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EXPERIMENTAL VERIFICATION OF FREE TRANSVERSE VIBRATIONS OF COLUMNS LOADED BY HEADS EQUIPPED WITH ROTARY NODES

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The main goal of this paper is to determine the influence of various elements mounted in rotary nodes of a forcing head, i.e.: rolling needle bearings or rigid elements on the natural transverse vibration frequency and value of the critical load of columns. The problem of stability and course of characteristic values against the external force loading the systems subjected to a specific load is discussed in the paper. A new constructional scheme of the forcing and receiving heads serving for the purpose of carrying out the generalised load by the force directed towards the positive pole and the load by a force directed towards the positive pole is presented. Theoretical considerations related to determination of boundary conditions using the method of mechanical energy variation are shown. Verification of the assumed goal is realised by experimental research.

Key words: elastic columns; divergence instability; natural transverse vibrations

1. Introduction

Divergence systems (Gajewski and Życzkowski, 1969; Leipholz, 1974; Timoshenko and Gere, 1963; Ziegler, 1968) or divergence pseudoflutter systems (Tomski *et al.*, 1996, 1998, 1999, 2004; Tomski and Szmidla, 2004a,b, 2006) are loaded by conservative forces. Only the conservative load is considered, without considering publications related to non-conservative systems. The systems subjected to conservative loads lose stability due to buckling, and the transition from the stable to unstable condition happens at frequencies of natural vibrations equal zero for the so-called divergence critical force P_{c1}. By solving the issue of free vibration of the mentioned systems, a particular course of the curves is obtained on the load (P) – natural frequency (ω) plane which is presented in Fig. 1.

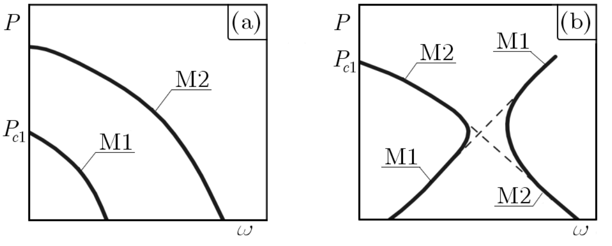


Fig. 1. (a) Divergence system; (b) divergence pseudoflutter system

All curves characteristic for the divergence systems (Fig. 1a) always have a negative slope. In the case of divergence pseudoflutter systems, for the load changing from 0 to the critical force (P_{c1}), the slope of the curves on the load (P) – natural frequency (ω) plane may be positive, zero or negative. For $P \approx P_{c1}$, the natural frequency curve slope is always negative. For such a system along the eigenvalue curves the natural vibration form change from the first form to the second one and inversely (where M1, M2 denote the first and the second form of vibrations, respectively).

Considering conservative loads of slender systems and their properties, the following cases can be distinguished:

- Euler's load (Timoshenko and Gere, 1963)
- loading by the force directed towards the positive pole – Timoshenko and Gere (1963), Gajewski and Życzkowski (1969) or the negative pole Gajewski and Życzkowski (1969)
- specific loads:
  - generalised load by the force directed towards the pole – Tomski *et al.* (1996, 1999), Tomski and Szmidla (2004a,b)
  - load by the follower force directed towards the pole – Tomski *et al.* (1998, 2004), Tomski and Szmidla (2004a)
  - load through a stretched element of finite bending rigidity – Tomski and Szmidla (2006).

For each of the listed types of specific loads there is a constructional scheme for the loading structure made of the forcing and receiving head.

Researches concerning the influence of components used in the receiving head (rolling bearing, plain bearing, rigid component) on natural vibrations of slender systems are included in works by Tomski and Podgórska-Brzdękiewicz (2006a,b). It was found that the use of rigid components results in a considerable increase in the basic experimental frequency of natural vibrations in the system with respect to the same quantity obtained from numerical simulations (for the applied mathematic model). Using the obtained characteristic values, the boundary conditions were modified by implementation of the equivalent rigidity of rotational spring modelling the rigidity of the free end of the system.

2. Statement of the problem

The main technical problem considered in this paper is the influence of various elements mounted in the rotary nodes of the forcing head, i.e.: rolling needle bearings or rigid elements, on the stability and the natural transverse vibration frequency of columns. A new constructional scheme of the forcing and receiving head serving for the purpose of carrying out the generalised load is given. Such a scheme can be implemented in the supporting systems for different applications.

This paper presents theoretical considerations, numerical calculations and experimental investigations concerning the stability and natural vibrations of two types of cantilever columns (**A** and **B**) differing in the way of realising the load. The schematic diagrams of the systems are shown in Fig. 2, where N is the value of external load necessary to obtain the compressive force of the rods P ($N = f(P)$).

Depending on geometric parameters of the forcing and receiving heads, the following cases can be distinguished (Fig. 2):

(a) generalised load by the force directed towards the positive pole – system **A**

(b) load by the force directed towards the positive pole – system **B**.

Boundary conditions are formed for the distinguished systems using the energetic method.

Examination of the systems concerns the influence that the elements used in the rotary nodes of the forcing head (rolling needle bearings or rigid elements) and geometry of this head have on the course of curves of the characteristic values and on the value of the critical force.

