transfer Functions for sinusoidal inputs

In practice, it is difficult to measure higher order FRFs of a system directly (Gifford and Tomlinson, 1989). Several techniques have been developed for measuring higher order Transfer Functions (TFs) of a system which can be related to the ideal higher order FRFs (Storer, 1991). The most fundamental technique uses a single sine wave input. This approach to measure TFs is straightforward, and the relationship between TF and FRF can be explained and interpreted.

Considering the output time signals in terms of the displacement $y(t)$, the response to the *sinusoidal forcing* $x(t)$, the n-th order TFs can be determined using the single, dimensional Fourier transform of the time signals

$$TF_n(\omega) = \frac{2^{n-1}Y(n\omega)}{[X(\omega)]^n} \tag{5.4}$$

The term $Y(\omega)$ is the fundamental output term at the input frequency ω, and $Y(n\omega)$ is the n harmonic term in the spectrum of the output. Each term in the spectra is a complex quantity, and the TFs convey both gain and phase information regarding the transfer of energy between frequencies. Equation (5.4), expressing higher order TFs, can be compared with equation (3.5) which defines the corresponding higher order FRFs. A stepped frequency sine test is a convenient way to measure these TFs both in simulations and in practical testing.

Comparison of equations (3.6) and (5.4) indicates the close relationship between the higher order FRFs defined from the Volterra series which are unique for the system, and the higher order TFs which can be measured easily in practice. The difference arises since the TFs are determined physically by inputting a sinewave to the system

$$x(t) = X\sin(\omega t) = \frac{X}{2}(\mathrm{e}^{\mathrm{j}\omega t} - \mathrm{e}^{-\mathrm{j}\omega t}) \tag{5.5}$$

rather than the ideal harmonic $x(t) = X\mathrm{e}^{\mathrm{j}\omega t}$. Two harmonic terms are present in the sinewave which can interact in a nonlinear system and give rise to 'degenerative' effects influencing the measurement of lower order TFs. These effects are thought to originate the classical distortion phenomenon observed on Transfer Functions measured during stepped-sine tests on nonlinear structures (Storer and Tomlinson, 1993).

In an analogous way to the case of a single sinusoidal input expressed in Eq. (5.4), when two sinusoidal excitations of frequencies ω_1 and ω_2 are applied, the second order TF is experimentally calculated as

$$TF_2(\omega_1, \omega_2) = \frac{Y(\omega_1 + \omega_2)}{X_1(\omega_1)X_2(\omega_2)} \tag{5.6}$$

This equation can be compared with equation (3.8) which represents the corresponding higher order FRF. Accordingly $TF_2(\omega_1, \omega_2)$ is proposed as a novel indicator of damage that can be measured during a stepped-sine test, again exploiting the fact that the two sinsoidal inputs combine to excite the sum of their frequencies as a result of the nonlinear behaviour caused by the presence of the crack.

5.3. Application and results

Using the nonlinear model described in a Section 5.1 to simulate the time domain response of the cracked beam, the higher order TFs were determined using the procedure outlined in Section 5.2. In order to simulate a stepped-sine test on the beam, the frequency of the sinusoidal excitation was varied over the range from 0.2 to $1.4\Omega_I$ (being Ω_I the first natural frequency of the equivalent linear system). The results are shown in Figs. 8 and 9 corresponding to damage conditions 1 and 2, and 2 and 3, respectively. Figures 8 and 9 can

