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VIBRATION OF NON-HOMOGENEOUS VISCO-ELASTIC CIRCULAR PLATE OF LINEARLY VARYING THICKNESS IN STEADY STATE TEMPERATURE FIELD

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An analysis is presented for free vibration of a non-homogeneous visco-elastic circular plate with linearly varying thickness in the radial direction subjected to a linear temperature distribution in that direction. The governing differential equation of motion for free vibration is obtained by the method of separation of variables. Rayleigh-Ritz's method has been applied. Deflection, time period and logarithmic decrement corresponding to the first two modes of vibrations of a clamped non-homogeneous visco-elastic circular plate for various values of non-homogeneity parameter, taper constant and thermal gradients are obtained and shown graphically for the Voigt-Kelvin model.

Key words: non-homogeneous, visco-elastic, circular plate, variable thickness, steady state temperature field

1. Introduction

In recent years, an interest towards the effect of temperature on vibration of plates of variable thickness are often encountered in engineering applications. Their use in machine design, nuclear reactor technology, naval structures and acoustical components is quite common. The reason for these is that during heating up periods, structures are exposed to high intensity heat fluxes and material properties undergo significant changes; in particular the thermal effect can not be taken as negligible.

Many analyses show that plate vibrations are based on non-homogeneity of materials. Non-homogeneity can be natural or artificial. Non-homogeneous materials such as plywood, delta wood, fiber-reinforced plastic, etc. are used in engineering design and technology to strengthen the construction. There are some artificial non-homogeneous materials such as glass epoxy and boron epoxy in steel alloys for making rods in nuclear reactors.

Consideration of visco-elastic behaviour of the plate material, together with its variation in thickness, of structural components not only ensures reduction in the rate and size but also meets desirability for high strength in various technological situations in the aerospace industry, ocean engineering and electronic and optical equipment.

In a survey of the recent literature, the authors have found that no work deals with vibration of non-homogeneous visco-elastic circular plates of variable thickness subject to thermal gradient. Several authors (Li and Zhou, 2001, Tomar and Gupta, 1983, 1985; Tomar and Tewari, 1981) studied the effect of thermal gradient on vibration of a homogeneous plate of variable thickness. Singh and Saxena (1995) discussed the transverse vibration of quarter of a circular plate with variable thickness. It is well known (Hoff, 1958) that in the presence of thermal gradient, the elastic coefficient of homogeneous materials becomes a function of space variables. Lal (2003) studied transverse vibrations of orthotropic non-uniform rectangular plates with continuously varying density. Warade and Deshmukh (2004) discussed thermal deflection of a thin clamped circular plate due to partially distributive heat supply. Sobotka (1971) discussed rheology of orthotropic visco-elastic plates. Gupta and Khanna (2007) studied the effect of linearly varying thickness on vibration of visco-elastic rectangular plates of variable thickness. Recently, Gupta and Kumar (2008) analysed vibration of non-homogeneous visco-elastic rectangular plates with linearly varying thickness.

The present work deals with vibration of clamped non-homogeneous visco-elastic circular plates with linearly varying thickness in the radial direction subjected to a linear temperature distribution in this direction for the Voigt-Kelvin model. The non-homogeneity is assumed to arise due to linear variation in density of the plate material in the radial direction. Rayleigh-Ritz's method has been applied to derive the frequency equation of the plate. The time period, deflection and logarithmic decrement for the first two modes of vibrations are calculated for various values of thermal constants, non-homogeneity parameter and taper constant at different points of a clamped non-homogeneous visco-elastic circular plate with linearly varying thickness.

2. Equation of transverse motion

The axisymmetric motion of a circular plate of the radius a is governed by the equation (Leissa, 1969)

$$r\frac{\partial}{\partial r}\Big[\frac{1}{r}\Big(\frac{\partial}{\partial r}(rM_r)-M_\theta\Big)\Big]=\rho h\frac{\partial^2 w}{\partial t^2} \tag{2.1}$$

The resultant moments M_r and M_θ for a polar visco-elastic material of the plate are

$$M_r=-\widetilde{D}D\Big(\frac{\partial^2 w}{\partial r^2}+\frac{\nu}{r}\frac{\partial w}{\partial r}\Big) \qquad M_\theta=-\widetilde{D}D\Big(\frac{1}{r}\frac{\partial w}{\partial r}+\nu\frac{\partial^2 w}{\partial r^2}\Big) \tag{2.2}$$

where

$$D=\frac{Eh^3}{12(1-\nu^2)} \tag{2.3}$$

and $\widetilde{D}$ is the visco-elastic operator.

