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STABILITY OF HYBRID ROTATING SHAFT WITH SIMPLY SUPPORTED AND/OR CLAMPED ENDS IN A WEAK FORMULATION

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In this paper, a technique of dynamic stability analysis proposed for the conventional laminated structures is extended to activated shape memory alloy hybrid rotating structures axially loaded by a time-dependent force. In the stability study, the hybrid shaft is treated as a thin-walled symmetrically laminated beam containing both the conventional fibers, and the activated shape memory alloy fibers parallel to the shaft axis. The stability analysis method is developed for distributed dynamic problems with relaxed assumptions imposed on solutions. The weak form of dynamical equations of the rotating shaft is obtained using Hamilton's principle. We consider the influence of activation through the change of temperature on the stability domains of the shaft in the case when the angular velocity is constant. The force stochastic component is assumed in the form of ergodic stationary processes with continuous realisations. The study of stability analysis is based on examining properties of Liapunov's functional along a weak solution. Solution to the problem is presented for an arbitrary combination of simply supported and/or clamped boundary conditions. Formulas defining dynamic stability regions are written explicitly.

Key words: weak equation, rotating shaft, thermal activation, almost sure stability analysis

1. Introduction

The dynamic stability of isotropic elastic simply supported shafts rotating with a constant speed has been studied for several years (cf., Bishop, 1959; Parks and Pritchard, 1969; Tylikowski, 1981). The increased use of advanced composite materials in various applications has caused a great research effort in the structural dynamic and acoustic analysis of composite materials. Composite materials find an increased range of applications for high-performance rotating shafts (e.g., see Napershin and Klimov, 1986; Bauchau, 1983; Song and Librescu, 1997). The uniform stability of laminated shafts modelled as composite shells rotating with a constant angular velocity under a combined axisymmetric loading was investigated by Tylikowski (1996). Thin-walled standard angle-ply laminated tubes meet relatively easy the requirements of torsional strength and stiffness but are more flexible to bending and have specific elastic and damping properties which depend on the system geometry, physical properties of plies, and on the laminate arrangement. Such systems are also sensitive to lateral buckling. Using the Liapunov method, Pavlović *et al.* (2008) investigated the effect of rotary inertia of the shaft cross-section on almost sure stability of a rotating viscoelastic shaft.

Shape Memory Alloy (SMA) hybrid composites are a class of materials capable of changing both their stiffnesses through the application of in-plane loads and their elastic properties. The stiffness modification occurs as a result of the thermally induced martensite phase transformation of SMA fibers which are embedded in standard laminated composite structures. Young's modulus of the nitinol (nickel-titanium alloys), which is an example of such a material, increases 3 to 4 times when the temperature changes from that below M_f (i.e. in the martensite phase) to that above A_f (i.e. in the austenite phase) (Cross *et al.*, 1970). The damping of vibrations in the SMA due to internal friction exhibits also important characteristics. The low-temperature martensic phase is characterised by a large damping coefficient while the high-temperature austenic phase shows a low damping coefficient. The decrease ratio is approximately equal to $1:10$. Comprehensive studies of eigen-frequencies and eigen-functions of SMA hybrid adaptive panels with uniformly and piecewise distributed actuation are presented in papers by Anders and Rogers (1991), Baz *et al.* (1995), Krawczuk *et al.* (1997).

One of the possible ways towards improving dynamic properties and smoothing rotary motion of shafts consists in the implementation of control and semiactive control methodology. The ones considered are based on the incorporation of adaptive materials such as piezoceramics (Przybyłowicz, 2004) and shape memory alloys (Tylikowski, 2005, 2007) into the structures. Stability of rotating shafts made of a functionally graded material with piezoelectric fibers was examined by Przybyłowicz (2005).

