JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
48, 2, pp. 415-433, Warsaw 2010

LINEAR-GRAPH AND CONTOUR-GRAPH-BASED MODELS OF PLANETARY GEARS

Józef Drewniak
Stanisław Zawiślak

University of Bielsko-Biala, Department of Mechanical Engineering Fundamentals, Bielsko-Biała, Poland; e-mail: jdrewniak@ath.bielsko.pl; szawislak@ath.bielsko.pl

Analysis and synthesis of mechanisms are basic engineering tasks. They can suffer from errors due to versatile reasons. The graph-based methods of analysis and synthesis of planetary gears can be alternative methods for accomplishing of the afore mentioned tasks, which additionally allow for checking of their correctness. In the paper, two graph-based methods of analysis of planetary gears are discussed. An exemplary planetary gear is analysed by means of the graph-based methods as well as the traditional Willis method. Force and torque analyses were performed as well. An algorithmic approach – which implies from the graph models – allows for checking of versatile variants of designs in an easy and schematic way, which can lead to optimisation of the design within a conceptual phase of the design procedure.

Key words: contour equation method, Hsu's graph, f-cycle equation, gear ratio variants

1. Introduction

Analysis and synthesis of mechanisms can be performed by means of versatile methods. These tasks can suffer from human errors. So, it is reasonable to have some alternative methods which allow for comparison of results and for detection of almost unavoidable mistakes. The graph-based methods deliver such alternative approaches for modelling of a wide class of mechanical systems. Two of them are considered in the present paper.

There were also some other attempts to model planetary gears via diagrams e.g. Wolf's pictograms (Müller and Wilk, 1996) or PKP-schemata and numerical codes (Ivančenko *et al.*, 1974), but these methods did not further evolve due to lack of generalization and lack of connections with other branches of mathematics. Also the method based on signal flow graph theory for modelling of gears (Bonnell and Hess, 1968; Wojnarowski and Lidwin, 1975; Uematsu, 1997) has not been too frequently used. On the contrary, the graph based methods have been independently, intensively developed for several recent years all over the world, see e.g. (Uyguroğlu and Demirel, 2005; Wojnarowski *et al.*, 2006). Besides the papers, there are some monographs where the graph methods were described in details and illustrated by means of representative examples. The linear graph-based methodology of modelling of mechanisms was extensively described in Tsai (2001). The word "linear" will be usually omitted in the rest of the present paper for simplicity. The contour based approach was discussed in Marghitu and Crocker (2001) as well as Marghitu (2005). The so-called contour graph is a symmetrical digraph without loops. The paper Wojnarowski *et al.* (2006) includes a review of the graph-based methods for gears and relevant references (58 items) almost totally different than these cited here, including topics connected with bond-graphs. Especially, the last mentioned work encloses a short list of Prof. Wojnarowski's achievements dedicated to this topic. He and Prof. Arczewski pioneered the graph-based modelling of mechanical systems in the 70s of the previous century in Poland. The present paper focuses on the graph-based analysis of planetary gears (Zawiślak, 2007, 2008) but the graph related modelling belongs to a wider family of the algebraic structure based approaches (Shai, 2001) which allow especially for classification of mechanisms (Davies, 1968), conceptual design (Zawiślak, 2006), enumeration of structures (Tuttle *et al.*, 1989) and synthesis of mechanical systems (Schmidt *et al.*, 2000) as well as determination of angular velocities of elements of multi-body systems (Arczewski and Dul, 1995). Powerfulness of the graph-based modelling of mechanical systems consists in the fact that a graph is inseparably connected with other algebraic structures like e.g. subgraphs (e.g. trees, cycles, cliques, paths), dual graph, matrices, vector spaces of cuts and cycles, structural numbers and matroids as well as algorithms connected with these structures. By interpreting the mechanical knowledge in terms of graphs, we solve mechanical problems via graph models (Shai and Preiss, 1999). The graphs applied by Shai had other rules of assignment than these discussed in the present paper.

Versatile linear graph models were considered by Tsai (2001) but Hsu's graph models of gears (Hsu, 2002) are utilized in the following considerations. In the case of analysis of car gearboxes, the graph-based methodology is also efficient using additionally transformations of a basic graph according to a particular gear drive (Zawiślak *et al.*, 2008; Zawiślak, 2008). An introductory comparison of the graph and contour methods for analysis of gears was done by the authors in Drewniak and Zawiślak (2010).

The application of the considered methods for analysis of an exemplary planetary gear is given underneath. This is a coupled gear (in German: Koppelgetriebe). The considered gear has an internal closed loop what causes that there is also a circle (loop) made of the stripped edges in the graph representation of the gear. Such structures were claimed (Tsai, 2001) as impossible to analyse via the graph method but the presented considerations, which lead to compatible results for all three discussed methods, denied this statement. The considered gear is suitable for designing a gearbox as an introductory layout where several inputs and outputs are available. If clutches and brakes are added, then some elements can be fixed and a respectable angular velocity is equal to zero.

Force and torque analyses were here performed in a traditional way. But this stage of analysis can also be done via the graph-based approach, which is especially useful for mechanisms (Marghitu, 2005). Dynamics of gears can be studied via the graph approach upon Andrews's methodology (Andrews *et al.*, 1997).

The general idea of the graph-based modelling of mechanical systems consists in the following steps:

- discretisation of a mechanical system. This means that appropriate simplifications have to be made. Some aspects are omitted. Some structural elements are considered as essential and they are interpreted as graph vertices. Some connections or relationships between these elements are abstracted. They are represented via graph edges, usually some system of weights can be assumed to the edges or vertices and edges,
- assignment of the graph to the mechanism (especially planetary gear) based upon special rules. There are several different rules depending on the object of modelling and problems solved via the graph based method,
- derivation of special subgraphs, e.g. f-cycles or contours. These subgraphs can be singled out based upon the graph-theoretical algorithms what causes that the approach is simple and algorithmic,
- listing the codes of these graph elements. The encoding rules are clearly defined what allows for avoidance of mistakes,
- generation of a system of equations in an algorithmic way using the codes. These codes allow for management (assignment) of the indicators of variables existed in the considered equations in a straightforward manner,
- solution of the obtained system in a chosen algebraic way to obtain needed angular velocities, ratios, forces, accelerations etc. Different sets of

unknowns are considered but if these unknowns are not essential for the solutions then they are excluded from the considerations via appropriate algebraic transformations.