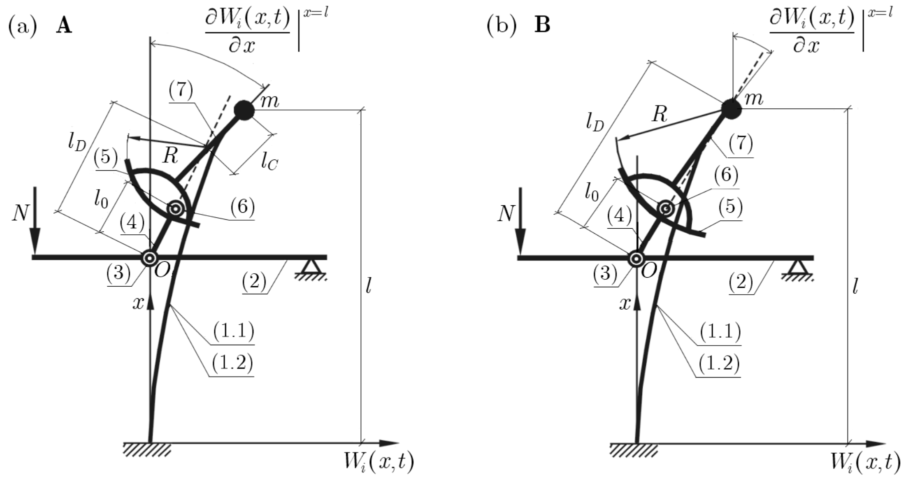


Fig. 2. Schematic diagrams of considered systems

2.1. Realisation of load

The structures loading the column (Fig. 2) comprise: beam (2), linear elements (4) and (7) and a circular element (5). Linear elements (2), (4) and (7) and structure (5) with radius R are characterised with infinite flexural rigidity. Nodes (3) and (6) are made of rotary elements transferring the load to the column which may comprise rolling needle bearings (F) or rigid elements (G) (Fig. 3). The radius of the element mounted in rotary node (6) has been marked as r. The following designations were introduced – compare Fig. 2:

- I – lower rotary node (3)
- II – upper rotary node (6).

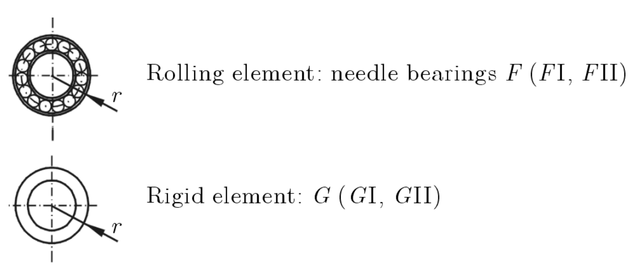


Fig. 3. Elements used in the rotary elements of the forcing head

The basic features of the column loading structures are as follows:

- the forcing structure comprises: linear elements (2) and (4) connected by a hinge joint and rotary nodes (3) and (6)

- element (7) connected by a rigid joint with circular cylinder (5) make the receiving head
- the centre of segment (5) having the circular cylinder outline is located at the distance l_C from the free end of the system, and at the distance l_D from node (3).

The direction of the loading force P is marked with a broken line (Fig. 2). It crosses the fixed point O (node (3)) located on the non-deflected axis of the column below its free end, node (6) and the point determining the curvature centre of a fragment of circular cylinder (5).

Values R and l_0 resulting from the construction of the forcing head are interrelated by the following relationship: $l_D = R - r + l_0$.

The real system is made as a planar frame made of two rods (1.1 and 1.2) with the bending rigidity $(EJ)_1$ and $(EJ)_2$, respectively, and the mass per unit length $(\rho_0 A)_1$ and $(\rho_0 A)_2$, and $(EJ)_1 = (EJ)_2$, $(\rho_0 A)_1 = (\rho_0 A)_2$, $(EJ)_1 + (EJ)_2 = EJ$, $(\rho_0 A)_1 + (\rho_0 A)_2 = \rho_0 A$, where: E denotes the longitudinal modulus of elasticity of the rod material, J – central axial moment of inertia of the column rod, ρ_0 – material density, A – cross-section area. The lengths of the column rods are equal to l. The rods of the column have the same cross-sections and are made of the same material. Such arrangement allows for forcing vibration in a privileged plane. There is symmetrical distribution of bending rigidity in this model. The rods and their physical and geometrical parameters are distinguished by (1.1), (1.2) indexes, which are only needed to calculate symmetrical natural frequencies and to determine corresponding forms of vibration (compare Tomski *et al.*, 1999). Hence, we can assume the global bending rigidity EJ and elementary mass of the column $\rho_0 A$ in the following considerations.

3. Column subjected to the generalised load by the force directed towards the positive pole – system A

3.1. Physical model and analysis of the system geometry

The physical model of the system subjected to a generalised load by the force directed towards the positive pole is shown in Fig. 2a (**A**). The system is loaded via beam (2) connected by a hinge joint with element (4) having the length of l_0. The column mounted rigidly $(x = 0)$ at the free end $(x = l)$ is jointed with linear elements (7). Displacement of the column end in relation to the axis x has the value of $W(x, t)$, and the deflection angle of this end compared to the non-deflected axis is $\partial W(x,t)/\partial x|^{x=l}$. Element (7) is rigidly jointed with circular cylinder (5) on which the second end of the element is moving (4). The total mass m comprises concentrated mass m_1 and corrected mass m_{zr} of elements (4), (5) and (7) in relation to the fixing point of mass m_1.

Figure 4 presents the physical model of the considered system, geometry and three options of externalising the internal forces (sections I–I, II–II also I–I and II–II). The formulas describing the total potential energy of the system depend on the place of externalising the load. The boundary conditions of the considered system can be obtained based on the total energy of the system for externalising the load in sections I–I or II–II. By externalising the internal forces in sections I–I and II–II, the potential energy of the system is equal to zero.