The present work investigates dynamic stability of thin-walled shafts rotating with a constant angular velocity and subjected to an axial force having fluctuations from the constant average value. The time-dependent force component introduces new terms to dynamic equations and lead to the parametric excitation. In this paper, a technique of dynamic stability analysis proposed for conventional laminated structures is extended to activated shape memory alloy hybrid rotating structures. The hybrid elements are treated as a thin-walled symmetrically laminated beam containing both the conventional (e.g., aramid, graphite or glass) fibers, arbitrarily oriented to the laminate coordinate axis, and the activated shape memory alloy fibers parallel to the shaft axis. We will consider the influence of activation through the change of temperature on the stability domains of the shaft in the case when the force stochastic component is an ergodic stationary process with continuous realisations. The structure buckles dynamically when the axial parametric excitation becomes so large that the structure does not oscillate about the unperturbed state described by $\boldsymbol{w}$, and a new increasing mode of oscillations occurs. In order to estimate the perturbed solutions of dynamic equations, we introduce a measure of distance $\|\cdot\|$, of the solution to dynamic equations with nontrivial initial conditions from the trivial solution. We say that the trivial solution $\boldsymbol{w} = \mathbf{0}$ of the dynamic equations is almost sure asymptotically stable (cf. Kozin, 1972) if the measure of distance between the perturbed solution and the trivial solution, $\|\cdot\|$, satisfies the condition

$$P(\lim_{t\to\infty} \|\boldsymbol{w}(t,\cdot)\| = 0) = 1 \tag{1.1}$$

where P denotes the probability. Using the appropriate energy-like Liapunov functional, the sufficient stability conditions for the almost sure asymptotic stability of the shaft equilibrium are derived. Finally, the influence of SMA activation on stability regions is examined. The action of distributed controllers is reduced, in the first approximation, to bending moments and transverse forces distributed on the actuator edges. The fourth order differential operators are present in classical strong dynamic equations. The use of the Heaviside generalised functions in an analytical description of loading leads to irregularities. In order to avoid the irregular terms resulting from differentiation of the force and moment terms, the dynamic equations are written in a weak form. The weak form of systems contains only the second order derivative of displacements, and there is no need to differentiate the terms describing the loading.

The first analysis of the stability of simply supported or clamped rectangular plates in a weak formulation was due to Tylikowski (2008). The problem here is focused on the stability analysis method of the equilibrium state of beams and plates with relaxed assumptions imposed on solutions. We consider dynamical systems described by partial differential equations that include time dependent coefficients implying parametric vibrations. The Hamilton principle is used to derive the weak form of rotating beam dynamic equations. Assuming the Lagrangian as a difference of kinetic and elastic energy, and taking the viscotic force and the destabilising force as external ones, we obtain the weak form of the equations. Due to elimination of the fourth order derivative, the solutions of weak equations are not so smooth as solutions of the strong one. The classical Liapunov technique for stability analysis of continuous elements is based on choosing or generating a functional which is positive definite in the class of functions satisfying boundary conditions of a structure. The time-derivative of Liapunov functional has to be negative in some defined sense. If the constant component of the axial force is smaller than the Euler critical force, the square root of energy-like Liapunov functional is assumed as a measure of distance between the disturbed solutions and the trivial one. Substituting functions related with beam displacements and velocities as the test functions, yields indentities making algebraic transformations of the functional time-derivative easy. The transformations are performed without the previous discretisation. Sufficient stability conditions of the compressed beam are derived for commonly applied boundary conditions.

2. Dynamic equations

The shaft, treated as a thin-walled symmetrically laminated beam, contains both the conventional (e.g. graphite or glass) fibers oriented at $+\Theta$ and $-\Theta$ to the shell axis and the activated shape memory alloy fibers parallel to the shell axis. The shaft rotates with a constant angular velocity ω.