The similar routine can be formulated for the reverse order of activities, i.e. synthesis: going from the graph generation towards creation of a gear functional structure (Tsai, 2001).

The considered planetary gear is presented in Fig. 1.

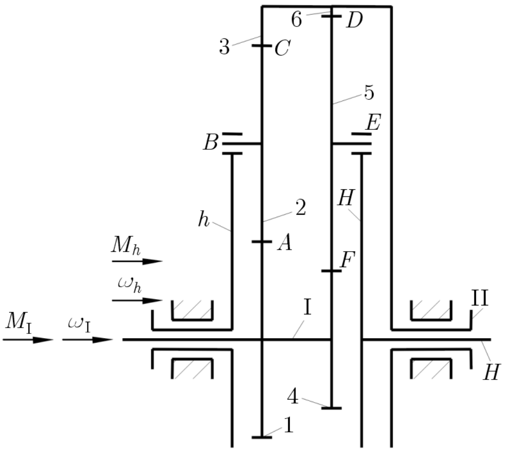

Fig. 1. Functional scheme of an exemplary planetary gear: $1, 2, \ldots, 6$ sun wheels, wheels with internal toothings and planets; h, H – arms; $A, B, \ldots, F$ – characteristic points; ω, M – angular velocities and torques

A special planetary gear is considered. It encloses an internal closed loop formed by wheels and planets. The mobility of the structure is equal to 2. It means that two elements have to be driven. Some considerations concerning possible inputs and outputs are presented in Section 4 (Tables 1 and 2). The mobility W (DOF, degree of freedom) for the considered mechanism, i.e. planetary gear, can be calculated upon the following formula (Grübler-Kutzbach's equation)

$$W = 3n - 2c_5 - c_4 = 18 - 12 - 4 = 2 \qquad (1.1)$$

where: n – number of links (movable elements), c_5 – number of full joints (one degree of freedom), e.g. rotational type; c_4 – number of half joints, e.g. meshing type; moreover c_5 and c_4 are equal to the total number of edges and diagonals of a polygon ($c_5 = 6$) and the number of stripped edges ($c_4 = 4$), respectively. We have also $n = 6$, i.e. number of graph vertices – in the case of linear graph representation of a planetary gear (see Fig. 2).

Nevertheless, it is possible to analyse this case. However, there are some restrictions on the number of teeth and dimensions to assure the possibility of assemblage of the gear and proper gearing in two parallel toothings. In Tsai (2001), there are several tables where the mechanical properties of mechanisms are expressed via characteristics of their assigned graphs.

For the considered gear, the following data for teeth numbers and the module are assumed: $z_1 = 15$; $z_2 = 24$; $z_3 = 63\,(-63)$; $z_4 = 18$; $z_5 = 21$ and $z_6 = 60\,(-60)$, $m = 2\,\mathrm{mm}$, where negative values of the teeth numbers are considered for the internal gearing in the case of the Willis method, and one common module for all meshings has been assumed.

The original approach discussed in the present paper consists in:

- usage of Tsai's derivation of the f-cycle equations method for Hsu's type graphs, which originally were used mainly for synthesis purposes,
- analysis of the closed-loop gear via the graph-based approach, which has also other novelty aspects. Generally, the contour method has been seldom utilized in mechanics till now, and for open layouts only. Besides the mentioned books, there are few published papers dealing with this approach for gears. Moreover, the examples of gears given in Prof. Marghitu's books are relatively simple. However, really complicated structures were analysed for versatile mechanisms without geared subsystems. In the present paper, a contour graph has been generalized for the analysed compound gear (closed-power-loop), i.e. double described vertices were introduced and additional arcs were considered to assure orientation of every contour, which means that the orientation of contours can be chosen arbitrarily,
- comparison of the detailed rules of assignment 'mechanical system-graph' for two considered graph approaches is given in the present paper – which were only roughly listed together in Drewniak and Zawiślak (2010),
- analysis of forces, torques and, especially, efficiency of the gear is made, which makes the presented considerations complex and comprehensive. Usage of a free body diagram makes the analysis very effective.

2. Graph based model of a planetary gear

Graph-based models of mechanisms, especially gears, have been developed for many years. Despite the fact that several different graph models of mechanisms were discussed in Tsai (2001) – the Hsu graphs have been chosen, and they are utilized as models of planetary gears in the present paper. It seems that this model is the most adequate.

The rough idea of modelling is as follows: only some general properties or aspects of a mechanism are taken into account, e.g. analysis of kinematics. Therefore, the main rotating elements of a mechanical system are represented by graph vertices and the relations among them are modelled via edges. The applied rules of assignment start from the idea: the singled out main gear elements like e.g. gear wheels, planets and arms are considered as vertices. Especially, all elements rotating around the main physical axis of a planetary gear are represented by the vertices of a polygon, moreover the mutual relations among them: 'rotation around the same axis' are coded via the polygon edges and its diagonals. The latter are not drawn for simplicity of the picture. However, they are used for determination of some f-cycles.