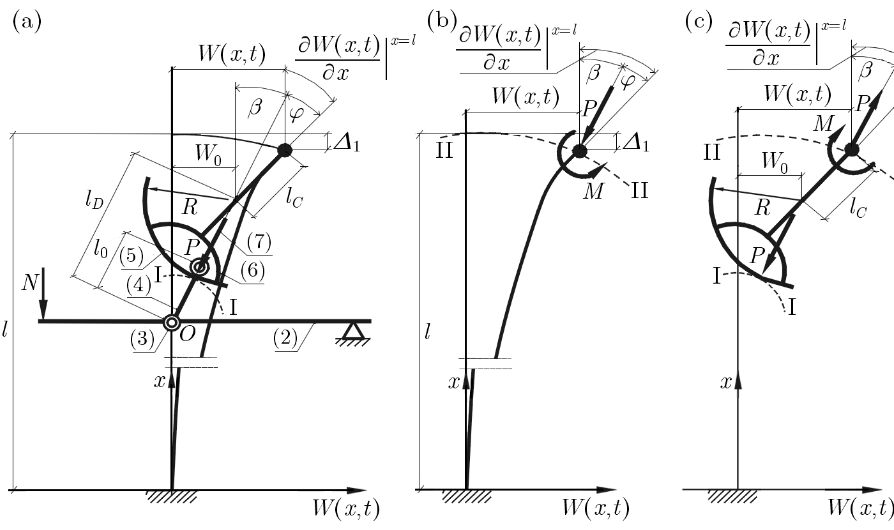


Fig. 4. Deflected form of column type **A** (see Fig. 2a), geometry and externalisation of internal forces

In order to unify the considerations presented in this work, the externalisation of the load in section II–II is examined (at the free end of the column).

Having examined geometry of the system (Fig. 4a), angle β formed by the direction of the force P with the non-deflected axis of the column is expressed by the following relations

$$\sin\beta = \frac{W_0}{l_D} \tag{3.1}$$

where W_0 is the displacement of the point determining the centre of the cylinder element of radius R, is

$$W_0 = W(l,t) - l_C \sin\Big[\frac{\partial W(x,t)}{\partial x}\Big|^{x=l}\Big] \tag{3.2}$$

Based on the relationships resulting from Fig. 4a, the condition of equality of the angels at the free end of the column looks as follows

$$\beta + \varphi = \frac{\partial W(x,t)}{\partial x}\Big|^{x=l} \tag{3.3}$$

where φ is the angle formed by the direction of the force P with the axis of rigid element (7).

The theory of small displacement was used in the description – if there are geometric relations between the elements of a given system, the following holds

$$\sin\varphi = \tan\varphi = \varphi \qquad\qquad \cos\varphi = 1 - \frac{\varphi^2}{2} \tag{3.4}$$

Using relations (3.1), (3.3) and $(3.4)_1$, the following is obtained after transformation

$$\frac{W_0}{l_D} = \frac{\partial W(x,t)}{\partial x}\Big|^{x=l} - \varphi \tag{3.5}$$

The final form of the expression determining the value of angle φ, determined based on formulas (3.2)-(3.5), was described in the following way

$$\varphi = \Big(1 + \frac{l_C}{l_D}\Big)\frac{\partial W(x,t)}{\partial x}\Big|^{x=l} - \frac{1}{l_D}W(l,t) \tag{3.6}$$

The $\varDelta_1$ value is a displacement resulting from shortening of the axis of the column caused by bending and amounts to

$$\varDelta_1 = \frac{1}{2}\int\limits_0^l \Big[\frac{\partial W(x,t)}{\partial x}\Big]^2\,dx \tag{3.7}$$

3.2. Mechanical energy of the system. Boundary conditions

Kinetic energy T of the system is the sum of kinetic energy of the column T_1 and energy of the concencrated mass $m - T_2$, and is expressed by the following formulas

$$T = T_1 + T_2 = \frac{\rho_0 A}{2}\int\limits_0^l \Big[\frac{\partial W(x,t)}{\partial t}\Big]^2\,dx + \frac{1}{2}m\Big[\frac{\partial W(l,t)}{\partial t}\Big]^2 \tag{3.8}$$

where $\rho_0 A$ is the mass density per column length unit, m – concentrated mass at the free end of the system (in point $x = l$).

Relations for potential energy of the system $V = \sum_{k=1}^{4} V_k$ (with externalising the internal forces in section II–II) are determined as follows:
— energy of elastic strain

$$V_1 = \frac{1}{2} EJ \int\limits_0^l \Big[\frac{\partial^2 W(x,t)}{\partial x^2}\Big]^2 \, dx \tag{3.9}$$

— potential energy of the vertical component of the P force

$$V_2 = -P\Delta_1 = -P\frac{1}{2} \int\limits_0^l \Big[\frac{\partial W(x,t)}{\partial x}\Big]^2 \, dx \tag{3.10}$$

— potential energy of the horizontal component of the P force

$$V_3 = \frac{1}{2} P\beta W(l,t) = \frac{1}{2}\frac{P}{l_D}\Big[W(l,t) - l_C \frac{\partial W(x,t)}{\partial x}\Big|^{x=l}\Big] W(l,t) \tag{3.11}$$