The deflection w can be sought in the form of product of two functions as follows

$$w(r,\theta,t)=W(r,\theta)T(t) \tag{2.4}$$

where $W(r,\theta)$ is the deflection function and $T(t)$ is the time function.

Using equations (2.2) and (2.4) in (2.1), one gets

$$\begin{aligned}&rD\frac{\partial^4 W}{\partial r^4}+\Big(D+2r\frac{\partial D}{\partial r}\Big)\frac{\partial^3 W}{\partial r^3}+\Big(-2\frac{D}{r}+(1+\nu)\frac{\partial D}{\partial r}+r\frac{\partial^2 D}{\partial r^2}\Big)\frac{\partial^2 W}{\partial r^2}+\\ &\qquad+\Big(2\frac{D}{r^2}-\frac{1+\nu}{r}\frac{\partial D}{\partial r}+\nu\frac{\partial^2 D}{\partial r^2}\Big)\frac{\partial W}{\partial r}-\rho hp^2W=0\\ &\frac{d^2T}{dt^2}+p^2\widetilde{D}T=0\end{aligned} \tag{2.5}$$

where p^2 is a constant.

These equations are expressions for transverse motion of a non-homogeneous circular plate with variable thickness and a differential equation of the time function for free vibration of the visco-elastic plate, respectively.

3. Analysis of equation of motion

Assuming a steady temperature field in the radial direction for a circular plate as

$$\tau=\tau_0\Big(1-\frac{r}{a}\Big) \tag{3.1}$$

where τ denotes the temperature excess above the reference temperature at any point at the distance r/a from the centre of the circular plate of the radius a and τ_0 denotes the temperature excess above the reference temperature at $r = 0$.

The temperature dependence of the modulus of elasticity for most structural materials is given as (Nowacki, 1962)

$$E(\tau) = E_0(1 - \gamma\tau) \tag{3.2}$$

where E_0 is the value of Young's modulus at the reference temperature, i.e. $\tau = 0$ and γ is the slope of variation of E with τ. The module variation, in view of expressions (3.1) and (3.2), becomes

$$E(r) = E_0\Big[1 - \alpha\Big(1 - \frac{r}{a}\Big)\Big] \tag{3.3}$$

where $\alpha = \gamma\tau_0$ $(0 \leqslant \alpha < 1)$ is a parameter known as thermal gradient.

The expression for the maximum strain energy V_{max} and maximum kinetic energy T_{max} in the plate, when it vibrates with the mode shape $W(r,\theta)$, are given as (Leissa, 1969)

$$\begin{aligned} V_{max} &= \frac{1}{2}\int\limits_0^{2\pi}\int\limits_0^{a} D\Big\{\Big(\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}\Big)^2 + \\ &\quad -2(1-\nu)\Big[\frac{\partial^2 W}{\partial r^2}\Big(\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}\Big) - \Big(\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial W}{\partial \theta}\Big)\Big)^2\Big]\Big\} r\,d\theta\,dr \\ T_{max} &= \frac{1}{2}p^2\int\limits_0^{2\pi}\int\limits_0^{a} \rho h W^2 r\,d\theta\,dr \end{aligned} \tag{3.4}$$

It is assumed that the thickness and non-homogeneity varies in the r-direction only, consequently the thickness h, non-homogeneity ρ and flexural rigidity D of the plate become a function of r only.