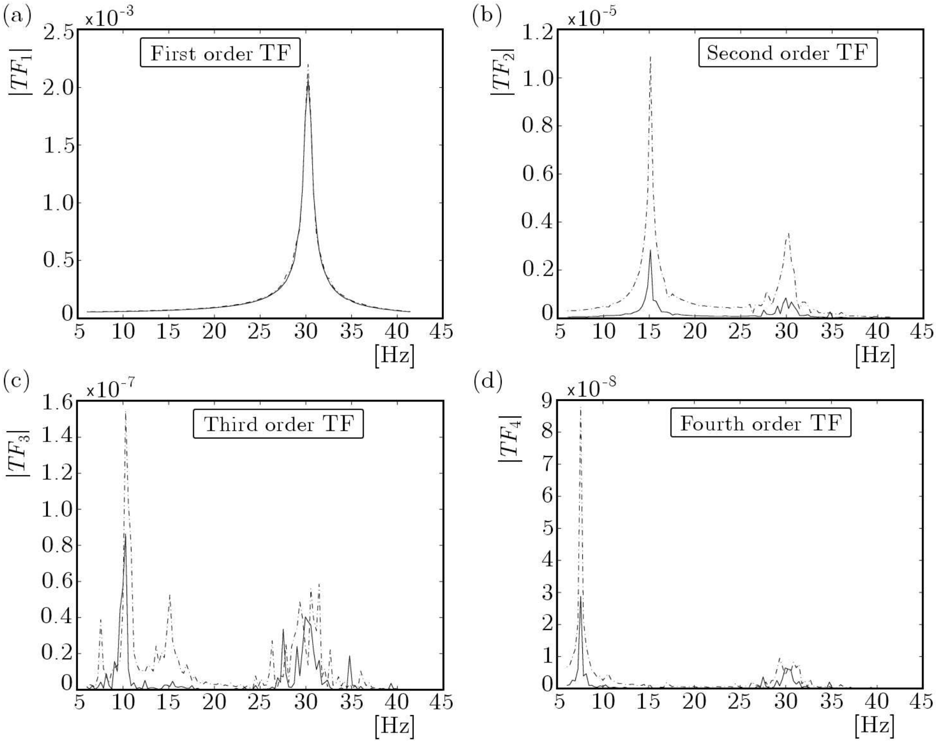

Fig. 8. Higher order FRFs for damage conditions 1 (—) and 2 (− · −)

be compared with the results shown in Fig. 5 and 6 which show the diagonals of higher order Frequency Response Functions obtained analytically via harmonic probing. The comparison indicates a generally close agreement between the corresponding functions. Focusing on the first order TF (Figs. 8a and 9a), very little variation can be observed as the depth of the crack increases, indicating that this particular function, which corresponds to that conventionally measured during a stepped-sine test on a structure, is not particularly sensitive to the presence of the damage. Instead, the second- and fourth-order TFs demonstrate a high degree of sensitivity to the crack size and position. Indeed, as shown in Fig. 8b, the peaks of the second-order function increase by a factor of between 4 and 5 in damage case 2 when compared to case 1.

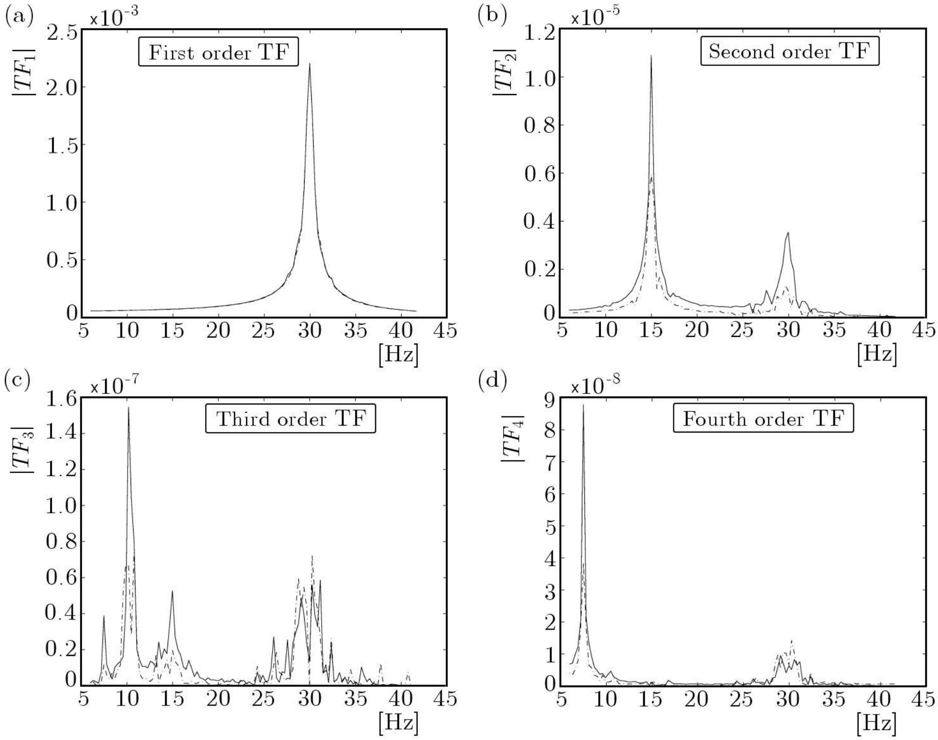

Fig. 9. Higher order TFs for damage conditions 2 (—) and 3 ($- \cdot -$)

(In cases 1 and 2 the damage is in the same location but the depth of the crack in case 2 is twice that of case 1.) The fourth-order TF also increases to a similar extent (see Fig. 8d). Moreover, in Fig. 9b, the peaks of the second-order function increase by a factor of 2 in damage case 2 when compared to case 3. (In cases 2 and 3 the crack depth is the same but the location of the crack in case 2 is closer to the clamped end than in case 3.)

The principal conclusion that can be drawn from these results is that higher order TFs and FRFs defined from the Volterra series may provide a highly-sensitive and practically useful indicator of the presence and extent of damage in a structure. In this context, the functions are basically detecting the nonlinear behaviour which, in this case, can be attributed to the presence of the breathing crack. Most importantly, the second- and fourth-order functions are already sensitive to the lowest level of damage, and increase significantly as the crack depth increases. The noise, which can be observed in the higher order TFs, in particular in the third- and fourth-order functions, can be attributed to the fact that the level of damping was low i.e. $\xi = 0.005$ (in order to represent the damping of a steel beam in a nominally realistic manner) causing the transient behaviour to be exhibited for a long duration each time the frequency of excitation is varied; correspondingly, the transient 'noise' on the simulated response pollutes the highly-sensitive higher order functions which are defined under the assumption that steady-state conditions have been reached (Storer and Tomlinson, 1993).