— potential energy of the bending moment

$$\begin{aligned} V_4 &= \frac{1}{2} P l_C \varphi \frac{\partial W(x,t)}{\partial x}\Big|^{x=l} = \\ &= \frac{1}{2} P l_C \Big[\Big(1 + \frac{l_C}{l_D}\Big) \frac{\partial W(x,t)}{\partial x}\Big|^{x=l} - \frac{1}{l_D} W(l,t)\Big] \frac{\partial W(x,t)}{\partial x}\Big|^{x=l} \end{aligned} \tag{3.12}$$

Formulation of the problem consisting in the determination of boundary conditions for the considered column is made using Hamilton's principle (Goldstein, 1950) that states

$$\delta \int\limits_{t_1}^{t_2} (T - V) \, dt = 0 \tag{3.13}$$

The commutation of integration (with respect to x and t) and variational calculation is used within Hamilton's principle (Eq. (3.13)). The equation of motion, after taking into account the commutation of variation and differentiation operators and after integrating the kinetic and potential energies of the system, is obtained in the form

$$EJ \frac{\partial^4 W(x,t)}{\partial x^4} + P \frac{\partial^2 W(x,t)}{\partial x^2} + \rho_0 A \frac{\partial^2 W(x,t)}{\partial t^2} = 0 \tag{3.14}$$

Using the *a priori* known geometric boundary conditions in the fixing point of the column $(x = 0)$

$$W(0,t) = 0 \qquad\qquad \frac{\partial W(x,t)}{\partial x}\Big|_{x=0} = 0 \tag{3.15}$$

the remaining boundary conditions necessary to solve the boundary problem are determined

$$\begin{aligned}
&EJ\frac{\partial^2 W(x,t)}{\partial x^2}\Big|^{x=l} + P\Big[\Big(1+\frac{l_C}{l_D}\Big)l_C\frac{\partial W(x,t)}{\partial x}\Big|^{x=l} - \frac{l_C}{l_D}W(l,t)\Big] = 0\\
&EJ\frac{\partial^3 W(x,t)}{\partial x^3}\Big|^{x=l} + P\Big[\Big(1+\frac{l_C}{l_D}\Big)\frac{\partial W(x,t)}{\partial x}\Big|^{x=l} - \frac{1}{l_D}W(l,t)\Big] - m\frac{\partial^2 W(l,t)}{\partial t^2} = 0
\end{aligned} \tag{3.16}$$

4. Column loaded by the force directed towards the positive pole – system B

4.1. Physical model and analysis of the system geometry

The physical model of the system loaded by the force directed towards the positive pole is presented in Fig. 5. The column is loaded by the force P applied to its free end and is jointed with rigid element (7) by means of a structure of mass m. The total mass m consists of the concentrated mass m_1 and reduced mass m_{zr} of elements (4), (5) and (7) in relation to the fixing point of the mass m_1. The location of linear element (4) agrees with the direction of the force P and makes an angle β with the axis x. Element (4) is connected by a hinge joint with beam (2); its second end is moving on circular cylinder (5) of radius R which is rigidly jointed with element (7).

Geometric relations between the elements and the loading structure lead to a relation between the transverse displacement and the angle in the direction of the force which, after considering some small displacement (3.4), is given in the following form

$$\beta = \frac{W(x,t)\big|^{x=l}}{l_D} \tag{4.1}$$

4.2. Total potential energy of the system. Boundary conditions

Potential energy of the system V, for externalisation of internal forces presented in Fig. 5b is the sum of elastic strain energy V_1, potential energy

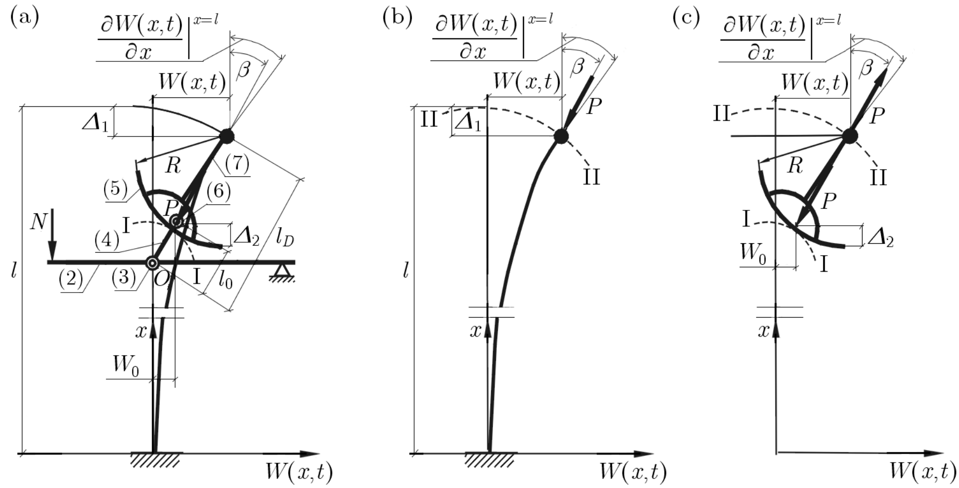


Fig. 5. Deflected form of column **B** (see Fig. 2b), geometry and externalisation of internal forces

of the components of the force P: vertical – V_2 and horizontal – V_3, and is expressed by the following formula

$$V = V_1 + V_2 + V_3 = \frac{1}{2}EJ\int\limits_0^l \Big[\frac{\partial^2 W(x,t)}{\partial x^2}\Big]^2\,dx - P\varDelta_1 + \frac{1}{2}P\beta W(l,t) \qquad (4.2)$$

$\varDelta_1$ is the longitudinal displacement described in formulas (3.7).