Assume the mode shape as (Ramaiah and Kumar, 1973)

$$W(r,\theta) = W_1(r) cos\theta \tag{3.5}$$

taking $W_1(r) = r\overline{W}_1(r)$ as the integration contains a negative power of r and introduce non-dimensional quantities

$$\begin{aligned} R &= \frac{r}{a} & \overline{h} &= \frac{h}{a} & \overline{W} &= \frac{\overline{W}_1}{a} \\ \overline{D} &= \frac{D}{a^3} & \overline{\rho} &= \frac{\rho}{a} \end{aligned} \tag{3.6}$$

Now let us assume the thickness and non-homogeneity of the plate to be

$$\overline{h}(R) = h_0(1-\beta R) \qquad\qquad \overline{\rho}(R) = \rho_0(1-\alpha_3 R) \tag{3.7}$$

where $h_0 = \overline{h}|_{R=0}$ and $\rho_0 = \overline{\rho}|_{R=0}$.

Using equations (3.5), (3.6) and (3.7) in equations (3.4), one gets

$$\begin{aligned}
V_{max} &= \frac{\pi a^5 E_0 h_0^3}{24(1-\nu^2)} \int\limits_0^1 (1-\alpha+\alpha R)(1-\beta R)^3 \Big\{ \Big(3\frac{d\overline{W}}{dR} + R\frac{d^2\overline{W}}{dR^2}\Big)^2 + \\
&\quad -2(1-\nu)\Big[\frac{d\overline{W}}{dR}\Big(\frac{d\overline{W}}{dR} + R\frac{d^2\overline{W}}{dR^2}\Big)\Big]\Big\} R\, dR \\
T_{max} &= \frac{\pi a^8 p^2 \rho_0 h_0}{2} \int\limits_0^1 (1-\alpha_3 R)(1-\beta R) R^3 \overline{W}^2\, dR
\end{aligned} \tag{3.8}$$

4. Solutions and frequency equation

Rayleigh-Ritz technique requires that the maximum strain energy must be equal to the maximum kinetic energy. It is, therefore, necessary for the problem under consideration that

$$\delta(V_{max} - T_{max}) = 0 \tag{4.1}$$

for arbitrary variation of W satisfying relevant geometric boundary conditions.

For a circular plate clamped at the edges $r = a$, i.e. $R = 1$, the boundary conditions are

$$\overline{W} = \frac{d\overline{W}}{dR} = 0 \qquad \text{at} \qquad R = 1 \tag{4.2}$$

and the corresponding two terms of deflection function is taken as

$$\overline{W}(R) = C_1(1-R)^2 + C_2(1-R)^3 \tag{4.3}$$

where C_1 and C_2 are undetermined coefficients.

Now using equations (3.8) in equation (4.1), one has

$$\delta(V_1 - p^2 \ell T_1) = 0 \tag{4.4}$$

where

$$V_1 = \int_0^1 (1-\alpha+\alpha R)(1-\beta R)^3 \Big\{ \Big(3\frac{d\overline{W}}{dR} + R\frac{d^2\overline{W}}{dR^2} \Big)^2 + -2(1-\nu)\Big[\frac{d\overline{W}}{dR}\Big(\frac{d\overline{W}}{dR} + R\frac{d^2\overline{W}}{dR^2} \Big) \Big] \Big\} R\,dR$$

$$T_1 = \int_0^1 (1-\alpha_3 R)(1-\beta R) R^3 \overline{W}^2\,dR \tag{4.5}$$

Here

$$\ell = \frac{12(1-\nu^2)\rho_0 a^3}{E_0 h_0^2} \tag{4.6}$$

Equation (4.4) involves the unknowns C_1 and C_2 arising due to substitution of $\overline{W}(R)$ from (4.3). These unknowns are to be determined from equation (4.4), for which

$$\frac{\partial}{\partial C_n}(V_1 - p^2 \ell T_1) = 0 \qquad n = 1,2 \tag{4.7}$$

Equation (4.7) simplifies to the form

$$b_{n1}C_1 + b_{n2}C_2 = 0 \qquad n = 1,2 \tag{4.8}$$

where b_{n1}, b_{n2} $(n = 1,2)$ involve the parametric constant and frequency parameter.

For a non-trivial solution, the determinant of the coefficient of equation (4.8) must be zero. Thus, one gets the frequency equation as

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = 0 \tag{4.9}$$

where

$$b_{11} = 2(F_1 + B_1 p^2) \qquad b_{12} = b_{21} = F_2 + B_2 p^2$$
$$b_{22} = 2(F_3 + B_3 p^2)$$

Here F_1, F_2, F_3 are functions of α, β and B_1, B_2, B_3 are functions of α_3.