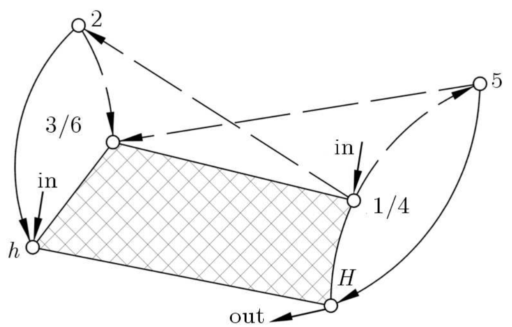

Fig. 2. Modified Hsu's linear graph of the planetary gear (see: variant – Table 2, row 2)

The relation for gear wheels: 'to be in gear' is represented by a stripped line. The relation 'to be an arm for a geared wheel (planet wheel)' is represented by a continuous line. An exemplary graph of the planetary gear (Fig. 1) is presented in Fig. 2. Some vertices are described in an unusual manner, e.g. 3/6 and 1/4. It is to be underlined that wheels 3 and 6 as well as 1 and 4 are connected by a stiff axis and they create a united element with two toothings. The set of f-cycles is as follows: $(1,2)h$; $(2,3)h$; $(4,5)H$ and $(5,6)H$ (Tsai, 2001), where only adequate descriptions of geared wheels are used, i.e. 1 instead of 1/4. Precisely, these notions are the codes of the f-cycles. Namely, first cycle $(1,2)h$ consists of three edges $\{(1,2),(2,h),(1,h)\}$. As it can be seen: edge $(1,h)$ is not drawn explicitly and it is hidden in the shaded polygon. Edge $(2,h)$ represents a pair 'planet and arm'. Edge $(1,2)$ represents gearing of two wheels. The rule for a code of the f-cycle is that we write a description of the stripped edge in brackets and the description of the arm outside the brackets. Moreover, the names of vertices are arranged in increasing order starting from numbers. The adequate detailed cycles can be written for the remaining three codes. Every f-cycle generates one equation.

The following system of equations can be derived based upon the mentioned f-cycles according to the formulated rules

$$\begin{aligned} \omega_1 - \omega_h &= -N_{21}(\omega_2 - \omega_h) \qquad & \omega_2 - \omega_h &= +N_{32}(\omega_3 - \omega_h) \\ \omega_4 - \omega_H &= -N_{54}(\omega_5 - \omega_H) \qquad & \omega_5 - \omega_H &= +N_{65}(\omega_6 - \omega_H) \end{aligned} \tag{2.1}$$

where: ω_i, $i = 1, 2, \ldots, 6$; ω_h and ω_H are angular velocities of respective gear elements; N_{ij} – ratios, $N_{ij} = z_i/z_j$, the signs '–' and '+' depend on the external and internal gearing, respectively, z_i, z_j – teeth numbers of the wheel and pinion, respectively.

Due to the layout of the considered planetary gear, the following equalities can be considered

$$\omega_3 = \omega_6 \qquad \omega_1 = \omega_4 \tag{2.2}$$

Solving the system of equations, the following solutions are derived

$$\omega_3 = \frac{(N_{54}N_{65} + 1)\omega_H - \omega_1}{N_{54}N_{65}} \qquad \omega_h = \frac{\omega_1 + N_{21}N_{32}\omega_3}{1 + N_{21}N_{32}} \tag{2.3}$$

Taking into account the input angular velocities

$$\omega_1 = 157\,\mathrm{s}^{-1} \qquad \omega_H = 87.5\,\mathrm{s}^{-1} \tag{2.4}$$

the numerical values of the output angular velocities are obtained

$$\begin{aligned} \omega_3 = \omega_6 &= \frac{87.5\Big(1 + \frac{21}{18}\frac{60}{21}\Big) - 157}{\frac{60}{18}} = 66.65\,\mathrm{s}^{-1} \\ \omega_h &= \frac{157 + 66.65\frac{24}{15}\frac{63}{24}}{1 + \frac{63}{15}} = 84.025\,\mathrm{s}^{-1} \end{aligned} \tag{2.5}$$

The achieved results will be compared with other methods of gear analysis.

3. Contour based model of the planetary gear

The contour method of modelling of a mechanical system consists in creation of a special graph enclosing the contours, i.e. closed circles built of arrows connecting the vertices representing the elements of the system. The contour graph of the gear presented in Fig. 1 is shown in Fig. 3. The rule is that the contour starts in a supporting system (body of a gear) and passes via elements on which angular (or linear) movement is passed by. We end the contour after returning to the support. All closed independent loops generated in this way have to be drawn. Then the list of codes of the derived contours is created. Next, in turn, the contour equations can be written in an algorithmic manner manipulating the indices in the following way: all relative velocities for every consecutive pair of element codes have to be inserted into the derived equations. There are relative angular velocities of the mechanical system elements inside the obtained system of equations. It is an unavoidable stage of the method. However, the unwanted quantities can be eliminated and the system can be solved in a step-by-step manner. A detailed explanation of the methodology can be studied upon (Marghitu and Crocker, 2001). Underneath, only the needed details tailored to planetary gear modelling will be given.

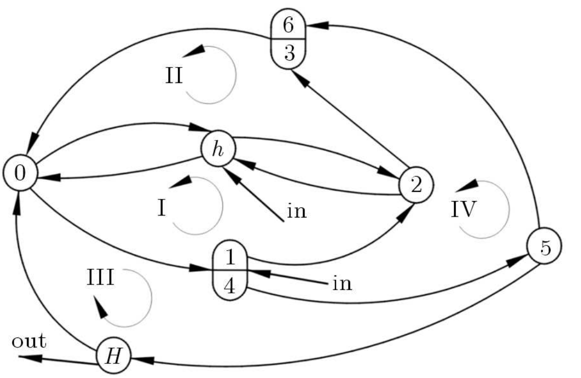

Fig. 3. Contour graph of the planetary gear

The following contours can be distinguished for the contour graph (Fig. 3) of the considered planetary gear:

(I) $0 \to 1 \to 2 \to h \to 0$
(II) $0 \to h \to 2 \to 3 \to 0$
(III) $0 \to 4 \to 5 \to H \to 0$
(IV) $0 \to 4 \to 5 \to 6 \to 0.$

Contour (I) generates the following pairs of indicators $\{(1,0); (2,1); (h,2); (0,h)\}$. They are used in the first two equations (1) of system (3.1). The contour can be interpreted as follows: we start it from 0, i.e. the support for the axis of wheel 1, which is geared with planet wheel 2, planet 2 rotates around its own axis mounted to h and rotates together with the arm h around the axis depicted by 0. This is the end of the contour: $0 \to 1 \to 2 \to h \to 0$. Analogous explanations can be given for the three remaining contours.