Taking into account Hamilton's principle (3.13) and considering relation (4.2), kinetic energy variation (3.8) and known boundary conditions (formulas (3.15)), the equation of motion is obtained (compare Eq. (3.14)) and the remaining boundary conditions necessary to solve the problem

$$\begin{aligned} &\frac{\partial^2 W(x,t)}{\partial x^2}\Big|^{x=l} = 0 \\ &\frac{\partial^3 W(x,t)}{\partial x^3}\Big|^{x=l} + \frac{P}{EJ}\Big[\frac{\partial W(x,t)}{\partial x}\Big|^{x=l} - \frac{1}{l_D}W(l,t)\Big] - \frac{m}{EJ}\frac{\partial^2 W(l,t)}{\partial t^2} = 0 \end{aligned} \qquad (4.3)$$

5. Solution to the boundary value problem

Considering the symmetrical distribution of flexural rigidity and mass per unit of length of the considered columns, then distributing variables of the function

$W_i(x,t)$ versus time t and space x in the following form

$$W_i(x,t) = y_i(x)\cos(\omega t) \qquad i = 1,2 \tag{5.1}$$

the equations of motion of the considered systems are obtained

$$(EJ)_i y_i^{IV}(x) + (S)_i y_i^{II}(x) - (\rho_0 A)_i \omega^2 y_i(x) = 0 \qquad (S)_1 + (S)_2 = P \tag{5.2}$$

where $(S)_i$ is the internal force in the i-th rod of the system.

The boundary conditions in the fixing point of systems **A** and **B** $(x = 0)$ were put in the following way

$$\begin{aligned} y_1(0) &= y_2(0) = 0 \qquad & y_1^I(0) &= y_2^I(0) = 0 \\ y_1(l) &= y_2(l) \qquad & y_1^I(l) &= y_2^I(l) \end{aligned} \tag{5.3}$$

and at the free end (after separation of variables), respectively for

— systems of type **A**

$$\begin{aligned} & y_1^{II}(l) + y_2^{II}(l) + \frac{P}{(EJ)_1} l_C \Big[\Big(1 + \frac{l_C}{l_D}\Big) y_1^I(l) - \frac{1}{l_D} y_1(l)\Big] = 0 \\ & y_1^{III}(l) + y_2^{III}(l) + \frac{P}{(EJ)_1} \Big[\Big(1 + \frac{l_C}{l_D}\Big) y_1^I(l) - \frac{1}{l_D} y_1(l)\Big] + \frac{m\omega^2}{(EJ)_1} y_1(l) = 0 \end{aligned} \tag{5.4}$$

— systems of type **B**

$$\begin{aligned} & y_1^{II}(l) + y_2^{II}(l) = 0 \\ & y_1^{III}(l) + y_2^{III}(l) + \frac{P}{(EJ)_1} \Big[y_1^I(l) - \frac{1}{l_D} y_1(l)\Big] + \frac{m\omega^2}{(EJ)_1} y_1(l) = 0 \end{aligned} \tag{5.5}$$

General solutions to equations $(5.2)_1$ are as follows

$$y_i(x) = D_{1i}\cosh(\alpha_i x) + D_{2i}\sinh(\alpha_i x) + D_{3i}\cos(\beta_i x) + D_{4i}\sin(\beta_i x) \tag{5.6}$$

where D_{ni} are integration constants $(n = 1-4)$, and

$$\alpha_i^2 = -\frac{1}{2}k_i^2 + \sqrt{\frac{1}{4}k_i^4 + \Omega_i^2} \qquad \beta_i^2 = \frac{1}{2}k_i^2 + \sqrt{\frac{1}{4}k_i^4 + \Omega_i^2} \tag{5.7}$$

where

$$\Omega_i^2 = \frac{(\rho_0 A)_i \omega^2}{(EJ)_i} \qquad k_i = \sqrt{\frac{(S)_i}{(EJ)_i}} \tag{5.8}$$

Substitution of solutions (5.6) into boundary conditions (5.3), and respectively for Eqs. (5.4) or Eqs. (5.5) yields a transcendental equation for eigenvalues of the considered system.

6. Constructional scheme of the loading head

Figure 6 presents a constructional scheme of the head used for loading the considered columns.

The structure consists of beam (1) fixed by a hinge joint to element (2) and equipped with rotary node (3), where the rolling element (FI) or rigid element (GI) can be mounted. The load is transferred through stretched bolt (4) mounted in sockets (4(1)) and ((4(2)), and pin (5) where a needle bearing is mounted (FII) or a rolling element (GII) – (6). The compressive force is carried by housing (7) jointed with element (9) which together with column rods (10(1)) and (10(2)) is mounted in block (11) of mass m. It is assumed that elements (1), (4), (4(1)), (4(2)), (7), (8), (9) and (11) are infinitely rigid, which is justified due to constructional reasons.

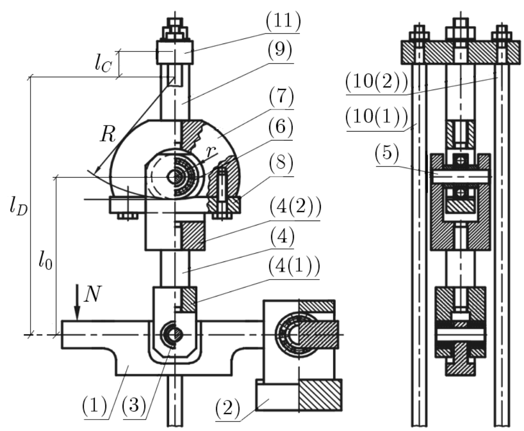


Fig. 6. Constructional scheme of the head loading the columns

Elements (1), (3), (4), (4(1)) and (4(2)) constitute the forcing head, and elements (7), (9), (11) – the receiving head. Intermediate element (8), and rotary node (6) comprise the forcing head.