By forcing the martensite austenite transformation of the SMA layer, we modify the basic mechanical properties such as Young's modulus and the internal damping coefficient. The shaft of length ℓ is assumed to have a constant circular cross-section of mean radius a, and thickness h without initial geometrical imperfections. The mean density is denoted by ρ and the area and the geometrical moment of inertia of the shaft cross-section are denoted by $A = 2\pi Ah$ and $J = \pi a^3 h$, respectively. A viscous model of external damping with a constant proportionality coefficient b_e is assumed to describe the dissipation of the shaft energy. The beam-like approach due to Bauchau (1981) is used in order to derive the shaft bending stiffness

$$EJ = \Big(A_{11} - \frac{A_{12}^2}{A_{22}}\Big)\pi a^3 \tag{2.1}$$

where A_{ij}, $i,j = 1,2$ denote in-plane stiffnesses of the thin-walled beam. More sophisticated considerations of thin-walled composite beams were performed by Song and Librescu (1990). Displacements of the center shaft line in the movable rotating coordinates are denoted by u and v. Introducing the dimensionless time with the time scale $k_t = \ell^2\sqrt{\rho A/EJ}$ and the dimensionless coordinate divided by ℓ, we obtain a shaft model with the unit mass density, unit bending stiffness, dimensionless angular velocity $\Omega = \omega k_t$, and modified damping coefficients of external and internal damping $\beta_e = b_e k_t$, $\beta_i = b_i k_t$, respectively. Starting from the rotating shaft without damping and axial loading, we write the action integral as a time integral of the difference between kinetic and bending energy

$$\mathcal{A}(\boldsymbol{w}) = \frac{1}{2}\int\limits_{t_o}^{t}\int\limits_{0}^{1}[(u_{,t} + \Omega v)^2 + (v_{,t} - \Omega u)^2 - (u_{,xx}^2 + v_{,xx}^2)]\;dx\;dt \tag{2.2}$$

where $\boldsymbol{w} = [u,v]^\top \in \mathcal{W} = [H_b^2(0,1)]^2$, the index b denotes the set of functions satisfying the essential boundary conditions, the time interval (t,t_o) is arbitrarily chosen. Consider

$$\widehat{w} = \boldsymbol{w} + \epsilon\boldsymbol{\theta} = \begin{bmatrix} u(t,x) \\ v(t,x) \end{bmatrix} + \epsilon\begin{bmatrix} \eta_1(t)\Phi(x) \\ \eta_2(t)\Psi(x) \end{bmatrix} \tag{2.3}$$

where $\boldsymbol{\eta}(t) = \boldsymbol{\eta}(t_o)$, $\boldsymbol{\theta} \in \mathcal{W}$. According to Hamilton's principle, of motion the shaft must have a stationary value to the action integral, therefore

$$\frac{d}{d\epsilon}\mathcal{A}(\boldsymbol{u} + \epsilon\boldsymbol{\theta})\Big|_{\epsilon=0} = 0 \tag{2.4}$$

Using equation (2.2) in equation (2.3) and integrating the time-derivatives of functions η_1 and η_2, by parts with respect to time we obtain dynamic equations of the shaft in a weak form

$$\begin{aligned} &\forall\Phi \qquad \int\limits_0^1 [(u_{,tt} - \Omega^2 u + 2\Omega v_{,t})\Phi + u_{,xx}\Phi_{,xx}]\;dx = 0 \\ &\forall\Psi \qquad \int\limits_0^1 [(v_{,tt} - \Omega^2 v - 2\Omega u_{,t})\Psi + v_{,xx}\Psi_{,xx}]\;dx = 0 \end{aligned} \tag{2.5}$$

Adding the internal viscous damping with the modified coefficient β_i, the external viscous damping with the modified coefficient β_e and the axial force

$f_o+f(t)$ as external works, the shaft dynamic equations can be written in the weak form as follows

$$
\begin{aligned}
&\forall \varPhi \qquad \int_0^1 [(u_{,tt} - \varOmega^2 u + 2\varOmega v_{,t})\varPhi + \beta_e(u_{,t} + \varOmega v)\varPhi + \beta_i u_{,t}\varPhi + \\
&\qquad\qquad + u_{,xx}\varPhi_{,xx} + (f_o + f(t))u_{,xx}\varPhi]\,dx = 0 \\
&\forall \varPsi \qquad \int_0^1 [(v_{,tt} - \varOmega^2 v - 2\varOmega u_{,t})\varPsi + \beta_e(v_{,t} - \varOmega u)\varPsi + \beta_i v_{,t}\varPsi + \\
&\qquad\qquad + v_{,xx}\varPsi_{,xx} + (f_o + f(t))v_{,xx}\varPsi]\,dx = 0
\end{aligned}
\tag{2.6}
$$