The contour graph is presented in Fig. 3 where the special drawing rules by Marghitu and Crocker (2001) were applied. It encloses four contours I-IV. The orientations of the contours are shown via crooked arrows inside particular contours I-IV.

Based upon the distinguished contours, the following system of equations can be written

$$\begin{array}{lll}
(1) & \boldsymbol{\omega}_{10}+\boldsymbol{\omega}_{21}+\boldsymbol{\omega}_{h2}+\boldsymbol{\omega}_{0h}=0 & \boldsymbol{r}_A\times\boldsymbol{\omega}_{21}+\boldsymbol{r}_B\times\boldsymbol{\omega}_{h2}=0 \\
(2) & \boldsymbol{\omega}_{h0}+\boldsymbol{\omega}_{2h}+\boldsymbol{\omega}_{32}+\boldsymbol{\omega}_{03}=0 & \boldsymbol{r}_B\times\boldsymbol{\omega}_{2h}+\boldsymbol{r}_C\times\boldsymbol{\omega}_{32}=0 \\
(3) & \boldsymbol{\omega}_{40}+\boldsymbol{\omega}_{54}+\boldsymbol{\omega}_{H5}+\boldsymbol{\omega}_{0H}=0 & \boldsymbol{r}_F\times\boldsymbol{\omega}_{54}+\boldsymbol{r}_E\times\boldsymbol{\omega}_{H5}=0 \\
(4) & \boldsymbol{\omega}_{40}+\boldsymbol{\omega}_{54}+\boldsymbol{\omega}_{65}+\boldsymbol{\omega}_{06}=0 & \boldsymbol{r}_F\times\boldsymbol{\omega}_{54}+\boldsymbol{r}_D\times\boldsymbol{\omega}_{65}=0
\end{array} \tag{3.1}$$

where $\boldsymbol{r}_k$ are the position vectors of the points $k = A, B, \dots, F$ shown in Fig. 1; $\boldsymbol{\omega}_{ij}$ – means vectors of relative angular velocity of the i-th element in relation to the element j.

Every contour generates two equations connected with angular velocities and some other for forces and torques. The latter are not used in the present paper. The first equation for every pair is a sum of angular velocities designated by a contour, the second one is a sum of cross products of the respective arm multiplied by the angular velocity. In the second case, some summands are omitted when the arm length is equal to zero. The following relations are used for simplification of the system:

— connected with the rule for exchanging of an order of the indicators

$$\begin{array}{ll}
\boldsymbol{\omega}_{i0}=-\boldsymbol{\omega}_{0i} & i=1,2,\dots,6 \\
\boldsymbol{\omega}_{0h}=-\boldsymbol{\omega}_{h0} & \boldsymbol{\omega}_{0H}=-\boldsymbol{\omega}_{H0}
\end{array} \tag{3.2}$$

— connected with the rule of transformation of the relative velocities into general ones, i.e. rotating around the main axis denoted by 0

$$\begin{array}{ll}
\omega_{i0}=\omega_i & i=1,2,\dots,6 \\
\omega_{h0}=\omega_h & \omega_{H0}=\omega_H
\end{array} \tag{3.3}$$

— connected with geometrical relations of the considered planetary gear

$$\begin{array}{ll}
r_A=r_1 & r_F=r_4 \\
r_B=r_1+r_2 & r_E=r_4+r_5 \\
r_C=r_1+2r_2 & r_D=r_4+2r_5
\end{array} \tag{3.4}$$

For example: r_A is equal to the pitch radius of geared wheel 1, so it is equal to r_1. An analogous explanation can be given for remaining equalities (3.4) for arms r_B to r_F. The transformed system can be written in the following form

$$\begin{aligned}
&-r_A\omega_1 - r_A\omega_{21} - r_A\omega_{h2} - r_A\omega_{0h} = 0 && r_A\omega_{21} + r_B\omega_{h2} = 0\\
&-r_C\omega_{h0} - r_C\omega_{2h} - r_C\omega_{32} - r_C\omega_{03} = 0 && r_B\omega_{2h} + r_C\omega_{32} = 0\\
&-r_E\omega_{40} - r_E\omega_{54} - r_E\omega_{H5} - r_E\omega_{0H} = 0 && r_F\omega_{54} + r_E\omega_{H5} = 0\\
&-r_D\omega_{40} - r_D\omega_{54} - r_D\omega_{56} - r_D\omega_{06} = 0 && r_F\omega_{54} + r_D\omega_{56} = 0
\end{aligned} \tag{3.5}$$

To solve system (3.1), the following actions have been undertaken: we omit the arrows, i.e. we turn the vector equations into scalar ones. It can be done because the velocities act as vectors along the same direction. Moreover, the senses (i.e. orientations along the given direction) will be established via the solution of the system. The vector multiplication (cross product) can be simplified to the scalar one because the angles between the arms and velocities in the case of gears with cylindrical wheels are always equal to $90°$. For other mechanisms enclosing cranks, pistons, cylinders, sliders, followers, etc., a detailed analysis of angles has to be done (Marghitu and Crocker, 2001) to simplify the system of equations correctly. Additionally, the first equation in (3.5) was multiplied by $-r_A$, the third by $-r_C$, etc., the resultant equations are gathered in system (3.5). Then the unwanted relative velocities can be eliminated.

Assuming the cylindrical gear wheels, we have the relationships

$$r_i = \frac{d_i}{2} = \frac{z_i m_i}{2} \tag{3.6}$$

where: m_i is the module of the i-th wheel, z_i – teeth number of the i-th wheel, $i = 1, 2, \ldots, 6$.