Frequency equation (4.9) is a quadratic one with respect to p^2 from which two values of p^2 can be found.

Choosing $C_1 = 1$, one obtains $C_2 = -F_4/F_5$ where $F_4 = 2(F_1 + p^2 B_1)$, $F_5 = F_2 + p^2 B_2$, therefore

$$\overline{W}(R) = (1-R)^2 - \frac{F_4}{F_5}(1-R)^3 \tag{4.10}$$

5. Time function of vibration of non-homogeneous visco-elastic plate

The time function of free vibration of the visco-elastic plate is defined by general ordinary differential equation $(2.5)_2$. Its form depends on the visco-elastic operator $\widetilde{D}$. For Kelvin's model, one has Gupta and Kumar (2008) and Sobotka (1978)

$$\widetilde{D} \equiv 1 + \frac{\eta}{G}\frac{d}{dt} \tag{5.1}$$

Taking the temperature dependence of shear modulus G and visco-elastic coefficient η in the same form as that of Young's modulus, one has

$$G(R) = G_0[1 - \alpha_1(1 - R)] \qquad\qquad \eta(R) = \eta_0[1 - \alpha_2(1 - R)] \tag{5.2}$$

where G_0 is the shear modulus and η_0 is the visco-elastic constant at some reference temperature, i.e. at $\tau = 0$.

Using equations (5.1) and (5.2) in equation $(2.5)_2$, one obtains

$$\frac{d^2T}{dt^2} + p^2q\frac{dT}{dt} + p^2T = 0 \tag{5.3}$$

where

$$q = \frac{\eta_0[1 - \alpha_2(1 - R)]}{G_0[1 - \alpha_1(1 - R)]}$$

Equation (5.3) is a differential equation of the second order for the time function T. Solving equation (5.3), one gets

$$T(t) = \mathrm{e}^{-\frac{p^2qt}{2}}(e_1 \cos st + e_2 \sin st) \tag{5.4}$$

where

$$s^2 = p^2 - \frac{1}{4}p^4q^2$$

and e_1, e_2 are integration constants.

Assuming that the initial conditions are

$$T = 1 \qquad \text{and} \qquad \frac{dT}{dt} = 0 \qquad \text{at} \quad t = 0 \tag{5.5}$$

and using condition (5.5) in equation (5.4), one gets

$$T(t) = \mathrm{e}^{-\frac{p^2qt}{2}}\Big(\cos st + \frac{p^2q}{2s}\sin st\Big) \tag{5.6}$$

Thus, the deflection $w(r,\theta,t)$ may be expressed as

$$w(r,\theta,t)=\overline{W}(R)\mathrm{e}^{-\frac{p^2qt}{2}}\Big(\cos st+\frac{p^2q}{2s}\sin st\Big)\cos\theta \tag{5.7}$$

The time period of vibration of the plate is given by

$$K=\frac{2\pi}{p} \tag{5.8}$$

where p is the frequency given by equation (4.9).

The logarithmic decrement of vibration is given by

$$\varLambda=\ln\frac{w_2}{w_1} \tag{5.9}$$

where w_1 is the deflection at any point of the plate at the time period $K=K_1$, and w_2 is the deflection at the same point at the time period succeeding K_1.

6. Results and discussion

The deflection w, time period K and logarithmic decrement $\varLambda$ are computed for a non-homogeneous clamped visco-elastic circular plate with linearly varying thickness for different values of taper constant β, thermal constants α, α_1, α_2 and non-homogeneity constant α_3 and different points for the first two modes of vibrations. The results are shown in Figs. 1-6.

For numerical computation, the following material parameters are used (Nagaya, 1977): $E_0=7.08\cdot 10^{10}\,\mathrm{n/m^2}$, $G_0=2.682\cdot 10^{10}\,\mathrm{n/m^2}$, $\eta_0=1.4612\cdot 10^{6}\,\mathrm{n.s/m^2}$, $\rho_0=2.8\cdot 10^{3}\,\mathrm{kg/m^3}$, $\nu=0.345$.