where $\varPhi$, $\varPsi$ are arbitrary sufficiently smooth test functions satisfying essential boundary conditions. There is no demand for the existence of derivatives higher than the second order. As reported by Banks *et al.* (1993), the usual integration by parts of the terms containing derivatives of the test functions with respect to the variable x and the assumption of sufficient smoothness of the components of shaft displacements lead to the commonly used strong formulation. The shaft is assumed to be simply supported or clamped on its ends. Therefore, the essential boundary conditions have the following form at its ends

$$
u(t,0) = u(t,1) = v(t,0) = v(t,1) = 0 \tag{2.7}
$$

It means that the displacements of the shaft in supporting bearings are small as compared with the displacements of a thin-walled flexible shaft. Weak linear equations (2.6) have the trivial solution (equilibrium state) $u = v = 0$.

3. Stability analysis

In order to determine conditions of smooth shaft motion corresponding to the Liapunov stability of the trivial solution $\boldsymbol{w} = \boldsymbol{0}$, we choose the positive-definite functional Liapunov as a sum of the modified kinetic and elastic energy of the shaft (Tylikowski, 2007)

$$
\begin{aligned}
V = \frac{1}{2}\int_0^1 &[(u_{,t} + \varOmega v + \beta u)^2 + (u_{,t} + \varOmega v)^2 + (v_{,t} - \varOmega u + \beta v)^2 + \\
&+ (v_{,t} - \varOmega u)^2 + 2(u_{,xx}^2 + v_{,xx}^2) - 2f_o(u_{,x}^2 + v_{,x}^2)]\,dx
\end{aligned}
\tag{3.1}
$$

where $\beta = \beta_i + \beta_e$. Functional (3.1) is positive-definite if the constant component of the axial force f_o fulfils the static buckling condition, i.e. is sufficiently small. Therefore, the measure of distance of solutions with nontrivial initial conditions from the trivial required in stability analysis can be chosen as a square root of the functional $\|\boldsymbol{w}\| = \sqrt{V}$. As trajectories of the solution to equations (2.6) are physically realisable, the classical calculus is applied to calculation of time-derivative of functional (3.1). Its time-derivative is given by

$$\begin{aligned}\frac{dV}{dt} = \int\limits_0^1 &[(u_{,t} + \Omega v + \beta u)(u_{,tt} + \Omega v_{,t} + \beta u_{,t}) + (u_{,t} + \Omega v)(u_{,tt} + \Omega v_{,t}) + \\ &+(v_{,t} - \Omega u + \beta v)(v_{,tt} - \Omega u_{,t} + \beta v_{,t}) + (v_{,t} - \Omega u)(v_{,tt} - \Omega u_{,t}) + \\ &+2(u_{,xx}u_{,xxt} + v_{,xx}v_{,xxt}) - 2f_o(u_{,x}u_{,xt} + v_{,x}v_{,xt})]\;dx\end{aligned} \tag{3.2}$$

In order to avoid integration by parts in equations (3.2) and generation the third and the fourth partial derivatives of displacements, we substitute $2u_{,t}$, $2\Omega v$, βu as the test functions in equation $(2.6)_1$. Therefore, we have three identities, respectively