Due to the fact that it is a universal design having multi-inputs and multi-outputs, all modules were assumed as equal. It causes that the module is not present in the formulas for the output angular velocities. In the case of different modules, these formulas would be a little more complicated but the system is also solvable. The solution to system of equations (3.5) is as follows

$$\begin{aligned}
\omega_6 = \omega_3 &= \frac{\omega_H(2r_4 + 2r_5) - r_4\omega_1}{2r_5 + r_4} = 66.65\,\mathrm{s}^{-1}\\
\omega_h &= \frac{r_1\omega_1 + (r_1 + 2r_2)\omega_3}{2(r_1 + r_2)} = 84.025\,\mathrm{s}^{-1}
\end{aligned} \tag{3.7}$$

We obtained the same numerical results (3.7) as in the case of the linear graph-based modelling – see formulas (2.5). It confirms the correctness of the performed modelling and analysis.

4. Analysis of an exemplary planetary gear by means of the Willis method and data comparisons

The correctness of kinematical analysis of the considered gear will be additionally checked via the classical Willis method. We also analyse several variants of constructional data. The second set of the test data is as follows: $z_1 = 18$; $z_2 = 21$; $z_3 = 60\,(-60)$; $z_4 = 15$; $z_5 = 24$ and $z_6 = 63\,(-63)$; $m_i = m = 2\,[\mathrm{mm}]$, where the negative values of teeth numbers are considered for internal gearing in the case of the Willis method. The given quantities are: $\omega_1 = 157\,\mathrm{s}^{-1}$; $\omega_h = 30\,\mathrm{s}^{-1}$; the unknowns are ω_h and $\omega_3 = \omega_6$. The same systems of equations derived based upon the two considered graph methods solved for the new unknowns give similar general solutions.

The ratio of an arbitrary planetary gear, including differential ones and those having mobility $W \geqslant 2$, can be calculated by means of the Willis formula. In the case of the considered gear, we have to consider the partial ratios i_{13}^h and i_{46}^H, where the upper indices h or H determine the virtually fixed link of the gear, after applying to all the gear links additional velocities $-\omega_h$ or $-\omega_H$, respectively. Then the relative velocity of the arm h or H is equal to zero. Therefore, the ratio i_{13}^h of the geared wheels, considering the order from 1 to 3, in the case of "theoretically fixed" arm h and $z_3 < 0$ can be expressed as

$$\frac{\omega_1 - \omega_h}{\omega_3 - \omega_h} = \frac{z_3}{z_1} \tag{4.1}$$

Similarly, the ratio i_{46}^H of the geared wheels, considering the order from 4 to 6, in the case of "theoretically fixed" arm H and $z_6 < 0$ can be written as

$$\frac{\omega_4 - \omega_H}{\omega_6 - \omega_H} = \frac{z_6}{z_4} \tag{4.2}$$

Taking into account the additional kinematic conditions: $\omega_{\mathrm{I}} = \omega_1 = \omega_4$ and $\omega_3 = \omega_6$, we can calculate the unknown angular velocities and the needed kinematic ratio $i_{\mathrm{I}H}$

$$\begin{aligned} &\omega_6 = \omega_3 = \omega_h\Big[\frac{z_1(\omega_{\mathrm{I}} - \omega_h)}{\omega_h z_3} + 1\Big] = -8.10\,\mathrm{s}^{-1} \\ &\omega_H = \frac{z_4\omega_{\mathrm{I}} - z_6\omega_3}{z_4 - z_6} = 23.65\,\mathrm{s}^{-1} \qquad i_{\mathrm{I}H} = \Big(\frac{\omega_{\mathrm{I}}}{\omega_H}\Big)_{\omega_h = 30\,\mathrm{s}^{-1}} \cong 6.638 \end{aligned} \tag{4.3}$$

The same values of these quantities were obtained for both graph-based methods. It confirms equivalence of them and the possibility of mutual checking of their correctness.

In Fig. 4, free body diagrams for forces and torques acting on the geared wheels and arms are presented. Some considerations on these quantities are given in the next Section.

Analysing different inputs and outputs, different ratios can be achieved. The results for the first set of data are presented in Table 1. Further analysis can be done for other values of the input velocities or numbers of teeth. The numerical values of respective velocities and ratios in the second set of data are given in Table 2.

Table 1. Gear ratios for the first set of data; outputs: ω_h and $\omega_3 = \omega_6$

No.	Input angular velocity $[\mathrm{s}^{-1}]$	Output angular velocity $[\mathrm{s}^{-1}]$	Ratio
1	$\omega_1 = 157$ ω_H – known ($\omega_H = 87.5$)	$\omega_6 = 66.65$	$(i_{16})_{\omega_H} = 2.36$
2	$\omega_1 = 157$ ω_H – known ($\omega_H = 87.5$)	$\omega_h = 84.03$	$(i_{1h})_{\omega_H} = 1.87$
3	$\omega_H = 87.5$ ω_1 – known ($\omega_1 = 157$)	$\omega_6 = 66.65$	$(i_{H6})_{\omega_1} = 1.31$
4	$\omega_H = 87.5$ ω_1 – known ($\omega_1 = 157$)	$\omega_h = 84.03$	$(i_{Hh})_{\omega_1} = 1.04$

Table 2. Gear ratios for the second set of data; outputs: ω_H and $\omega_3 = \omega_6$

No.	Input angular velocity $[\mathrm{s}^{-1}]$	Output angular velocity $[\mathrm{s}^{-1}]$	Ratio
1	$\omega_1 = 157$ ω_h – known ($\omega_h = 30$)	$\omega_6 = -8.10$	$(i_{16})_{\omega_h} = -19.38$
2	$\omega_1 = 157$ ω_h – known ($\omega_h = 30$)	$\omega_H = 23.65$	$(i_{1H})_{\omega_h} = 6.638$
3	$\omega_h = 30$ ω_1 – known ($\omega_1 = 157$)	$\omega_6 = -8.10$	$(i_{h6})_{\omega_1} = -3.70$
4	$\omega_h = 30$ ω_1 – known ($\omega_1 = 157$)	$\omega_H = 23.65$	$(i_{hH})_{\omega_1} = 1.26$