The thickness of the plate at the center is taken as $h_0=0.01\,\mathrm{m}$.

Figure 1 shows that the time period K of the first two modes of vibration decreases with an increase in the non-homogeneity parameter α_3, and whenever the taper constant β and thermal constant α increase then the time period increases for the first two modes of vibrations.

Figures 2 and 3 show that the deflection w starts from the maximum value to decrease to zero for the first mode of vibration, but for the second mode of vibration the deflection starts from the minimum value to grow and decrease again to finally become zero for a fixed value of θ and increasing R for the initial time $0K$ and $5K$ and uniform thickness.

Figures 4 and 5 show that when the taper constant β increases, the deflection for the first mode of vibration firstly increases to maximum then decreases

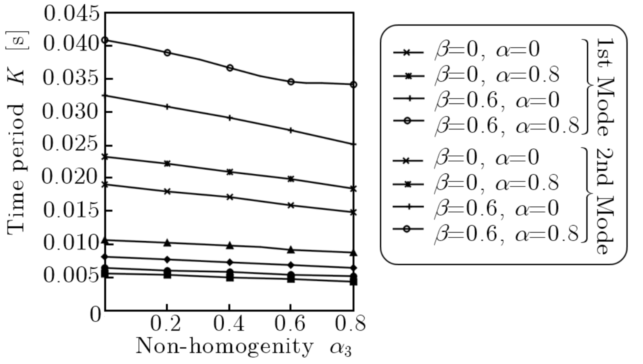

Fig. 1. Variation of time period with non- homogeneity constant of visco-elastic non-homogeneous circular plate with linearly varying thickness

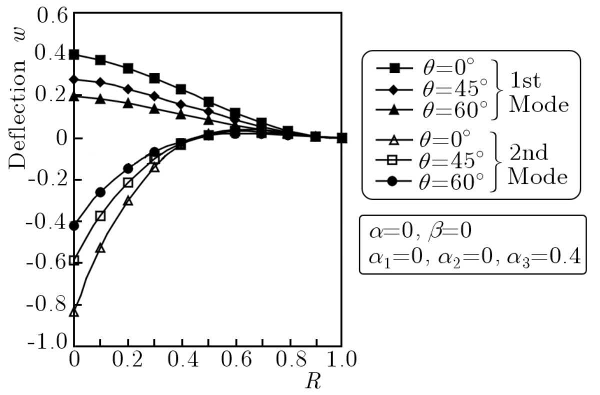

Fig. 2. Transverse deflection w vs. R of visco-elastic non-homogeneous circular plate with linearly varying thickness at initial time $0K$

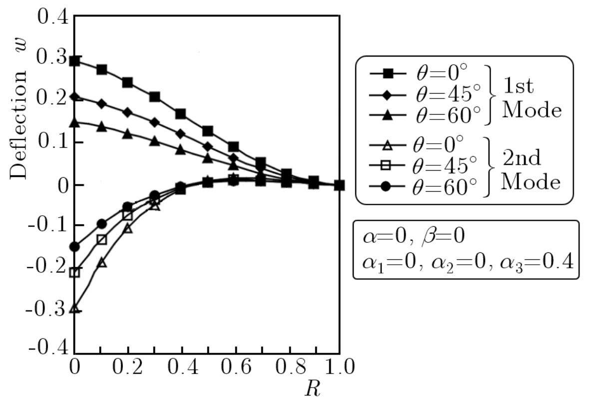

Fig. 3. Transverse deflection w vs. R of visco-elastic non-homogeneous circular plate with linearly varying thickness at initial time $5K$

and finally becomes zero, but for the second mode of vibration the deflection starts from the minimum value to increase and then decrease again finally reaching zero for a fixed value of θ and increasing R for the initial time $0K$ and $5K$.

