JOURNAL OF THEORETICAL AND APPLIED MECHANICS
53, 3, pp. 629-642, Warsaw 2015
DOI: 10.15632/jtam-pl.53.3.629

ON FREE VIBRATIONS OF THIN FUNCTIONALLY GRADED PLATE BANDS RESTING ON AN ELASTIC FOUNDATION

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In this note free vibrations of plate bands with functionally graded properties, resting on an elastic foundation, are investigated. On the micro-level, these plate bands have a tolerance-periodic structure. It can be shown that in dynamic problems of those objects, the effect of the microstructure size plays a role. This effect can be described in the framework of the tolerance model, which is presented here for these bands. Obtained results are evaluated introducing the asymptotic model. Both fundamental and higher free vibration frequencies of these plate bands are calculated using the Ritz method. The effects of differences of material plate properties in the cell on the microlevel and of the foundation are shown.

Keywords: thin functionally graded plate band, microstructure size, free vibrations, material properties, elasic foundation

1. Introduction

In the civil engineering, plates interacting with the subsoil are often used as elements of building foundations or reinforcements of roads foundations. In many cases, the first approximation of the subsoil can be a model of Winkler's foundation.

In this paper, free vibrations of thin functionally graded plate bands with span L (along the x_1-axis) interacting with elastic Winkler's foundation are considered. It is assumed that these plate bands have the functionally graded structure along their span on the macrolevel, but on the microlevel their structure is, so called, tolerance-periodic in x_1, i.e. nearly periodic, cf. Jędrysiak (2010), Jędrysiak and Michalak (2011), Kaźmierczak and Jędrysiak (2010, 2011, 2013). Hence, the plate bands are called thin functionally graded plate bands, cf. Jędrysiak (2010). The plate material properties are assumed to be independent of the x_2-coordinate. The size of microstructure is determined by length l of "the cell", being very small compared to the plate span L. A fragment of the plate band is shown in Fig. 1.

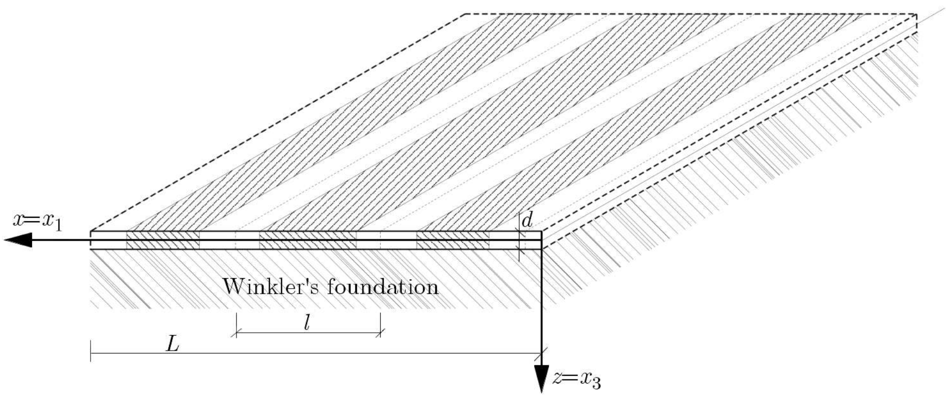

Fig. 1. A fragment of a thin functionally graded plate band interacting with Winkler's foundation

Vibrations of such plates are described by a partial differential equation with highly oscillating, tolerance-periodic, non-continuous coefficients. Because analysis of these plates is too complicated using the equation of the plate theory, different averaged models have been proposed. These models are usually described by partial differential equations with smooth, slowly-varying coefficients.

Functionally graded structures can be described by approaches applied to analyse macroscopically homogeneous media, e.g. periodic, cf. Suresh and Mortensen (1998). Between these models it can be mentioned those based on the asymptotic homogenization method for periodic solids, cf. Bensoussan *et al.* (1978). Models of such kind for periodic plates can be found in a series of papers, e.g. Kohn and Vogelius (1984). Other models are based on the microlocal parameters approach, cf. Matysiak and Nagórko (1989), or the nonstandard homogenization, cf. Nagórko (1998). However, the effect of the microstructure size on the dynamic plate behaviour is neglected in the governing equations of those models. Composite plates can be also parts of more complicated structures such as thin-walled composite columns or beams, cf. Kołakowski (2009, 2012), Królak *et al.* (2009), Kubiak (2006). Moreover, nonhomogeneous Winkler's type solids can be approximations of foam cores in three layered composite plates, cf. Magnucka-Blandzi (2011). Theoretical and numerical results of different problems of functionally graded structures are presented in many papers. Jha *et al.* (2013) analysed free vibrations of functionally graded thick plates with shear and normal deformations effects. The static response of functionally graded plates and shells was investigated using higher order deformation theories by Oktem *et al.* (2012). Vibrations of FG-type plates were analysed using a collocation method with higher-order plate theories by Roque *et al.* (2007). Free vibrations of shells were presented by Tornabene *et al.* (2011). Problems of functionally graded plates resting on a foundation were also considered, e.g. by Tahouneh and Naei (2014) with using the three-dimensional elasticity theory by Yajuvindra Kumar and Lal (2012), where vibrations of nonhomogeneous plates with varying thickness interacting with a foundation were analysed. A list of papers of various theoretical and numerical results of thermomechanical problems of functionally graded structures can be found in Jędrysiak (2010), Woźniak *et al.* (2008, 2010). Unfotunately, the effect of the microstructure size is usually neglected in the governing equations of those models.

This effect can be taken into account in the governing equations in the framework of the tolerance modelling, cf. Woźniak and Wierzbicki (2000), Woźniak *et al.* (2008, 2010). Various thermomechanical problems of periodic structures were investigated in a series of papers applying this method, e.g. dynamics of periodic plane structures by Wierzbicki and Woźniak (2000), vibrations of medium-thickness plates by Baron (2006), static problems of thin plates with moderately large deflections by Domagalski and Jędrysiak (2012), nonlinear vibrations of beams by Domagalski and Jędrysiak (2014), vibrations of thin plates resting on an elastic nonhomogeneous foundation by Jędrysiak (1999, 2003), vibrations of medium-thickness plates resting on an elastic foundation by Jędrysiak and Paś (2005, 2014), vibrations with damping of plate strips with a periodic system of concentrated masses by Marczak and Jędrysiak (2014), vibrations of wavy-type plates by Michalak (2002), vibrations of thin plates with stiffeners by Nagórko and Woźniak (2002), vibrations of thin cylindrical shells by Tomczyk (2007, 2013). These papers show that the effect of the microstructure size plays a crucial role in nonstationary (and some stationary) problems of periodic structures.