$$\begin{aligned}&\int\limits_0^1 [2(u_{,tt} - \Omega^2 u + 2\Omega v_{,t})u_{,t} + 2\beta u_{,t}^2 + 2\beta_e \Omega v u_{,t} + 2u_{,xx}u_{,xxt} + \\ &\qquad +2(f_o + f(t))u_{,xx}u_{,t}]\;dx = 0 \\ &\int\limits_0^1 [2\Omega(u_{,tt} - \Omega^2 u + 2\Omega v_{,t})v + 2\Omega\beta_e(u_{,t} + \Omega v)v + 2\Omega\beta_i u_{,t}v + \\ &\qquad +2\Omega u_{,xx}v_{,xx} + 2\Omega(f_o + f(t))u_{,xx}v]\;dx = 0 \\ &\int\limits_0^1 [\beta(u_{,tt} - \Omega^2 u + 2\Omega v_{,t})u + \beta\beta_e(u_{,t} + \Omega v)u + \beta\beta_i u_{,t}u + \\ &\qquad +\beta u_{,xx}^2 + \beta(f_o + f(t))u_{,xx}u]\;dx = 0\end{aligned} \tag{3.3}$$

In a similar way we substitute $2v_{,t}$, $2\Omega u$, βv as the test functions in equation $(2.6)_2$

$$\begin{aligned}&\int\limits_0^1 [2(v_{,tt} - \Omega^2 v - 2\Omega u_{,t})v_{,t} + 2\beta v_{,t}^2 - 2\beta_e \Omega u v_{,t} + \\ &\qquad +2v_{,xx}v_{,xxt} + 2(f_o + f(t))v_{,xx}v_{,t}]\;dx = 0\end{aligned}$$

$$\int_0^1 [2\Omega(v_{,tt} - \Omega^2 u - 2\Omega u_{,t})v + 2\Omega\beta_e(v_{,t} - \Omega u)u + 2\Omega\beta_i v_{,t}u +$$
$$+2\Omega v_{,xx}u_{,xx} + 2\Omega(f_o + f(t))v_{,xx}u]\, dx = 0 \tag{3.4}$$
$$\int_0^1 [\beta(v_{,tt} - \Omega^2 v - 2\Omega u_{,t})v + \beta\beta_e(v_{,t} - \Omega u)v + \beta\beta_i v_{,t}v +$$
$$+\beta v_{,xx}^2 + \beta(f_o + f(t))v_{,xx}v]\, dx = 0$$

Subtracting identities (3.3), $(3.4)_1$ and $(3.4)_3$ from equation (3.2) and adding identity $(3.4)_2$ we obtain the following form

$$\frac{dV}{dt} = -\int_0^1 \{\beta(u_{,t}^2 + v_{,t}^2) + \beta(u_{,xx}^2 + v_{,xx}^2) - \beta f_o(u_{,x}^2 + v_{,x}^2) +$$
$$+(\beta_e - \beta_i)\omega^2(u^2 + v^2) + 2\Omega\beta_e(vu_{,t} - uv_{,t}) + \tag{3.5}$$
$$+[(2u_{,t} + \beta u)u_{,xx} + (2v_{,t} + \beta v)v_{,xx}]f(t)\}\, dx$$

After rewriting, we receive

$$\frac{dV}{dt} = -\beta_e V + U \tag{3.6}$$

where the auxiliary functional U is known. Now, we look for a function χ which satisfies the following inequality

$$U \leqslant \chi V \tag{3.7}$$

Substituting inequality (3.7) into equation (3.6) yields the first order differential inequality

$$\frac{dV}{dt} \leqslant (\chi - \beta_e)V \tag{3.8}$$

which implies

$$\|w(t,x)\| \leqslant \|w(0,x)\| \exp\Big[-\Big(\beta_e - \frac{1}{t}\int_0^t \chi(\tau)\, d\tau\Big)t\Big] \tag{3.9}$$

The ergodicity of the axial loading leads to the following almost sure stochastic stability condition

$$\overline{\mathrm{E}}\chi(t) \leqslant \beta_e \tag{3.10}$$

where $\overline{\mathrm{E}}$ denotes the mean value operator. It should be noticed that the way to obtain equation (3.5) is purely algebraic contrary to systems described by strong equations, where the derivation of stability conditions is based on integrations by parts and manipulations with higher order partial derivatives. Usually, the Liapunov stability analysis of shafts is performed for both ends simply supported (cf. Bishop, 1959; Tylikowski, 1996). In order to extend the field of possible applications, let us assume the following combinations of boundary conditions: a) s-s, b) c-s, c) c-c, where s denotes the simply supported end, and c denotes the clamped end.