7. Experimental stand

The experimental research on natural frequencies of the considered systems have been carried out on an experimental stand designed and built at the Technical University of Czestochowa, Institute of Mechanics and Machine Design Fundamentals. Its constructional scheme is shown in Fig. 7 (Tomski *et al.*, 1998).

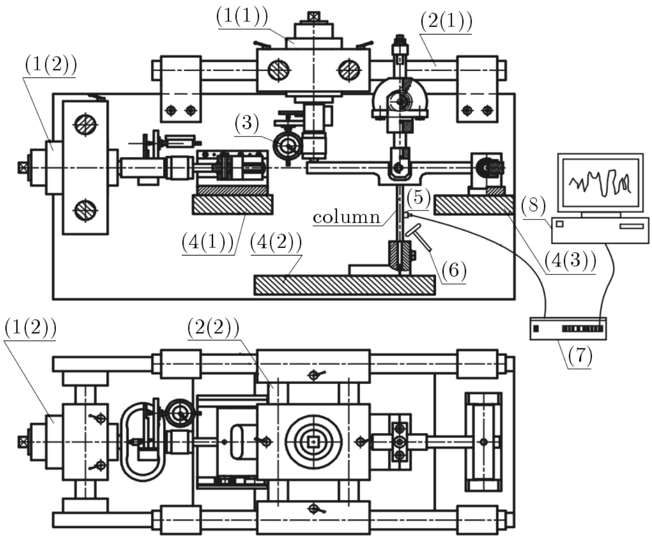


Fig. 7. Test stand for experimental research of the considered columns

The stand presented in Fig. 7 has two loading heads 1(1) and 1(2) and can be used to examine free vibration of the columns and frames located vertically or horizontally. Head 1(1) was used for experimental research of the considered systems, which can move on guides 2(1) and 2(2) both in the longitudinal (horizontal) and transverse direction. Head 1(1) is equipped with bolt systems whose movement load the tested column. The measurement of the loading force was made by dynamometer 3. The supports realising the set boundary conditions of the columns were installed on plates 4(1-3).

Experimental research of the natural vibration frequencies were carried out by the measuring system presented in Fig. 7. The system is comprised of accelerometer (5) (Brüel & Kjær – 4508 B), two-channel vibration analyser (7) by Brüel & Kjaer and computer (8). Vibration of the column fixed on the experimental stand was induced by hammer (6) and then accelerations in individual measuring points were measured by accelerometric sensor (5). The signal from the sensor was transmitted to analyser (7), where it was processed and sent to computer (8).

8. Experimental and numerical results

Based on the solution to the boundary value problem, numerical calculations of the natural vibration frequency against the external load were made. Then they were verified on the experimental stand for natural vibration frequency measurements of the columns and flat frames (Fig. 7). The physical and geometrical parameters of the systems as well as the values characterising the forcing and receiving heads are presented in Table 1.

Table 1. Physical and geometrical parameters of the considered columns and parameters the forcing and receiving heads

Column	$(EJ)_i$ [Nm^2]	$(\rho_0 A)_i$ [kg/m]	l [m]	l_C [m]	l_D [m]	m [kg]	R [m]
A	71.76	0.315	0.61	0.084	0.244	1.4	0.085
B	71.76	0.315	0.61	–	0.325	1.4	0.085

In order to verify the mathematical model, experimental research was carried out determining the characteristic values of the considered systems when the rolling bearings (F) and/or slide bearings (G) were used in rotary nodes (3), (6) of the loading heads (see Fig. 3).

The measurements were made for the columns loaded in the following way:

a) generalised load by the force directed towards positive pole **A** – Fig. 8

b) load by the force directed towards the positive pole **B** – Fig. 9.