The tolerance modelling method is also applied to similar thermomechanical problems of functionally graded structures, e.g. Jędrysiak (2010), Woźniak *et al.* (2010). Some applications to dynamic and stability problems for thin transversally graded plates with the microstructure size bigger than the plate thickness were shown by Jędrysiak and Michalak (2011), Kaźmierczak and Jędrysiak (2010, 2011, 2013, 2014); for thin functionally graded plates with the microstructure size of an order of the plate thickness by Jędrysiak (2013, 2014), Jędrysiak and Pazera (2014); for functionally graded skeletonal shells by Michalak (2012); for heat conduction in functionally graded hollow cylinders by Ostrowski and Michalak (2011); for thin longitudinally graded plates by Michalak and Wirowski (2012), Wirowski (2012). An extended list of papers can be found in the books by Woźniak *et al.* (2008, 2010).

The main aims of this paper are four. The first of them is to formulate the tolerance and the asymptotic models of vibrations for thin transversally graded plate bands. The second aim is to apply these models to calculate free vibration frequencies of a simply supported plate band interacting with Winkler's foundation using the Ritz method. The third is to analyse the effect of various distribution functions of material properties and the effect of the foundation on the frequencies. The fourth aim is to show the effect of differences in the cell between material properties (Young's modulus and mass densities) on the frequencies.

2. Formulation of the problem

Considerations are assumed to be independent of the x_2-coordinate. Let us denote $x = x_1$, $z = x_3$, $x \in [0, L]$, $z \in [-d/2, d/2]$, with d as a constant plate thickness. The plate band is described in the interval $\Lambda = (0, L)$, with "the basic cell" $\Delta \equiv [-l/2, l/2]$ in the interval $\overline{\Lambda}$, where l is the length of the basic cell, satisfying conditions: $d \ll l \ll L$. By $\Delta(x) \equiv (x - l/2, x + l/2)$ a cell with the centre at $x \in \Lambda$ is denoted. It is assumed that the plate band is made of two elastic isotropic materials, perfectly bonded across interfaces. The materials are characterised by Young's moduli E', E'', Poisson's ratios ν', ν'' and mass densities ρ', ρ''. Similarly, the elastic foundation is made of two various materials characterised by Winkler's coefficients k', k''. Let $E(x)$, $\Delta(x)$, $k(x)$, $x \in \Lambda$, be tolerance-periodic, highly-oscillating functions in x, but Poisson's ratio $\nu \equiv \nu' = \nu''$ constant. Assuming $E' \neq E''$ and/or $\rho' \neq \rho''$, the plate material structure can be treated as functionally graded in the x-axis direction. Similarly, for $k' \neq k''$, the foundation structure is functionally graded. Let ∂ denote the derivative with respect to x, and $w(x, t)$ ($x \in \overline{\Lambda}$, $t \in (t_0, t_1)$) be deflection of the plate band.

Tolerance-periodic functions in x describe the plate band properties – the mass density per unit area of the midplane μ, the rotational inertia ϑ and the bending stiffness B

$$\mu(x) \equiv d\rho(x) \qquad \vartheta(x) \equiv \frac{d^3}{12}\rho(x) \qquad B(x) \equiv \frac{d^3}{12(1-\nu^2)}E(x) \tag{2.1}$$

respectively. Free vibrations of thin functionally graded plate bands, on the well known Kirchhoff plate theory assumptions, can be described by the partial differential equation of the fourth order with respect to the deflection w

$$\partial\partial[B\partial\partial w] + \mu\ddot{w} - \partial(\vartheta\partial\ddot{w}) + kw = 0 \tag{2.2}$$

with highly-oscillating, non-continuous, tolerance-periodic coefficients.

3. Foundations of the modelling

3.1. Introductory concepts

Following the book edited by Woźniak *et al.* (2010) and the book by Jędrysiak (2010), some introductory concepts of the tolerance modelling are used, i.e. the averaging operator, tolerance system, slowly-varying function $SV_\xi^\alpha(\Lambda, \Delta)$, tolerance-periodic function $TP_\xi^\alpha(\Lambda, \Delta)$, highly oscillating function $HO_\xi^\alpha(\Lambda, \Delta)$, fluctuation shape function $FS_\xi^2(\Lambda, \Delta)$, where ξ is the tolerance parameter and α is a positive constant determining kind of the function. Some of these concepts are reminded below.

A cell at $x \in \Lambda_\Delta$ is denoted by $\Delta(x) = x + \Delta$, $\Lambda_\Delta = \{x \in \Delta : \ \Delta(x) \subset \Lambda\}$. The basic concept of the modelling technique is the averaging operator, for an integrable function f defined by

$$\langle f \rangle(x) = \frac{1}{l} \int\limits_{\Delta(x)} f(y)\, dy \qquad x \in \Lambda_\Delta \qquad y \in \Delta(x) \tag{3.1}$$

If f is a function tolerance-periodic in x, its averaged value by (3.1) is slowly-varying in x.

Let $h(\cdot)$ be a highly oscillating function defined on $\overline{\Lambda}$, $h \in HO^2_\xi(\Lambda, \Delta)$, continuous together with the gradient $\partial^1 h$ and with a piecewise continuous and bounded gradient $\partial^2 h$. The function $h(\cdot)$ is the fluctuation shape function of the 2-nd kind, $FS^2_\xi(\Lambda, \Delta)$, if it depends on l as a parameter and the condition $\langle \mu h \rangle(x) \approx 0$ holds for every $x \in \Lambda_\Delta$, where $\mu > 0$ is a certain tolerance-periodic function, l is the microstructure parameter.

3.2. Fundamental assumptions

Following the books by Woźniak *et al.* (2010), Jędrysiak (2010) and applying the introductory concepts, the following fundamental modelling assumptions can be formulated.

The micro-macro decomposition is the first assumption, in which the deflection w appears in the form

$$w(x, t) = W(x, t) + h^A(x) V^A(x, t) \qquad A = 1, \dots, N \qquad x \in \Lambda \tag{3.2}$$

with $W(\cdot, t)$, $V^A(\cdot, t) \in SV^2_\xi(\Lambda, \Delta)$ (for every t) as basic kinematic unknowns (W is called the macrodeflection; V^A are called the fluctuation amplitudes) and $h^A(\cdot) \in FS^2_\xi(\Lambda, \Delta)$ as the known fluctuation shape functions.

In the tolerance averaging approximation, being the second modelling assumption, the terms of an order of $O(\xi)$ are treated as negligibly small in the course of modelling.

4. The tolerance modelling procedure

The modelling procedure of the tolerance averaging technique was shown by Woźniak *et al.* (2010) and for thin functionally graded plates by Jędrysiak (2010). Below, it is outlined.