In order to find χ effectively, we use the expansions of the shaft displacements

$$\begin{bmatrix} u(t,x) \\ v(t,x) \end{bmatrix} = \sum_{n=1}^{\infty} \mathcal{W}_n(x) \begin{bmatrix} S_n(t) \\ T_n(t) \end{bmatrix} \tag{3.11}$$

where $\mathcal{W}_n$ are the beam functions (cf. Graff, 1975) depending on the assumed boundary conditions. In a similar way, velocities of transverse shaft motion are given by

$$\begin{bmatrix} u_{,t}(t,x) \\ v_{,t}(t,x) \end{bmatrix} = \sum_{n=1}^{\infty} \mathcal{W}_n(x) \begin{bmatrix} \dot{S}_n(t) \\ \dot{T}_n(t) \end{bmatrix} \tag{3.12}$$

Integrating, we have the following equality (Tylikowski, 2008)

$$\begin{aligned} \int_0^1 \mathcal{W}_{n,xx}^2 \, dx &= \gamma_n \alpha_n^2 \int_0^1 \mathcal{W}_{n,x}^2 \, dx \\ \int_0^1 \mathcal{W}_{n,x}^2 \, dx &= \frac{\alpha_n^2}{\gamma_n} \int_0^1 \mathcal{W}_n^2 \, dx \end{aligned} \tag{3.13}$$

where α_n is an eigen-value of the corresponding boundary problem and the sequence $\{\gamma_n\}$ is known. Due to the existence of even-order space derivatives in functional (3.1) and in its time-derivative (3.5), the value of functionals can be calculated as a sum of suitable quadratic terms

$$V = \sum_{n=1}^{\infty} V_n \qquad\qquad U = \sum_{n=1}^{\infty} U_n \tag{3.14}$$

where V_n and U_n are calculated for a single term of expansions (3.11) and (3.12). If χ_n, which satisfies a single term inequality, is known

$$\frac{dV_n}{dt} \leqslant (\chi_n - \beta_e) V_n \tag{3.15}$$

then the function χ can be effectively calculated

$$\chi = \max_{n=1,2,\dots} \chi_n \tag{3.16}$$

Denoting $\kappa_n = \chi_n - \beta_e$ and substituting the n-th terms of expansions (3.11) and (3.12) into inequality (3.15), we obtain the second order quadratic inequality with respect to the four variables $\dot{S}_n$, S_n, $\dot{T}_n$, T_n

$$\begin{aligned}
&\Big[\dot{S}_n^2 + T_n^2\Omega^2 + 2\dot{S}_n T_n \Omega + \frac{1}{2}\beta^2 S_n^2 + \Big(\beta - \frac{2\alpha_n^2}{\gamma_n} f(t)\Big)\dot{S}_n S_n + \dot{T}_n^2 + S_n^2\Omega^2 + \\
&-2\dot{T}_n S_n + \frac{1}{2}\beta^2 T_n^2 + \Big(\beta - \frac{2\alpha_n^2}{\gamma_n} f(t)\Big)\dot{T}_n T_n + \Big(\alpha_n^2 - \frac{f_o}{\gamma_n}\Big)\alpha_n^2 (S_n^2 + T_n^2)\Big]\kappa_n + \\
&+(\dot{S}_n^2 + \dot{T}_n^2)\beta + 2\beta_e \Omega(\dot{S}_n T_n - S_n \dot{T}_n) + \\
&+\Big[(\beta_e - \beta_i)\Omega^2 + \Big(\alpha_n^2 - \frac{f_o}{\gamma_n}\Big)\alpha_n^2 - \beta\frac{2\alpha_n^2}{\gamma_n} f(t)\Big](S_n^2 + T_n^2) \geqslant 0
\end{aligned} \tag{3.17}$$