For the third data set, we assume that the output arm h is fixed via a brake ($\omega_h = 0\,\mathrm{s}^{-1}$) and the other data from the second case remain. It is a particular version of the considered gear, which can also have a particular practical meaning. In this case DOF of the planetary gear is equal to 1. Then the ratio $i_{13} = i_{13}^h$ is determined by means of a formula adequate for the fixed axis and in the case when $z_3 < 0$

$$\frac{\omega_1}{\omega_3} = \frac{z_3}{z_1} \tag{4.4}$$

The ratio i_{46}^H can be determined similarly as in the general approach to the considered gear, see $(4.3)_3$. The following values of angular velocities and the ratio i_{IH} have been obtained

$$\begin{aligned} \omega_3 &= \frac{\omega_1 z_1}{z_3} = -47.10\,\mathrm{s}^{-1} \qquad & \omega_H &= \frac{z_4\omega_1 - z_6\omega_3}{z_4 - z_6} = -7.85\,\mathrm{s}^{-1} \\ i_{\mathrm{I}H} &= \Big(\frac{\omega_{\mathrm{I}}}{\omega_H}\Big)_{\omega_h=0} = -20 \end{aligned} \tag{4.5}$$

Also in this case, the same results were obtained upon the graph-based models. The designer can analyse several possibilities to change the input velocities or to fix some gear elements using brakes in the conceptual phase of gear design, e.g. within the three data sets analysed above.

5. Analysis of forces and torques

Forces and torques acting in the gear can be analysed. The analysis is performed in a step-by-step manner viewing the considerations from an input and an output.

In Fig. 4, the free body diagram is presented. The fact that toothings 1 and 4 as well as 3 and 6 are fixed on the same elements (respectively) is taken into account especially as a condition of equilibrium of the corresponding forces $F_{1,2}$ and $F_{4,5}$.

In Figs. 5, 6, 7, the third case is analysed. The scheme of the gear with an additional brake (placed at the input) is given in Fig. 5. In Fig. 6, analysis of velocities is shown. In Fig. 7, the free body diagram for the considered gear in the third case is given. This type of diagram enables us to clearly analyse the forces and to spot the points where equilibrium of forces has to be achieved. Analysing the directions and senses of the angular velocities and torques, we can deduct which quantities are passive or active.

Based upon Fig. 7, the forces and torques acting on the geared wheels and the arm H are analysed. The output torque of the gear is equal to

$$M_H = M_{\mathrm{I}} i_{\mathrm{I}H} \eta_{\mathrm{I}H} \tag{5.1}$$

where $\eta_{\mathrm{I}H}$ is the efficiency.

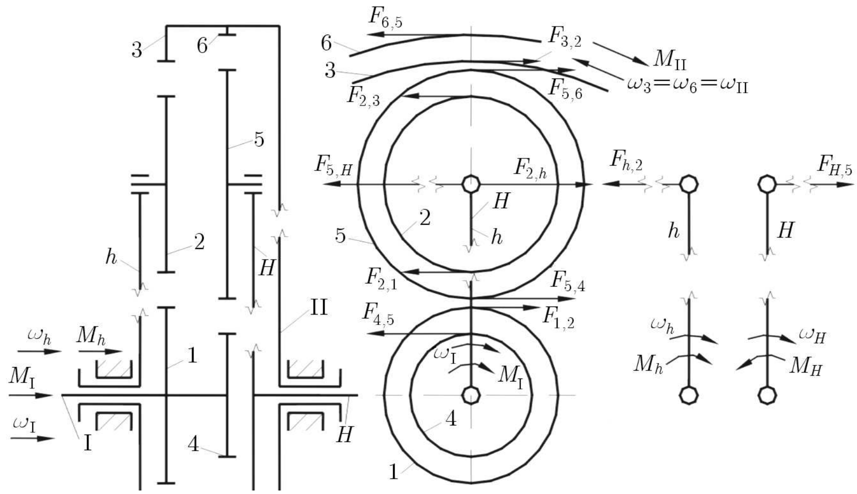

Fig. 4. Free body diagram of the considered gear with analysis of forces for the first and second analysed case

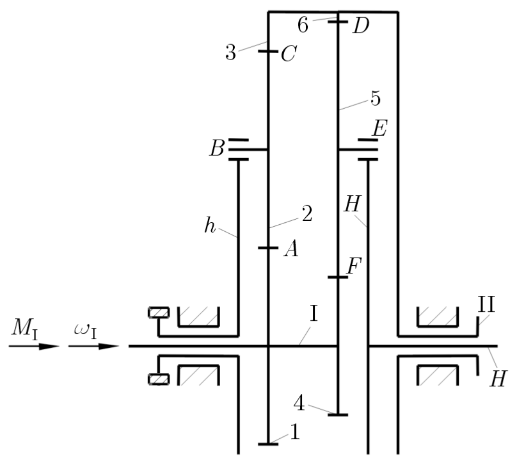

Fig. 5. Scheme of the gear for the third case, i.e. with fixing of the arm h via a brake

Therefore, it is necessary to calculate the efficiency of the gear η_{IH} in the case of the given efficiency of meshing of some pairs of the geared wheels, i.e. the external ones $(1,2$ and $4,5)$ as well as internal $(2,3$ and $5,6)$. The relative velocities of geared wheels 6 and 4 (in relation to the arm H) for a theoretically considered planetary gear are as follows (Abramov, 1976; Müller and Wilk, 1996)

$$\omega_6^H = \omega_6 - \omega_H = -39.25\,\mathrm{s}^{-1} < 0 \qquad\qquad \omega_4^H = \omega_4 - \omega_H = 164.85\,\mathrm{s}^{-1} > 0 \tag{5.2}$$