As mentioned before, this paper proposes a novel characteristic function as a means for detecting the crack. Figure 10 illustrates the second order function of Eq. (5.6), i.e. in which the two sinusoidal inputs combine due to the second-order nonlinear behaviour to excite the first natural frequency of the beam. Figure 10, obtained again by simulating a physically realisable stepped-sine test, can be compared to Fig. 7, the nominally equivalent function determined analytically through harmonic probing; although differences do occur, probably as a result of higher order contributions or 'interference' to the function shown in Fig. 10, the same overall characteristic form can be observed in both cases. Most importantly, the functions shown in Figs. 7 and 10 are again highly sensitive to the position and extent of the breathing crack (of course these functions are zero when the beam is linear i.e. undamaged).

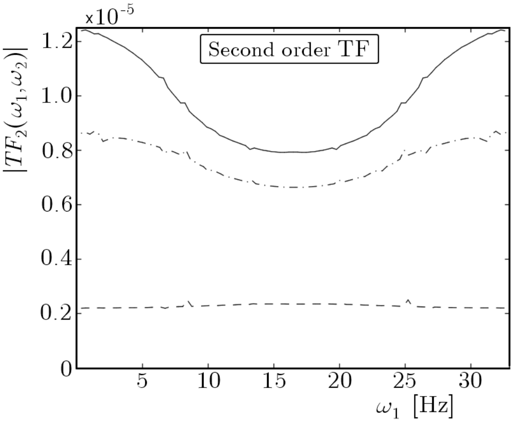

Fig. 10. Second order TF with two sinusoidal inputs for damage conditions 1 ($--$), 2 (—), and 3 ($-\cdot-$)

6. Conclusions

This research proposes the use of higher order Frequency Response Functions (FRFs) derived from the Volterra series to detect nonlinear behaviour which can be attributed to the presence of damage in structural systems. In the previous research, it was observed that the higher order FRFs are extremely sensitive indicators of nonlinear dynamic behaviour in general. Using the example of a cracked cantilever beam with a transverse one-edge non-propagating closing crack, the results present in this article serve to demonstrate that the shape of higher order functions, both when defined analytically directly from a single dof model using harmonic probing and when determined using a numerical simulation on a nonlinear finite element model, depend upon the size and the position of the crack. Thus, the higher order FRFs the principal diagonals of which can be estimated experimentally in a stepped- or swept sine test, may form the basis for a relatively sensitive structural damage-identification procedure.

Furthermore, this paper proposes a new characteristic function, again based on the higher order FRFs, for detecting damage by exploiting the fact that due to the second-order nonlinear behaviour, in this case caused by the presence of damage, two sinusoidal inputs combine to excite the sum of their frequencies. In particular, by selecting the frequencies of the inputs accordingly, it is possible to excite the first natural frequency of the beam through this second-order nonlinear combination. Based on this principle, a characteristic function has been proposed which, in a similar way to the diagonals of the higher order FRFs, may provide a useful indicator of the presence and extent of damage in a structure by detecting its nonlinear behaviour.

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Wykrywanie nieliniowych zjawisk za pomocą szeregu Volterry podczas oceny uszkodzeń konstrukcji belkowych

Streszczenie

Na przykładzie belki wspornikowej z karbem, przedstawiono metodę identyfikacji uszkodzenia za pomocą tzw. funkcji częstości odpowiedzi (*Frequency Response Function* – FRF) wyższego rzędu opartych na szeregu Volterry. Ogólnie wiadomo, że belka z „oddychającym” karbem poddana wymuszeniu dynamicznemu drga w sposób nieliniowy poprzez zmiany stanu (pomiędzy otwarciem i zamknięciem karbu). W pracy omówiono prosty model belki wspornikowej z tego typu uszkodzeniem pozwalający na analizę pierwszej postaci drgań. W zakresie częstości obejmujących pierwszą postać własną belki możliwe jest określenie charakterystyk dynamicznych układu poprzez zamodelowanie go jako względnie prostego oscylatora bi-liniowego. Wykazano, że zastosowanie funkcji FRF wyższego rzędu pozwala na identyfikację nieliniowych zjawisk w uszkodzonej belce oraz jakościową ocenę rodzaju tego uszkodzenia i jego lokalizacji. W pierwszej części pracy scharakteryzowano drgania belki w przypadku wymuszenia harmonicznego. Następnie wprowadzono nową funkcję charakterystyczną – także opartą na FRF – do wykrywania karbu na podstawie nieliniowej odpowiedzi drugiego rzędu, w której dwie składowe harmoniczne wymuszenia nakładają się, pobudzając częstość wynikającą z ich sumy. Przeprowadzono analizę porównawczą pomiędzy wynikami otrzymanymi z zaproponowanego modelu oraz modelu MES symulującego badania eksperymentalne układu.

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