After some reduction, we obtain an auxiliary matrix of quadratic form (3.17)

$$\begin{bmatrix} a & b & 0 & d \\ b & c & -d & 0 \\ 0 & -d & a & b \\ d & 0 & b & c \end{bmatrix} \tag{3.18}$$

where

$$a = \kappa_n + \beta \qquad b = \frac{1}{2}\beta\kappa_n - \frac{2\alpha_n^2}{\gamma_n} f(t) \qquad d = \Omega(\beta_e + \kappa_n)$$

$$c = \kappa_n\Big[\alpha_n^2\Big(\alpha_n^2 - \frac{f_o}{\gamma_n}\Big) + \Omega^2 + \frac{\beta^2}{2}\Big] + \beta\alpha_n^2\Big(\alpha_n^2 - \frac{f_o}{\gamma_n}\Big) + \Omega^2(\beta_e - \beta_i) - \frac{\beta\alpha_n^2}{\gamma_n} f(t)$$

We recall Sylvester's conditions for the positive-definiteness of matrix (3.18)

$$\begin{aligned} &a > 0 && ac - b^2 > 0 \\ &a(ac - b^2 - d^2) > 0 && (b^2 + d^2 - ac)^2 > 0 \end{aligned} \tag{3.19}$$

As latest inequality $(3.19)_4$ is satisfied, it is easy to notice that third Sylvester inequality $(3.19)_3$ is essential from the stability point of view. It is equivalent to the elementary second order inequality with respect to κ_n

$$\kappa_n^2 + 2\beta\kappa_n + 4\frac{\beta^2\alpha_n^2(\alpha_n^2 - f_o/\gamma_n) - \beta_i^2\Omega^2}{\beta^2 + 4\alpha_n^2(\alpha_n^2 - f_o/\gamma_n)} > 0 \tag{3.20}$$

which leads to the determination of κ_n and finally to the almost sure stability condition from equations (3.16) and (3.10)

$$\overline{\mathrm{E}}\chi = \overline{\mathrm{E}} \max_{n=1,2,\dots} \sqrt{\frac{\beta^4 + 4(\beta^2 + \alpha_n^2 f(t)/\gamma_n)\alpha_n^2 f(t)/\gamma_n + 4\Omega^2\beta_i^2}{\beta^2 + 4\alpha_n^2(\alpha_n^2 - f_o/\gamma_n)}} \leqslant \beta \qquad (3.21)$$

Estimating the limit behaviour of χ_n as n tends to ∞, we find the critical value of angular velocity for a constant axial force f_o

$$\Omega^2 < \Big(1 + \frac{\beta_e}{\beta_i}\Big)^2 \alpha_1^2\Big(\alpha_1^2 - \frac{f_o}{\gamma_1}\Big) \qquad (3.22)$$

The function χ in inequality (3.21) is random due to randomness of the axial force $f(t)$. Therefore, the probability distribution must be known in order to calculate the average in equation (3.21).

4. Results

Numerical calculations based on formula (3.21) are performed for s-s boundary conditions with the changing time-dependent component of the axial force and the coefficient β_e of external damping. A number of iterative steps are performed in order to determine the value of β_e. The dimensions of hybrid shafts are: length $\ell = 1\,\mathrm{m}$, radius $r = 0.04\,\mathrm{m}$, total thickness $h = 0.004\,\mathrm{m}$. The material data are given in Table 1.

Table 1. SMA hybrid shaft specification

Material	Nitinol-Epoxy NiTi – 40%, Epoxy – 60%		Glass--Epoxy	Graphite--Epoxy
	activated	unactivated		
Density [kg/m^3]	2350	2350	1790	1560
E_{11} [GPa]	41.93	19.31	53.98	211.0
E_{22} [GPa]	20.93	17.25	17.93	5.30
G_{12} [GPa]	7.54	6.43	8.96	2.60
ν_{12}	0.25	0.25	0.25	0.25
β_i	0.01	0.012	0.01	0.01

The shaft consists of seven layers of equal thickness: of two external layers with activated SMA fibers parallel to the shaft axis and of five internal conventional layers symmetrically arranged with the lamination angle $\pi/4$.