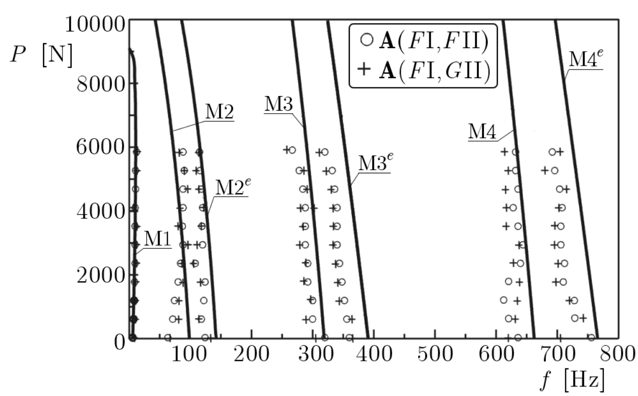


Fig. 8. Curves in the plane: load – natural frequency for columns **A**

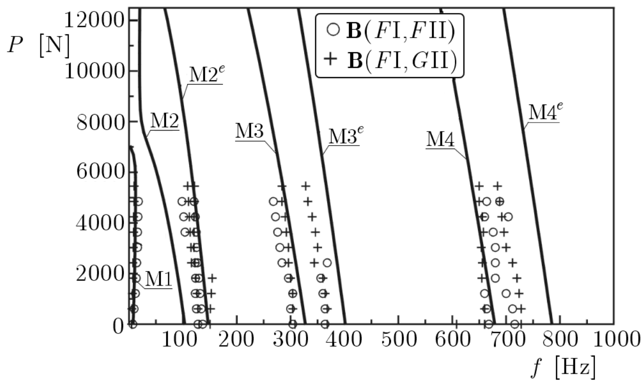


Fig. 9. Curves in the plane: load – natural frequency for columns **B**

The results of numerical calculations were marked with lines. The obtained results of experimental results were shown with points, while rotary nodes I and/or II of loading head were equipped with the rolling bearings (F) and slide bearings (G). The results are limited to the first four basic natural frequencies (M1-M4) and three additional frequencies (M2^e-M4^e) characterised by symmetry of vibrations.

An arithmetic average of the three conducted measurements is presented in Figs. 8 and 9 by dots

$$f^e = \frac{1}{3}\sum_{i=1}^{3} f_i^e \tag{8.1}$$

where f_i^e denotes the results of consecutive measurements.

Analysis of the error of the proposed mathematical model is carried out. The arithmetic average of experimental measurement (Eq. (8.1)) was treated as an actual value of the measured quantity that is the natural frequency. The relative percentage error of the theoretical model is given by the expression

$$\Delta f^e = \left|\frac{f^t - f^e}{f^t}\right| \cdot 100\% \tag{8.2}$$

where f^t is the value of the measured quantity, obtained theoretically.

There is very good consistence of the results of experimental investigations with numerical calculations in regards to the basic natural vibration frequency. In the case of remaining natural vibration frequencies, between the results obtained in the experimental research and numerical computations the value of the relative percentage error Δf does not exceed 10% for over 250 points. Only for the second natural vibration frequency for $P = 0$ (Fig. 8) it differs from the rest results. The lowest value of Δf of all accomplished measurements equals 0.24%, while the highest 31%.

Differences between the experimental and numerical results may be caused by the following factors:

- in the mathematical model – not taking into account the resistance to motion – deformation of the contact surface,
- certain inaccuracy in the determination of Young's modulus and the material density,
- possibility of inaccurate mounting of the accelerometer in the plane of vibrations,
- error in accurate determination of the loading force of the system when using the dynamometer,
- assumption of the infinite rigidity of mounting of the system in the mathematical model,
- influence of the test stand on the tested column.

It was found that the loading of the column by the head within which rigid elements transfer the load (slide bearings) (GI and/or GII) does not influence the course of characteristic values of the system against the external load.

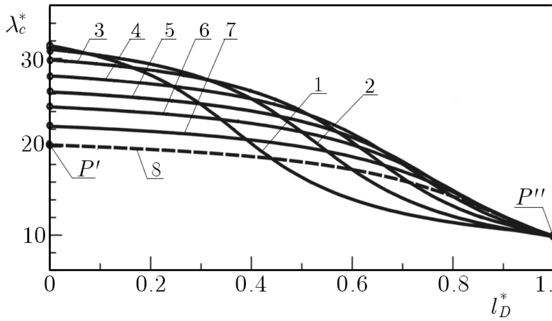
The results of numerical simulations related to the determination of the critical load P_c and the course of natural vibration frequency against the external force loading the column were presented. The critical force, length l_C and distance l_D as well as concentrated mass m located at the free end of the column were expressed in non-dimensional coordinates

$$\lambda_c^* = \frac{P_c l^2}{EJ} \qquad l_C^* = \frac{l_C}{l} \qquad l_D^* = \frac{l_D}{l} \qquad m^* = \frac{m}{\rho_0 A l} \tag{8.3}$$

The change of the critical parameter of load λ_c^* against the parameter l_D^* for some selected values of non-dimensional length l_C^* was presented in Fig. 10.

The curves and points presented and listed in Fig. 10 correspond to the critical parameter of load or the course of its change for columns loaded respectively by:

- generalised load by the force directed towards the positive pole (column **A**) – curves 1-7,
- force directed towards the positive pole (column **B**, $l_C^* = 0$) – curve 8,
- follower force directed towards the positive pole ($l_D^* = 0$) – points (°) marked on the axis of ordinates (e.g. Tomski *et al.*, 2004; Tomski and Szmidla, 2004a),



	l_C^*
1	0.5
2	0.4
3	0.3
4	0.2
5	0.15
6	0.1
7	0.05
8	0

Fig. 10. Change of the critical parameter of load λ_c^* against the parameter l_D^* for various values of l_C^*

- Euler's load for the system hinged at the free end ($l_C^* = 0$ and $l_D^* = 0$) – point P',
- Euler's load for the system hinged at both ends ($l_D^* = 1$) – point P''.

For the columns loaded with the considered types of load the courses of natural vibration frequency against the external load were determined. Only the character of change of the first two basic frequencies in non-dimensional form were determined Ω_t^* ($t = 1, 2$) as well as an additional symmetrical natural frequency Ω_2^{*s} against the non-dimensional parameter of load λ^*, where

$$\lambda^* = \frac{Pl^2}{EJ} \qquad\qquad \Omega^* = \Omega^2 l^4 = \frac{\rho_0 A \omega^2 l^4}{EJ} \tag{8.4}$$

The calculations were made for selected values of non-dimensionally expressed geometric parameters characterising the forcing and receiving heads and for a constant value of the concentrated mass m located at the free end of the system (Eqs. (8.3)). The influence of the change of parameter l_D^* on the course of natural vibration frequency against the external load parameter for selected values of l_C^* is shown in Fig. 11 and Fig. 12. The numerical simulations were carried out for the system **A** – Fig. 11.