The formulation of the action functional is the first step

$$\mathcal{A}(w(\cdot)) = \int\limits_{\Lambda} \int\limits_{t_0}^{t_1} \mathcal{L}(y, \partial\partial w(y, t), \partial \dot{w}(y, t), \dot{w}(y, t), w(y, t))\, dt\, dy \tag{4.1}$$

where the lagrangean $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{1}{2}(\mu \dot{w}\dot{w} + \vartheta \partial\dot{w}\partial\dot{w} - B\partial\partial w \partial\partial w - kww) \tag{4.2}$$

From the principle of stationary action to functional (4.1) combined with (4.2), after some manipulations, known equation (2.2) of free vibrations for thin functionally graded plate bands interacting with Winkler's foundation is derived.

The next step of the tolerance modelling is substituting micro-macro decomposition (3.2) into lagrangean (4.2). Applying averaging operator (3.1) and the tolerance averaging approximation, the tolerance averaged form $\langle \mathcal{L}_h \rangle$ of lagrangean (4.2) is obtained in the third step

$$\begin{aligned} \langle \mathcal{L}_h \rangle = -\frac{1}{2}\{ & (\langle B \rangle \partial\partial W + 2\langle B\partial\partial h^B \rangle V^B)\partial\partial W + \langle k \rangle WW + 2W\langle kh^B \rangle V^B + \langle \vartheta \rangle \partial\dot{W}\partial\dot{W} \\ & + (\langle kh^A h^B \rangle + \langle B\partial\partial h^A \partial\partial h^B \rangle)V^A V^B - \langle \mu \rangle \dot{W}\dot{W} + (\langle \vartheta \partial h^A \partial h^B \rangle - \langle \mu h^A h^B \rangle)\dot{V}^A \dot{V}^B \} \end{aligned} \tag{4.3}$$

where the macrodeflection W and the fluctuation amplitudes V^A, $A = 1, \ldots, N$, are new basic kinematic unknowns. The known fluctuation shape functions V^A are introduced in micro-macro decomposition (3.2).

Using the principle of stationary action to the averaged functional $\mathcal{A}_h$ combined together with lagrangean (4.3), the system of governing equations is derived.

5. Model equations

From the principle of stationary action applied to the averaged functional $\mathcal{A}_h$ with lagrangean (4.3), after some manipulations, the following system of equations for W and V^A is obtained

$$\begin{aligned}&\partial\partial(\langle B\rangle\partial\partial W + \langle B\partial\partial h^B\rangle V^B) + \langle k\rangle W + \underline{\langle kh^A\rangle} V^A + \langle \mu\rangle \ddot{W} - \langle\vartheta\rangle\partial\partial\ddot{W} = 0\\ &\langle B\partial\partial h^A\rangle\partial\partial W + \underline{\langle kh^A\rangle} W = -(\langle B\partial\partial h^A\partial\partial h^B\rangle + \underline{\langle kh^A h^B\rangle})V^B\\ &\qquad - (\underline{\langle \mu h^A h^B\rangle} + \underline{\langle \vartheta\partial h^A\partial h^B\rangle})\ddot{V}^B\end{aligned} \tag{5.1}$$

The above equations are a system of $N + 1$ differential equations constituting the tolerance model of thin functionally graded plate bands. The underlined terms in these equations depend on the microstructure parameter l. Hence, this model allows one to take into account the effect of the microstructure size on free vibrations of these plates. The coefficients of equations (5.1) are slowly-varying functions in x. It can be observed that boundary conditions for these plate bands (in $\Lambda = (0, L)$) are formulated only for the macrodeflection W (on edges $x = 0, L$) but not for the fluctuation amplitudes V^A, $A = 1, \ldots, N$.

It can be observed that after neglecting terms with the parameter l in equations $(5.1)_2$, the algebraic equations for fluctuation amplitudes V^A are obtained

$$V^A = -(\langle B\partial\partial h^A\partial\partial h^B\rangle)^{-1}\langle B\partial\partial h^B\rangle\partial\partial W \tag{5.2}$$

Substituting formula (5.2) into $(5.1)_1$, the following equation for W is derived

$$\partial\partial\Big((\langle B\rangle - \langle B\partial\partial h^A\rangle(\langle B\partial\partial h^A\partial\partial h^B\rangle)^{-1}\langle B\partial\partial h^B\rangle)\partial\partial W\Big) + \langle k\rangle W + \langle\mu\rangle\ddot{W} = 0 \tag{5.3}$$

The above equation together with micro-macro decomposition (3.2) represents the asymptotic model of thin functionally graded plate bands. Governing equation (15) with equations (5.2) of this model can be obtained using also the formal asymptotic modelling procedure, cf. Woźniak *et al.* (2010), Kaźmierczak and Jędrysiak (2011, 2013). It can be observed that this procedure leads to model equations without terms describing the effect of the microstructure size on free vibrations of these plates. Hence, in the framework of the asymptotic model, the macrobehaviour of these plate bands can be only investigated.

6. Example: free vibrations of plate bands

6.1. Introduction

Free vibrations of a simply supported thin plate band with span L along the x-axis interacting with Winkler's foundation are considered. The properties of the plate band are

$$\rho(\cdot, z), E(\cdot, z) = \begin{cases} \rho', E' & \text{for} \quad z \in ((1-\gamma(x))l/2, (1+\gamma(x))l/2) \\ \rho'', E'' & \text{for} \quad z \in [0, (1-\gamma(x))l/2] \cup [(1+\gamma(x))l/2, l] \end{cases} \tag{6.1}$$

with a distribution function of material properties $\gamma(x)$, see Fig. 2. Moreover, it is assumed that the foundation is homogeneous with the Winkler's coefficient $k = \text{const}$.

Our considerations are restricted only to one fluctuation shape function, i.e. $A = N = 1$. Denote $h \equiv h^1$, $V \equiv V^1$. Hence, micro-macro decomposition (3.2) of the deflection $w(x,t)$ has the form

$$w(x,t) = W(x,t) + h(x)V(x,t)$$

where $W(\cdot,t), V(\cdot,t) \in SV_\xi^2(\Lambda,\Delta)$ for every $t \in (t_0,t_1)$, $h(\cdot) \in FS_\xi^2(\Lambda,\Delta)$.