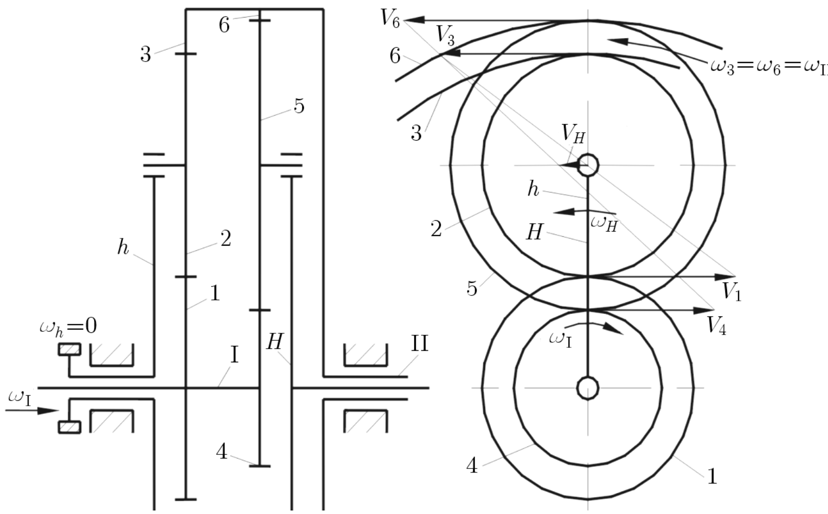

Fig. 6. Analysis of velocities in the third case (with the arm h via a brake)

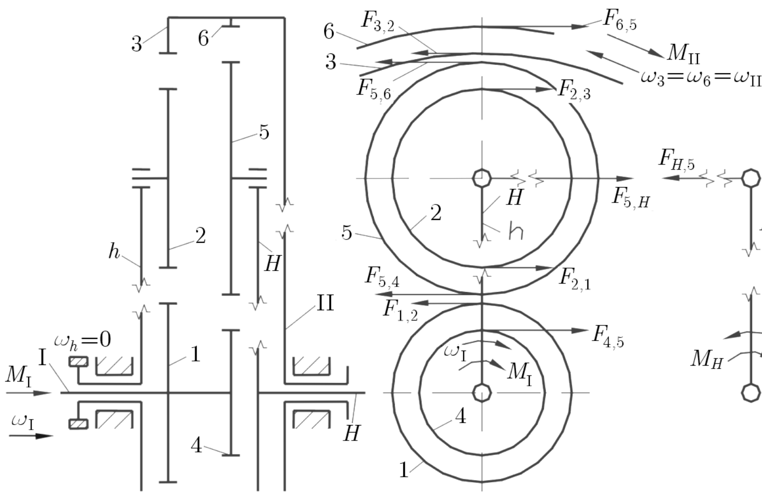

Fig. 7. Free body diagram for the third case (arm h fixed via a brake)

Because the senses of directions of the velocity ω_6^H and the torque M_6 are compatible (Fig. 7), then geared wheel 6 is an active gear in the series of wheels 4-5-6, whereas wheel 4 is passive, thus

$$M_6|\omega_6^H|\eta_{64}^H = M_4|\omega_4^H| \tag{5.3}$$

From the condition of equilibrium for the arm H, we have

$$M_H = M_4 + M_6 \tag{5.4}$$

The relative velocities of geared wheels 3 and 1 (in relation to the arm h) for a theoretically considered planetary gear (after introducing that the angular velocity $\omega_h = 0$) are as follows

$$\omega_1^h = \omega_1 = 157\,\mathrm{s}^{-1} > 0 \qquad\qquad \omega_3^h = \omega_3 = -47.10\,\mathrm{s}^{-1} < 0 \tag{5.5}$$

hence wheel 1 is active and wheel 3 is passive

$$M_1|\omega_1^h|\eta_{13}^h = M_3|\omega_3^h| \tag{5.6}$$

The equilibrium condition for shaft I is expressed by the equation

$$M_{\mathrm{I}} = -M_1 + M_4 \tag{5.7}$$

Moreover, taking into account the equilibrium condition for wheels 3 and 6 (in the case when the total power is passed outside via the arm H), i.e. upon the assumption

$$M_{\mathrm{II}} = M_6 - M_3 = 0 \tag{5.8}$$

we obtain the equality

$$M_3 = M_6 \tag{5.9}$$

Summarizing the above considerations, it is possible to calculate the efficiency of the considered gear

$$\eta_{\mathrm{I}H} = \frac{M_H}{M_{\mathrm{I}} i_{\mathrm{I}H}} = \frac{1 + \frac{\omega_4^H}{|\omega_6^H|\eta_{64}^H}}{\Big(\frac{\omega_4^H \omega_3^h}{|\omega_6^H||\omega_1^h|\eta_{64}^H \eta_{13}^h} - 1\Big) i_{\mathrm{I}H}} \cong 0.79 \tag{5.10}$$

where

$$\eta_{13}^h = \eta_{12}^h \eta_{23}^h \qquad\qquad \eta_{64}^H = \eta_{65}^H \eta_{54}^H \tag{5.11}$$

moreover, it was assumed that $\eta_{12}^h = \eta_{54}^H = 0.99$ (for the external meshing) and $\eta_{23}^h = \eta_{65}^H = 0.98$ (for the internal meshing).

6. Final remarks

Based upon the above-described considerations, the following conclusions can be drawn: two graph-based methods of modelling of mechanical systems have been used for analysis of an exemplary planetary gear. The obtained ratios (for two applied methods) as well as the results of Willis method are equal, which confirms the equivalence of these approaches. The usage of graph-based methods for analysis of the exemplary gear with a closed internal loop has been described and performed step by step giving a detailed explanation to all activities. The methods are relatively simple, algorithmic and general. This confirms usefulness of these methods for checking of correctness of gear analysis.

Both the applied graph-based methods were slightly modified, i.e. Tsai's and Hsu's approaches were joined as well as Marghitu's graph was tailored to a scheme having a closed loop.

Additionally, it has been shown (by a counter example) that Tsai's claim that the case when a graph has a circuit built of stripped lines is impossible for analysis is not a general rule.

Moreover, the graph-based models give a powerful tool for modelling and representation of the knowledge about mechanical systems (Andrews *et al.*, 1997; Shai and Preiss, 1999; Tsai, 2001; Zawiślak, 2007) what is needed for Artificial Intelligence based methods. The graph based models of gears and versatile mechanical systems are very effective in realisation of some other engineering tasks like e.g. synthesis and enumeration of design solutions, which is actually beyond the scope of the present paper but can be studied based upon the given references.