The course of curves on the plane P-ω presented in Fig. 12 corresponds to the system **B**. The range of changes of eigenvalues is limited with the curves obtained for the systems loaded by the force directed towards the positive pole (curve 1) and supported by hinge joints at both ends and exposed to the axial compressive force (curve 7).

The influence of change of parameter l_C^* on the course of natural vibration frequency against the external load parameter λ^* for constant values l_D^* and m^* is shown in Fig. 13.

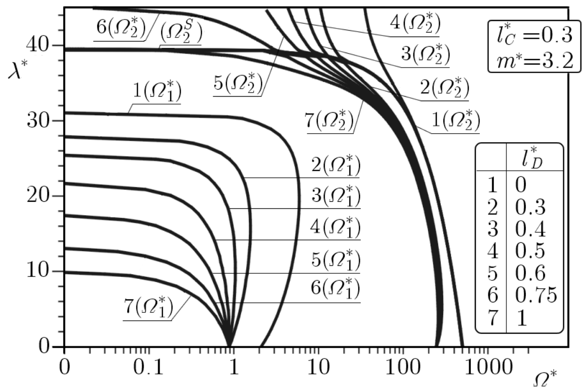


Fig. 11. Curves on the plane: loading parameter – parameter of natural vibration frequency of column **A**

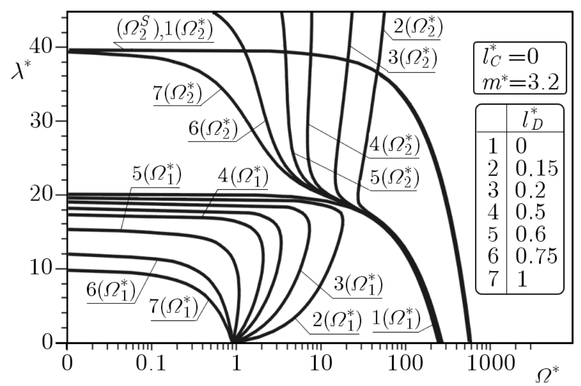


Fig. 12. Curves on the plane: loading parameter – parameter of natural vibration frequency of column **B**

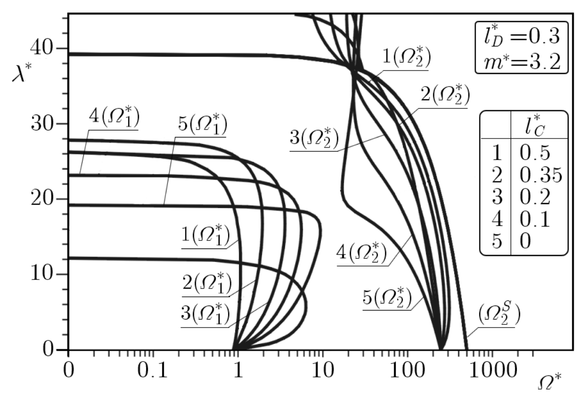


Fig. 13. Curves on the plane: loading parameter – parameter of natural vibration frequency of columns for various parameters l_C^*

With the considered value of the parameter l_D^* for the system **B** $(l_C^* = 0)$, the course of natural vibration frequency against the external load allows one to classify these systems among the divergence pseudoflutter ones. The system **A** may be of the divergence pseudoflutter or divergence type depending on l_C^*.

9. Final remarks

- Depending on the value of geometric parameters of the forcing heads, characteristic cases were obtained for columns subjected to generalised load by the force directed towards the positive pole and columns loaded with the force directed to positive pole.
- Based on natural vibration frequency measurements and numerical simulations, it was demonstrated that the type of elements used in the rotary nodes of the forcing head does not influence the experimental values of natural vibration frequency.
- For the columns subjected to the generalised load by the force directed towards the positive pole, an increase in the critical force was obtained compared to the system fixed with an articulated joint for $x = l$, where the force maintains the constant direction (Euler's load).
- The obtained results of numerical computations and experimental investigations regarding the course of natural frequencies in relation to the external load showed good agreement.
- Appropriate selections of geometric parameters of the forcing heads allow obtaining an increase or decrease in the natural vibration frequency of the system with an increase in the column loading force (divergence pseudoflutter system).

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Eksperymentalna weryfikacja swobodnych drgań poprzecznych kolumn obciążonych poprzez głowice o węzłach obrotowych

Streszczenie

Głównym celem pracy jest określenie wpływu różnych elementów zamocowanych w węzłach obrotowych głowicy obciążającej, tj. łożysk tocznych igiełkowych lub elementów sztywnych na częstość drgań własnych oraz wartość obciążenia krytycznego kolumn. W pracy rozważa się zagadnienia stateczności i przebiegu wartości własnych w funkcji zewnętrznej siły obciążającej układ, poddany obciążeniu swoistemu. Prezentuje się nowe rozwiązanie konstrukcyjne głowicy obciążającej, służącej do realizacji obciążenia siłą uogólnioną skierowaną do bieguna dodatniego oraz siłą skierowaną do bieguna dodatniego. Przedstawia się rozważania teoretyczne dotyczące sformułowania warunków brzegowych metodą wariacji energii mechanicznej poszczególnych układów. Weryfikacja założonego celu jest realizowana na drodze badań eksperymentalnych.

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