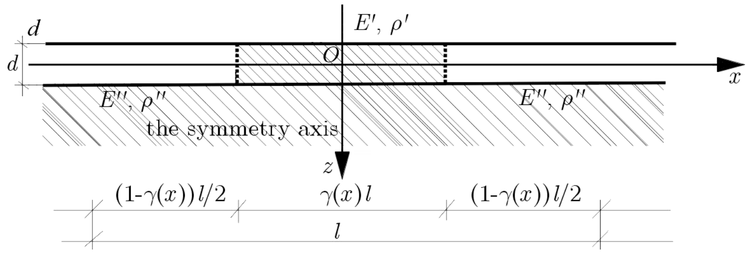

Fig. 2. "Basic cell" of the functionally graded plate band interacting with Winkler's foundation

The cell structure is shown in Fig. 2. Thus, the periodic approximation of the fluctuation shape function $h(x)$ has the form

$$\widetilde{h}(x,z) = \Lambda^2[\cos(2\pi z/l) + c(x)] \qquad z \in \Delta(x) \qquad x \in \Lambda$$

where the parameter $c(x)$ is a slowly-varying function in x and is determined by $\langle \widetilde{\mu}\widetilde{h}\rangle = 0$

$$c = c(x) = \{\sin[\pi\widetilde{\gamma}(x)](\rho' - \rho'')\}\Big\{\pi\{\rho'\widetilde{\gamma}(x) + \rho''[1 - \widetilde{\gamma}(x)]\}\Big\}^{-1}$$

where $\widetilde{\gamma}(x)$ is the periodic approximation of the distribution function of the material properties $\gamma(x)$. The parameter $c(x)$ is treated as constant in the calculations of derivatives $\partial\widetilde{h}$, $\partial\partial\widetilde{h}$.

Under denotations:

$$\begin{array}{llll} \breve{B} = \langle B\rangle & \widehat{B} = \langle B\partial\partial h\rangle & \overline{B} = \langle B\partial\partial h\partial\partial h\rangle & \breve{K} = \langle k\rangle \\ \widetilde{K} = l^{-2}\langle kh\rangle & \overline{K} = l^{-4}\langle khh\rangle & \breve{\mu} = \langle \mu\rangle & \overline{\mu} = l^{-4}\langle \mu hh\rangle \\ \breve{\vartheta} = \langle \vartheta\rangle & \overline{\vartheta} = l^{-2}\langle \vartheta\partial h\partial h\rangle & & \end{array} \tag{6.2}$$

tolerance model equations (5.1) can be written as

$$\begin{aligned} &\partial\partial(\breve{B}\partial\partial W + \widehat{B}V) + \breve{K}W + l^2\widetilde{K}V + \breve{\mu}\ddot{W} - \breve{\vartheta}\partial\partial\ddot{W} = 0 \\ &\widehat{B}\partial\partial W + l^2\widetilde{K}W + (\overline{B} + l^4\overline{K})V + l^2(l^2\overline{\mu} + \overline{\vartheta})\ddot{V} = 0 \end{aligned} \tag{6.3}$$

however, plate band equation (5.3) has the form

$$\partial\partial[(\breve{B} - \widehat{B}^2/\overline{B})\partial\partial W] + \breve{K}W + \breve{\mu}\ddot{W} - \breve{\vartheta}\partial\partial\ddot{W} = 0 \tag{6.4}$$

Equation (6.4) describes free vibrations of this plate band within the asymptotic model. All coefficients of model equations (6.3) and (6.4) are slowly-varying functions in x.

6.2. The Ritz method applied to the model equations

Equations (6.3) or (6.4) have slowly-varying functional coefficients. Analytical solutions to them are too difficult to find. Hence, approximate formulas fo free vibrations frequencies can be derived using the known Ritz method, cf. Kaźmierczak and Jędrysiak (2010). In order to obtain these formulas, relations of the maximum strain energy $\mathcal{A}_{max}$ and the maximal kinetic energy $\mathcal{V}_{max}$ have to be determined.

Solutions to equation (6.4) and equations (6.3) are assumed in form satisfying the boundary conditions for the simply supported plate band

$$W(x,t) = A_W \sin(\alpha x)\cos(\omega t) \qquad V(x,t) = A_V \sin(\alpha x)\cos(\omega t) \tag{6.5}$$

where α is the wave number, ω is the free vibrations frequency. Introducing denotations

$$\begin{aligned}
\breve{B} &= \frac{d^3}{12(1-\nu^2)} \int\limits_0^L \{E''[1-\widetilde{\gamma}(x)] + \widetilde{\gamma}(x)E'\}[\sin(\alpha x)]^2\, dx \\
\widehat{B} &= \frac{\pi d^3}{3(1-\nu^2)}(E'-E'') \int\limits_0^L \sin(\pi\widetilde{\gamma}(x))[\sin(\alpha x)]^2\, dx \\
\overline{B} &= \frac{(\pi d)^3}{3(1-\nu^2)} \int\limits_0^L \{(E'-E'')[2\pi\widetilde{\gamma}(x) + \sin(2\pi\widetilde{\gamma}(x))] + 2\pi E''\}[\sin(\alpha x)]^2\, dx \\
\breve{\mu} &= d \int\limits_0^L \{[1-\widetilde{\gamma}(x)]\rho'' + \widetilde{\gamma}(x)\rho'\}[\sin(\alpha x)]^2\, dx \\
\breve{\vartheta} &= \frac{d^3}{12} \int\limits_0^L \{[1-\widetilde{\gamma}(x)]\rho'' + \widetilde{\gamma}(x)\rho'\}[\cos(\alpha x)]^2\, dx \\
\overline{\mu} &= \frac{d}{4\pi} \int\limits_0^L \{(\rho'-\rho'')[2\pi\widetilde{\gamma}(x) + \sin(2\pi\widetilde{\gamma}(x))] + 2\pi\rho''\}[\sin(\alpha x)]^2\, dx \\
&\quad + \frac{d}{\pi}(\rho'-\rho'') \int\limits_0^L c(x)[\pi c(x)\widetilde{\gamma}(x) - 2\sin(\pi\widetilde{\gamma}(x))][\sin(\alpha x)]^2\, dx \\
&\quad + d\rho'' \int\limits_0^L [c(x)]^2[\sin(\alpha x)]^2\, dx \\
\overline{\vartheta} &= \frac{\pi d^3}{12} \int\limits_0^L \{(\rho'-\rho'')[2\pi\widetilde{\gamma}(x) - \sin(2\pi\widetilde{\gamma}(x))] + 2\pi\rho''\}[\sin(\alpha x)]^2\, dx \\
\breve{K} &= k \int\limits_0^L [\sin(\alpha x)]^2\, dx \\
\widetilde{K} &= k \int\limits_0^L c(x)[\sin(\alpha x)]^2\, dx = \frac{k(\rho'-\rho'')}{\pi} \int\limits_0^L \frac{\sin(\pi\widetilde{\gamma}(x))}{\rho'\widetilde{\gamma}(x) + \rho''[1-\widetilde{\gamma}(x)]}[\sin(\alpha x)]^2\, dx \\
\overline{K} &= \frac{1}{2}k \int\limits_0^L c(x)[\sin(\alpha x)]^2\, dx = \frac{k(\rho'-\rho'')}{2\pi} \int\limits_0^L \frac{\sin[\pi\widetilde{\gamma}(x)]}{\rho'\widetilde{\gamma}(x) + \rho''[1-\widetilde{\gamma}(x)]}[\sin(\alpha x)]^2\, dx
\end{aligned} \tag{6.6}$$