Concerning the obtained results presented in Tables 1 and 2, it can be stated that the achieved ratios – shown in Table 1 – are approximately one, but in Table 2 the ratios above six and approximately nineteen are listed. Therefore, the first set of data should be rejected as useless. It is interesting to see that the sets of teeth data differ only by the number of teeth that have been exchanged between two parallel series of toothings.

References

1. Abramov B.M., 1976, *Typical Problems in Theory of Mechanisms and Machines*, Kharkov [in Russian]

2. Andrews G.C., Lo L., Wong C., 1997, Graph theory - the key to AI applications in dynamics, *Proceedings IEEE Conference on Systems, Man and Cybernetics*, Florida, **3**, 2939-2944

3. Arczewski K., Dul F., 1995, Determination of angular velocities within a multi-body system by means of graphs, *ZAMM Z. Angew. Math. Mech.*, **75**, SI, S105-S106

4. Bonnell R., Hess R., 1968, Applications of signal-flow graph theory to epicyclic gear trains, *IEEE Transactions on Automatic Control*, **13**, 5, 594-595

5. Davies T.H., 1968, An extension of Manolescu's classification of planar kinematic chains and mechanisms of mobility $m > 1$, using graph theory, *J. Mech.*, **3**, 87-100

6. Drewniak J., Zawiślak S., 2010, Kinematical and dynamical analysis of closed kinematic chains using graphs and contour equations, *PAMM*, after the Conference GAMM2009 in Gdańsk

7. Hsu C.H., 2002, An analysis methodology for the kinematic synthesis of epicyclic gear mechanisms, *ASME Journal of Mechanical Design*, **124**, 574-576

8. Ivančenko L.N., Suškov J.A., Wašec A.D., 1974, *Avtomatizaciya vybora skhem planetarnykh korobok peredachi*, Mashinostroenie, Leningrad [in Russian]

9. Marghitu D.B., 2005, *Kinematic Chains and Machine Components Design*, Academic Press, London as well as Elsevier, San Diego

10. Marghitu D.B., Crocker M.J., 2001, *Analytical Elements of Mechanisms*, Cambridge University Press, Cambridge

11. Müller L., Wilk A., 1996, *Toothed Epicyclic Gears*, PWN, Warszawa [in Polish]

12. Schmidt L.C., Shetty H., Chase S., 2000, A graph grammar approach to mechanism synthesis, *ASME Journal of Mechanical Design*, **122**, 371-376

13. Shai O., 2001, The multidisciplinary combinatorial approach (MCA) and its applications in engineering, *Artificial Intelligence for Engineering Design, Analysis and Manufacturing*, **15**, 2, 109-144

14. Shai O., Preiss K., 1999, Graph theory representation of engineering systems and their embedded knowledge, *Artificial Intelligence in Engineering*, **13**, 273-285

15. Tsai L.W., 2001, *Mechanism Design: Enumeration of Kinematic Structures According to Function*, CRC, Boca Raton, USA

16. Tuttle E.R., Peterson S.W., Titus J.E., 1989, Enumeration of basic kinematic chains using the theory of finite groups, *ASME Journal of Mechanisms, Transmissions, and Automation in Design*, **111**, 498-503

17. Uematsu S., 1997, An application of graph theory to the kinematic analysis of planetary gear trains, *International Journal of the Japan Society for Precision Engineering*, **31**, 2, 141-146

18. Uyguroğlu M., Demirel H., 2005, TSAI-TOKAD (T-T) graph: The combination of non-oriented and oriented graphs for the kinematics of articulated gear mechanisms, *Meccanica*, **40**, 3, 223-232

19. Wojnarowski J., Kopeć J., Zawiślak S., 2006, Gears and graphs, *Journal of Theoretical and Applied Mechanics*, **44**, 1, 139-162

20. Wojnarowski J., Lidwin A., 1975, The application of signal flow graphs – the kinematic analysis of planetary gear trains, *Mechanism and Machine Theory*, **10**, 1-B, 17-31

21. Zawiślak S., 2006, Graph-based methodology as a support of conceptual design of planetary gears, *Proc. XX Conference on TMM in Zielona Góra*, 405-411

22. Zawiślak S., 2007, Artificial intelligence aided design of gears based on graph-theoretical models, *Proc. IFToMM2007 Twelfth World Congress in Mechanism and Machine Science*, Besancon, France, 1-6 (http://130.15.85.212/proceedings/WorldCongress07/articles/article_cd.htm, see Zawislak S.)

23. Zawiślak S., 2008, Graph theoretic based models of planetary gears, in: *Theory of Machines and Mechanisms*, J. Wojnarowski and I. Adamiec-Wójcik (Edit.), Wydawnictwo AT-H, Bielsko-Biała, 2, 67-74

24. Zawiślak S., Szypuła Ł., Myśliwiec M., Jagosz A., 2008, Some applications of graph transformations in modelling of mechanical systems, *Pre-Proceedings Seventh International Workshop on Graph Transformation and Visual Modelling Techniques (GT-VMT 2008)*, Budapest, Hungary, 332-345

Grafowe i konturowe modele przekładni planetarnych

Streszczenie

Analiza i synteza mechanizmów są podstawowymi działaniami inżyniera. Narażone są one z różnych powodów na niezamierzone błędy. Metody analizy i syntezy przekładni planetarnych oparte na teorii grafów mogą być metodami alternatywnymi dla realizacji tych zadań, które pozwalają sprawdzić poprawność przeprowadzonych rozważań. W artykule rozważa się dwie metody grafowe modelowania przekładni. Przykładową przekładnię planetarną analizuje się tymi metodam, a wyniki porównuje się z metodą Willis'a. Przeprowadzono także analizę sił, momentów i sprawności. Ujęcie algorytmiczne – które wynika z modelowania grafami – pozwala na łatwe sprawdzanie wielu wariantów rozwiązań w łatwym ujęciu schematycznym, co wpływa na optymalizację rozważanych rozwiązań konstrukcyjnych w fazie koncypowania.

Manuscript received July 17, 2009; accepted for print October 19, 2009