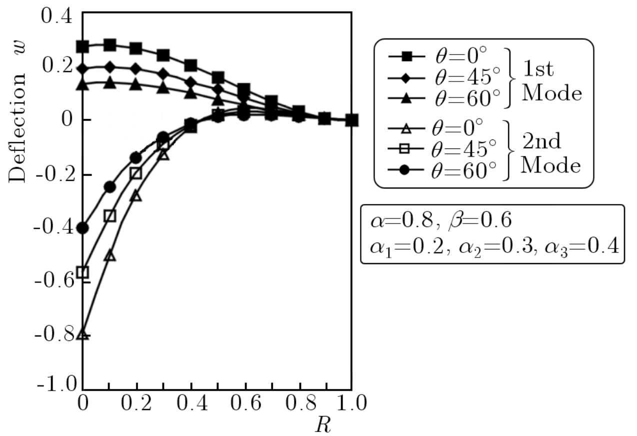

Fig. 4. Transverse deflection w vs. R of visco-elastic non-homogeneous circular plate with linearly varying thickness at initial time $0K$

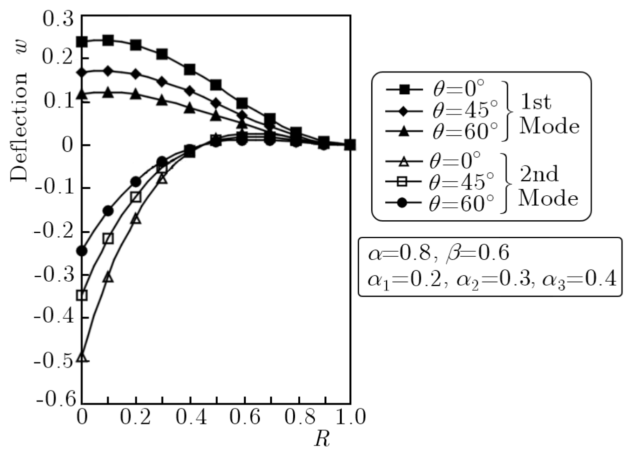

Fig. 5. Transverse deflection w vs. R of visco-elastic non-homogeneous circular plate with linearly varying thickness at initial time $5K$

Figure 6 shows that the logarithmic decrement Λ decreases with an increase in R but it remains the same for a fixed value of R and different values of θ. It can be seen in Fig. 6 that as the non-homogeneity parameter α_3 increases, the logarithmic decrement Λ decreases for the first two modes of vibration.

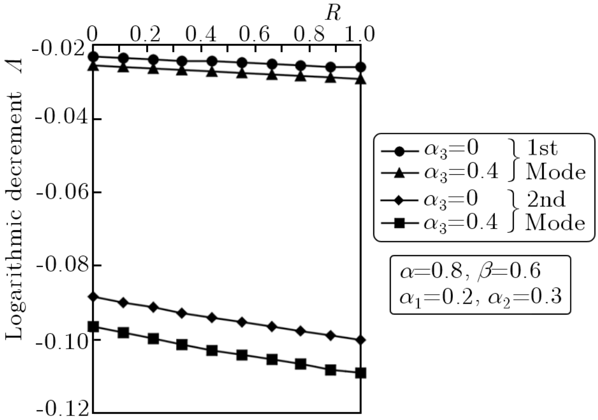

Fig. 6. Logarithmic decrement Λ vs. R of visco-elastic non-homogeneous circular plate with linearly varying thickness

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Drgania niejednorodnej lepko-sprężystej płyty kołowej o liniowo zmiennej grubości i ustalonym polu temperatury

Streszczenie

W pracy przedstawiono analizę drgań swobodnych niejednorodnej lepko-sprężystej płyty kołowej o liniowo zmiennej grubości w kierunku promieniowym i poddanej polu temperatury o liniowym rozkładzie w tym kierunku. Konstytutywne równanie różniczkowe ruchu dla drgań swobodnych otrzymano poprzez separację zmiennych. Zastosowano metodę Rayleigha-Ritza. W wyniku analizy wyznaczono ugięcie płyty, okres drgań i logarytmiczny dekrement tłumienia dwóch pierwszych postaci drgań dla warunków brzegowych odpowiadających zamocowaniu niejednorodnej płyty na brzegu. Wyniki przedstawiono graficznie w funkcji parametru niejednorodności, stałej zawężania grubości oraz zmiennego gradientu temperatury przy wykorzystaniu modelu reologicznego Kelvina-Voigta opisującego właściwości materiału płyty.

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