and using (6.5), the formulas of the maximum energies – strain $\mathcal{E}_{max}$ and kinetic $\mathcal{V}_{max}$ – in the framework of the tolerance model, take the form

$$\begin{aligned}
\mathcal{E}_{max}^{TM} &= \frac{1}{2}[(\breve{B}A_W^2\alpha^2 - 2\widehat{B}A_W A_V)\alpha^2 + \breve{K}A_W^2 + 2l^2\widetilde{K}A_W A_V + (\overline{B} + l^4\overline{K})A_V^2] \\
\mathcal{V}_{max}^{TM} &= \frac{1}{2}[A_W^2(\breve{\mu} + \breve{\vartheta}\alpha^2) + A_V^2 l^2(l^2\overline{\mu} + \overline{\vartheta})]\omega^2
\end{aligned} \tag{6.7}$$

For the asymptotic model, they can be written as

$$\mathcal{E}^{AM}_{max} = \frac{1}{2}[(\breve{B}A_W^2\alpha^2 - 2\widehat{B}A_W A_V)\alpha^2 + \breve{K}A_W^2 + \overline{B}A_V^2] \qquad \mathcal{V}^{AM}_{max} = \frac{1}{2}A_W^2(\breve{\mu} + \breve{\vartheta}\alpha^2)\omega^2 \tag{6.8}$$

The conditions of the Ritz method take the form

$$\frac{\partial(\mathcal{E}_{max} - \mathcal{V}_{max})}{\partial A_W} = 0 \qquad \frac{\partial(\mathcal{E}_{max} - \mathcal{V}_{max})}{\partial A_V} = 0 \tag{6.9}$$

Using (6.9) to relations (6.7), after some manipulations, the following formulas are obtained

$$(\omega_{-,+})^2 \equiv \frac{l^2(l^2\overline{\mu} + \overline{\vartheta})(\alpha^4\breve{B} + \breve{K}) + (\breve{\mu} + \alpha^2\breve{\vartheta})(\overline{B} + l^4\overline{K})}{2(\breve{\mu} + \alpha^2\breve{\vartheta})l^2(l^2\overline{\mu} + \overline{\vartheta})} \tag{6.10}$$

$$\mp\frac{\sqrt{[l^2(l^2\overline{\mu} + \overline{\vartheta})(\alpha^4\breve{B} + \breve{K}) - (\breve{\mu} + \alpha^2\breve{\vartheta})(\overline{B} + l^4\overline{K})]^2 + 4(l^2\widetilde{K} - \alpha^2\widehat{B})^2 l^2(\breve{\mu} + \alpha^2\breve{\vartheta})(l^2\overline{\mu} + \overline{\vartheta})}}{2(\breve{\mu} + \alpha^2\breve{\vartheta})l^2(l^2\overline{\mu} + \overline{\vartheta})}$$

for the lower ω_- and the higher ω_+ free vibrations frequencies, respectively, in the framework of the tolerance model.

For asymptotic model conditions (6.9) applied to equations (6.8) lead, after some manipulations, to the following formula

$$\omega^2 \equiv [(\alpha^4\breve{B} + \breve{K})\overline{B} - \alpha^4\widehat{B}^2][(\breve{\mu} + \breve{\vartheta}\alpha^2)\overline{B}]^{-1} \tag{6.11}$$

of the lower free vibration frequency ω.

6.3. Results

Calculations are made for the following distribution functions of the material properties $\gamma(x)$

$$\begin{aligned}
&\widetilde{\gamma}(x) = \sin^2\frac{\pi x}{L} \qquad \widetilde{\gamma}(x) = \cos^2\frac{\pi x}{L} \qquad \widetilde{\gamma}(x) = \Big(\frac{x}{L}\Big)^2 \\
&\widetilde{\gamma}(x) = \sin\frac{\pi x}{L} \qquad \widetilde{\gamma}(x) = \frac{1}{2}
\end{aligned} \tag{6.12}$$

where formula $(6.12)_5$ determines an example of a periodic plate band.

Let us also introduce dimensionless frequency parameters for the free vibration frequencies ω and ω_-, ω_+ determined by equations (6.11) and (6.10), respectively

$$\begin{aligned}
&\Omega^2 \equiv 12(1-\nu^2)(E')^{-1}l^2\omega^2 \qquad (\Omega_-)^2 \equiv 12(1-\nu^2)(E')^{-1}l^2(\omega_-)^2 \\
&(\Omega_+)^2 \equiv 12(1-\nu^2)(E')^{-1}l^2(\omega_+)^2
\end{aligned} \tag{6.13}$$

Moreover, a dimensionless parameter of the foundation is introduced

$$\kappa \equiv 12(1-\nu^2)(E')^{-1}kd$$

Results of calculations are shown in Figs. 3-6, where the results obtained by the tolerance or asymptotic models for plate bands with the simply supported edges are presented. Calculations are made for Poisson's ratio $\nu = 0.3$, wave number $\alpha = \pi/L$, ratio $l/L = 0.1$, ratios of plate thickness $d/l = 0.1$, 0.01 and ratios of the foundation $\kappa = 5 \cdot 10^{-5}$, 0.05. Figures 3 and 4 show plots of the lower frequency parameters versus both ratios $E''/E' - \rho''/\rho'$, but Figs. 5 and 6 present diagrams of the higher frequency parameters versus these both ratios. Plots in Figs. 3a, 4a, 5a, 6a are made for $\kappa = 5 \cdot 10^{-5}$, but in Figs. 3b, 4b, 5b, 6b for $\kappa = 0.05$. Moreover, in Figs. 3 and 4, a comparison of the lower frequency parameters versus both ratios $E''/E' - \rho''/\rho'$ calculated in the framework of the tolerance model (formulas $(6.13)_2$ and $(6.10)_1$) and of the asymptotic model (formulas $(6.13)_1$ and (6.11)) is presented. Plots shown in Figs. 3 and 5 are made for $d/l = 0.1$, but in Figs. 4 and 6 they are for $d/l = 0.01$.

From the results shown in Figs. 3-6 some remarks and comments are formulated.