Thus, the laminate configuration can be uniquely defined by the following notation: $[0 \ / \ (\pi/4) \ / \ (-\pi/4) \ / \ (\pi/4) \ / \ (-\pi/4) \ / \ (\pi/4) \ /0]$. We can calculate the in-plane stiffnesses A_{ij} (c.f. Whitney, 1987) and then also from Eq. (2.1) the reduced Young modulus E of the beam-like cylindrical shell.

The main results are shown in Fig. 1. The figure compares the stochastic stability domains in the plane (β_e, σ^2) for the zero and nearly critical axial force. The critical variance of the time-dependent component of the axial force strongly depends on the external damping coefficient β_e. The time scale in the activated and unactivated state is $k_t = 1.2697 \cdot 10^{-2}\,\mathrm{s}$, $k_t = 1.0635 \cdot 10^{-2}\,\mathrm{s}$, respectively. An increase of angular velocity significantly decreases the stability region. The thermal activation significantly increases the stability regions due to increasing stiffness of the nitinol fibers. The influence of the shaft material is not observed in Fig. 1 due to the dimensionless quantities.

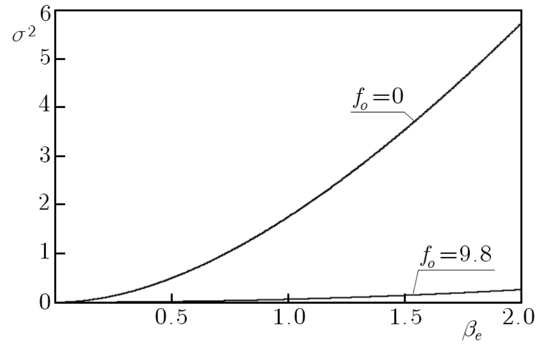

Fig. 1. Boundaries of stability domains for zero and nearly critical axial force components

5. Conclusions

A technique has been presented for the analysis of dynamic stability of an activated simply supported hybrid shaft rotating with a constant angular velocity. The shaft consists of classical symmetrically angle-plied layers and symmetrically laminated active plies with axially oriented SMA fibers. The dynamic stability and the stochastic stability problem is reduced to the problem of positive definiteness of the auxiliary matrix. The explicit criteria derived in the paper define stability regions in terms of geometrical and material properties, lamination angle as well as constant values and variances of the axial force. For a constant axial force, the criterion asssumes a closed form of an algebraic inequality. If the axial force is time-dependent, the almost sure stability criterion has a form of a transcendental equation involving the axial force probability distribution. Analytical results are presented to demonstrate how the thermal activation affects critical parameters. The influence of fluctuation class (Gaussian or harmonic) is not significant. The influence of boundary conditions on stability domains is negligible when the constant component of the axial force is small as compared with the critical loading.

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Słabe sformułowanie stabilności hybrydowych obracających się wałów swobodnie podpartych i sztywnie utwierdzonych

Streszczenie

W pracy rozszerzono możliwości analizy stabilności układów ciągłych na obracający się wał hybrydowy poddany czasowo zmiennej sile osiowej przy osłabionych założeniach spełnianych przez rozwiązania. Kompozytowy wał hybrydowy obracający się ze stałą prędkością kątową traktowany jest jako cienkościenna belka zawierającą

obok klasycznych włókien również włókna wykonane z materiału z pamięcią kształtu. Słabą postać równań ruchu wału wyprowadzono z zasady Hamiltona. Rozpatrzony jest wpływ aktywacji termicznej na obszar stabilości wału przy założeniu nie tylko swobodnego podparcia obu końców wału, lecz również przy podparciu utwierdzonym i mieszanym. Podczas wyprowadzania warunków stabilności korzysta się z badania właściwości funkcjonału Lapunowa wzdłuż rozwiązania słabych równań ruchu wału. Wyprowadzono jawną postać warunków stabilności.

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