1° The lower frequency parameters calculated by the asymptotic model, (6.11), and the tolerance model, $(6.10)_1$, depend on the plate thickness ratio d/l and the parameter of foundation κ, see Figs. 3 and 4:

- The lower frequency parameters calculated by the asymptotic model, (6.11), and the tolerance model, $(6.10)_1$, are nearly identical for thicker plates, e.g. $d/l = 0.1$, and weaker foundations, e.g. $\kappa = 5 \cdot 10^{-5}$, see Fig. 3a.
- However, higher values of these parameters are found from the tolerance model for smaller thickness of plates, $d/l < 0.1$, and for stronger foundations, $\kappa > 5 \cdot 10^{-5}$, see Figs. 3b and 4.
- Differences between these frequency parameters depend on the plate thickness ratio d/l and the parameter of foundation κ. They increase with a decrease in the plate thickness, $d/l > 0$, and an increase in the stiffness of foundation, $\kappa > 5 \cdot 10^{-5}$, e.g. $d/l = 0.01$, $\kappa = 0.05$, see Fig. 3b.

2° The effect of distribution functions of the material properties $\gamma(x)$ on the lower frequency parameters for various ratios $E''/E' \in [0, 1]$, $\rho''/\rho' \in [0, 1]$ for the simply supported plate band can be observed in Figs. 3 and 4:

- The highest values of these frequency parameters are obtained for all pairs of ratios $(E''/E', \rho''/\rho')$ from the above intervals of the function $\gamma(x)$ by $(6.12)_2$ and for smaller thickness of the plates, $d/l < 0.1$, or for stronger foundations, cf. $\kappa > 5 \cdot 10^{-5}$, see Figs. 3b and 4.
- The highest values of these frequency parameters for thicker plates, e.g. $d/l = 0.1$, or for weaker foundations, e.g. $\kappa = 5 \cdot 10^{-5}$, see Fig. 3a, are obtained:
  - for $\gamma(x)$ by $(6.12)_2$ and for pairs of ratios $(E''/E', \rho''/\rho')$ such that $E''/E' > (E''/E')_0 > 0$, $\rho''/\rho' < (\rho''/\rho')_0((E''/E')_0) > 0$, where $(\rho''/\rho')_0$ depends on $(E''/E')_0$,
  - for $\gamma(x)$ by $(6.12)_4$ and for pairs of ratios $(E''/E', \rho''/\rho')$ such that $E''/E' < (E''/E')_0 > 0$, $\rho''/\rho' > (\rho''/\rho')_0((E''/E')_0) > 0$, where $(\rho''/\rho')_0$ depends on $(E''/E')_0$ (unfortunately, it is not visible in this form of these diagrams)

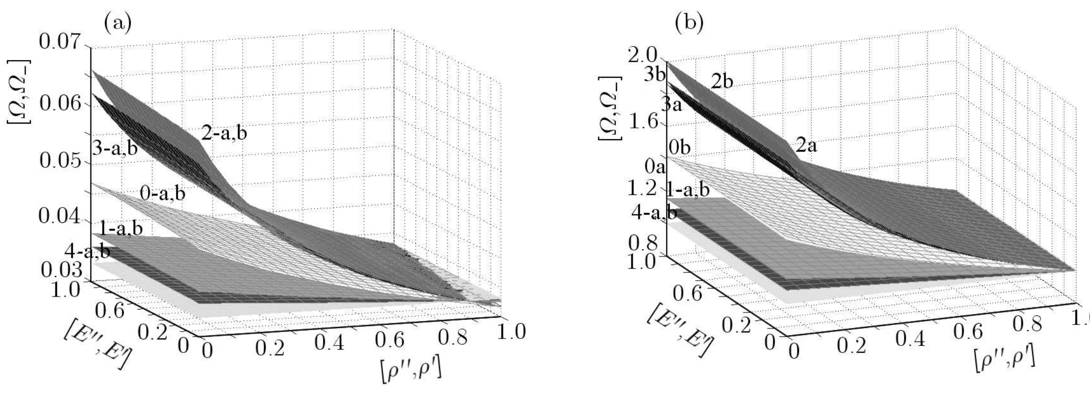

Fig. 3. Plots of the dimensionless frequency parameters Ω and Ω_- of lower free vibration frequencies versus ratios $E''/E' - \rho''/\rho'$ by the asymptotic model (surfaces a), the tolerance model (surfaces b), made for: (a) $d/l = 0.1$, $\kappa = 5 \cdot 10^{-5}$; (b) $d/l = 0.1$, $\kappa = 0.05$ ($1 - \gamma$ by $(6.12)_1$; $2 - \gamma$ by $(6.12)_2$; $3 - \gamma$ by $(6.12)_3$; $4 - \gamma$ by $(6.12)_4$; $0 - \gamma$ by $(6.12)_5$; the grey plane is related to the frequency parameter for the homogeneous plate band, i.e. $E''/E' = \rho''/\rho' = 1$)

- The smallest values of these frequency parameters are obtained for all pairs of ratios $(E''/E', \rho''/\rho')$ from the above intervals of the function $\gamma(x)$ by $(6.12)_4$ and for smaller thickness of the plates, $d/l < 0.1$, or for stronger foundations, cf. $\kappa > 5 \cdot 10^{-5}$, see Figs. 3b and 4.

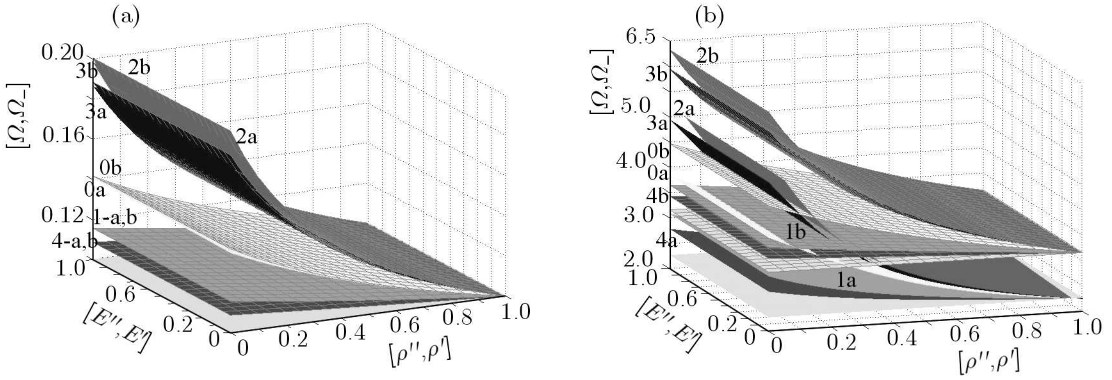

Fig. 4. Plots of the dimensionless frequency parameters Ω and Ω_- of lower free vibration frequencies versus ratios $E''/E' - \rho''/\rho'$ by the asymptotic model (surfaces a), the tolerance model (surfaces b), made for: (a) $d/l = 0.01$, $\kappa = 5 \cdot 10^{-5}$; (b) $d/l = 0.01$, $\kappa = 0.05$ (1 – γ by $(6.12)_1$; 2 – γ by $(6.12)_2$; 3 – γ by $(6.12)_3$; 4 – γ by $(6.12)_4$; 0 – γ by $(6.12)_5$; the grey plane is related to the frequency parameter for the homogeneous plate band, i.e. $E''/E' = \rho''/\rho' = 1$)

- The smallest values of these frequency parameters for thicker plates, e.g. $d/l = 0.1$, or for weaker foundations, e.g. $\kappa = 5 \cdot 10^{-5}$, see Fig. 3a, are obtained:
  - for $\gamma(x)$ by $(6.12)_4$ and for pairs of ratios $(E''/E', \rho''/\rho')$ such that $E''/E' > (E''/E')_1 > 0$, $\rho''/\rho' < (\rho''/\rho')_1((E''/E')_1) > 0$, where $(\rho''/\rho')_1$ depends on $(E''/E')_1$,
  - for $\gamma(x)$ by $(6.12)_2$ and for pairs of ratios $(E''/E', \rho''/\rho')$, such that $E''/E' > (E''/E')_2 > 0$, $\rho''/\rho' > (\rho''/\rho')_2((E''/E')_2) > 0$, where $(\rho''/\rho')_2$ depends on $(E''/E')_2$ (unfortunately, it is not visible in this form of diagrams),
  - for $\gamma(x)$ by $(6.12)_3$ and for pairs of ratios $(E''/E', \rho''/\rho')$, such that $(E''/E')_2 > E''/E' < (E''/E')_3 > 0$, $(\rho''/\rho')_2((E''/E')_2) > \rho''/\rho' > (\rho''/\rho')_3((E''/E')_3) > 0$, where $(\rho''/\rho')_2$, $(\rho''/\rho')_3$ depend on $(E''/E')_2$, $(E''/E')_3$, respectively (not visible in this form of diagrams),
  - for $\gamma(x)$ by $(6.12)_5$ (periodic plate band) and for pairs of ratios $(E''/E', \rho''/\rho')$, such that $(E''/E')_1 > E''/E' > (E''/E')_3 > 0$, $(\rho''/\rho')_1((E''/E')_1) < \rho''/\rho' < (\rho''/\rho')_3((E''/E')_3) > 0$, where $(\rho''/\rho')_1$, $(\rho''/\rho')_3$ depend on $(E''/E')_1$, $(E''/E')_3$, respectively (not visible in this form of diagrams).

3° Figure 3 shows also an interesting feature that for the distribution functions of the material properties $\gamma(x)$ used and for rather thicker plates, e.g. $d/l = 0.1$, and weaker foundations, e.g. $\kappa = 5 \cdot 10^{-5}$, the lower frequency parameters are higher or smaller than this parameter for the homogeneous plate band made of a stronger material, i.e. $\rho''/\rho' = E''/E' = 1$ (the grey plane in Fig. 3a).

4° The effect of distribution functions of the material properties $\gamma(x)$ on higher frequency parameters for various ratios $E''/E' \in [0, 1]$, $\rho''/\rho' \in [0, 1]$ for the simply supported plate band can be observed in Figs. 5 and 6:

- The highest values of these frequency parameters for rather very thin plates, e.g. d/l=0.01, and for stronger foundations, e.g. $\kappa = 0.05$, see Fig. 6a, are obtained:
  - for $\gamma(x)$ by $(6.12)_3$ and for pairs of ratios $(E''/E', \rho''/\rho')$ such that $E''/E' > (E''/E')_0 > 0$, $\rho''/\rho' < (\rho''/\rho')_0((E''/E')_0) > 0$, where $(\rho''/\rho')_0$ depends on $(E''/E')_0$,
  - for $\gamma(x)$ by $(6.12)_2$ and for pairs of ratios $(E''/E', \rho''/\rho')$ such that $E''/E' < (E''/E')_0 > 0$, $\rho''/\rho' > (\rho''/\rho')_0((E''/E')_0) > 0$, where $(\rho''/\rho')_0$ depends on $(E''/E')_0$.

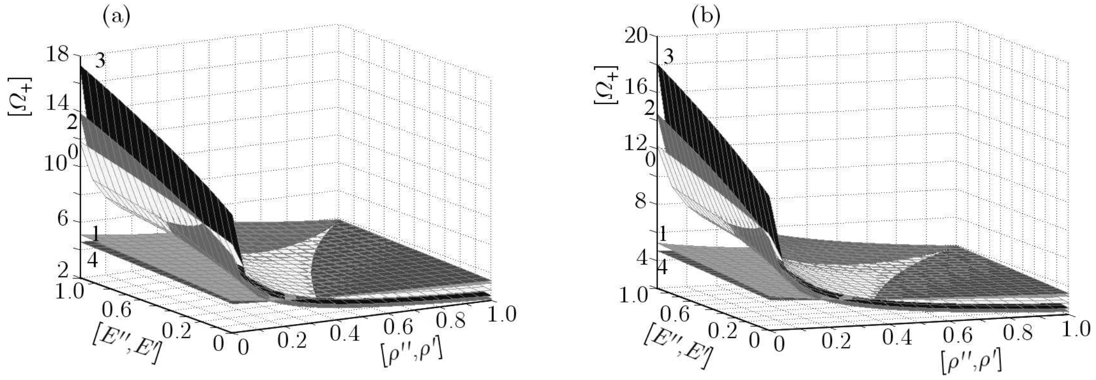

Fig. 5. Plots of the dimensionless frequency parameters Ω_+ of higher free vibration frequencies versus ratios $E''/E' - \rho''/\rho'$, made for: (a) $d/l = 0.1$, $\kappa = 5 \cdot 10^{-5}$; (b) $d/l = 0.1$, $\kappa = 0.05$ ($1 - \gamma$ by $(6.12)_1$; $2 - \gamma$ by $(6.12)_2$; $3 - \gamma$ by $(6.12)_3$; $4 - \gamma$ by $(6.12)_4$; $0 - \gamma$ by $(6.12)_5$)

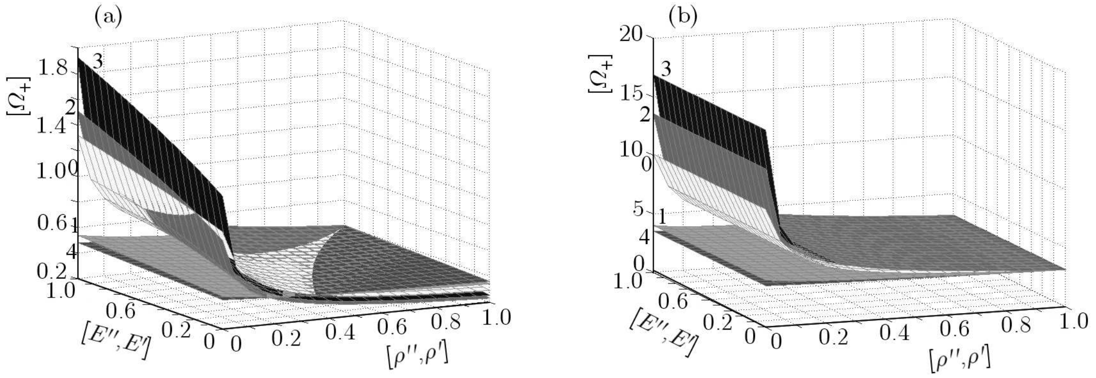

Fig. 6. Plots of the dimensionless frequency parameters Ω_+ of higher free vibration frequencies versus ratios $E''/E' - \rho''/\rho'$, made for: (a) $d/l = 0.01$, $\kappa = 5 \cdot 10^{-5}$; (b) $d/l = 0.01$, $\kappa = 0.05$ ($1 - \gamma$ by $(6.12)_1$; $2 - \gamma$ by $(6.12)_2$; $3 - \gamma$ by $(6.12)_3$; $4 - \gamma$ by $(6.12)_4$; $0 - \gamma$ by $(6.12)_5$)

- The highest values of these frequency parameters for thicker plates, e.g. $d/l > 0.01$, and for weaker foundations, e.g. $\kappa \leqslant 0.05$, see Figs. 5 and 6a, are obtained:
  - for $\gamma(x)$ by $(6.12)_4$ and for pairs of ratios $(E''/E', \rho''/\rho')$ such that $E''/E' < (E''/E')_0 > 0$, $\rho''/\rho' > (\rho''/\rho')_0((E''/E')_0) > 0$, where $(\rho''/\rho')_0$ depends on $(E''/E')_0$,
  - for $\gamma(x)$ by $(6.12)_5$ (periodic plate band) and for pairs of ratios $(E''/E', \rho''/\rho')$ such that $(E''/E')_1 > E''/E' > (E''/E')_0 > 0$, $(\rho''/\rho')_1((E''/E')_1) < \rho''/\rho' < (\rho''/\rho')_0((E''/E')_0) > 0$, where $(\rho''/\rho')_0$, $(\rho''/\rho')_1$ depend on $(E''/E')_0$, $(E''/E')_1$, respectively,
  - for $\gamma(x)$ by $(6.12)_2$ and for pairs of ratios $(E''/E', \rho''/\rho')$ such that $E''/E' > (E''/E')_1 > 0$, $(\rho''/\rho')_2((E''/E')_2) < \rho''/\rho' < (\rho''/\rho')_1((E''/E')_1) > 0$, where $(\rho''/\rho')_1$, $(\rho''/\rho')_2$ depend on $(E''/E')_1$, $(E''/E')_2$, respectively,
  - for $\gamma(x)$ by $(6.12)_3$ and for pairs of ratios $(E''/E', \rho''/\rho')$ such that $(E''/E')_2 > E''/E' > (E''/E')_1 > 0$, $(\rho''/\rho')_2((E''/E')_2) > \rho''/\rho' < (\rho''/\rho')_1((E''/E')_1) > 0$, where $(\rho''/\rho')_1$, $(\rho''/\rho')_2$ depend on $(E''/E')_1$, $(E''/E')_2$, respectively.
- The smallest values of the higher frequency parameters for rather very thin plates, e.g. $d/l = 0.01$, and for stronger foundations, e.g. $\kappa = 0.05$, see Fig. 6b, are obtained for $\gamma(x)$ by $(6.12)_4$ and for all pairs of ratios $(E''/E', \rho''/\rho')$.
- The smallest values of these frequency parameters for thicker plates, e.g. $d/l > 0.01$, and for weaker foundations, e.g. $\kappa \leqslant 0.05$, see Figs. 5 and 6a, are obtained:

– for $\gamma(x)$ by $(6.12)_3$ and for all pairs of ratios $(E''/E', \rho''/\rho')$ such that $E''/E' > (E''/E')_3 > 0$, $\rho''/\rho' < (\rho''/\rho')_3((E''/E')_3) > 0$, where $(\rho''/\rho')_3$ depends on $(E''/E')_3$,
– for $\gamma(x)$ by $(6.12)_2$ and for pairs of ratios $(E''/E', \rho''/\rho')$ such that $E''/E' < (E''/E')_3 > 0$, $\rho''/\rho' > (\rho''/\rho')_3((E''/E')_3) > 0$, where $(\rho''/\rho')_3$ depends on $(E''/E')_3$.

7. Remarks

Using the tolerance modelling to the known differential equation of thin plates resting on Winkler's foundation, the averaged tolerance model equations of functionally graded plate bands are obtained. From the differential equation with non-continuous, tolerance-periodic coefficients, a system of differential equations with slowly-varying coefficients is derived. It should be noted that the tolerance model equations have terms dependent on the microstructure parameter l, and describe the effect of the microstructure size on the behaviour of these plates. However, the asymptotic model equation neglects this effect.

Free vibration frequencies of the simply supported plate band have been analysed in the example for various distribution functions of the material properties $\gamma(x)$, different ratios of material properties E''/E', ρ''/ρ' and of the plate thickness d/l as well as various parameters of foundation κ.

Analysing results of this example, it can be observed that:

1° Using both the presented models – the tolerance and the asymptotic one, lower free vibrations frequencies can be analysed.

2° Lower and higher free vibrations frequencies decrease with an increase in the ratio ρ''/ρ', but they increase with the increasing ratio E''/E'.

3° The asymptotic model cannot be applied to analyse lower free vibrations frequencies of rather very thin plates (with the ratio $d/l = 0.01$) and rather strong foundations (with the foundation parameter $\kappa = 0.05$), see Fig. 4b.

4° For thicker plates (e.g. $d/l = 0.1$) and weaker foundations ($\kappa = 5 \cdot 10^{-5}$), microstructured plates can be made applying different distribution functions of the material properties $\gamma(x)$ such that their lower fundamental free vibrations frequencies are smaller or higher than these frequencies for the homogeneous plate made of a stronger material (plates with the ratios $E''/E' = \rho''/\rho' = 1$) for different pairs of the ratios $(E''/E', \rho''/\rho')$.

Hence, the tolerance model can be used as a certain tool to analyse various vibration problems of thin functionally graded plates under consideration, for instance – higher order vibrations related to the microstructure of the plates. Other dynamic problems of such plates will be shown in forthcoming papers.

Acknowledgements

This contribution has been supported by the National Science Centre of Poland under grant No. 2011/01/N/ST8/07758.

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Manuscript received August 28, 2014; accepted for print February 10, 2015