JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
45, 1, pp. 161-169, Warsaw 2007

ON THE STABILITY BEHAVIOUR OF ELASTIC BEAMS UNDER INTERNAL PRESSURE

Lena Zentner
Valter Böhm
Hartmut Witte

Faculty of Mechanical Engineering, Technical University of Ilmenau, Germany
e-mail: Lena.Zentner@tu-ilmenau.de; Valter.Boehm@tu-ilmenau.de; Hartmut.Witte@tu-ilmenau.de

Scientific reception of the term "stability" stresses steady adaptation to its changing fields of application. Nevertheless, the determination of critical forces remains one of the main tasks of stability theories. We exemplify some classes of the stability loss in beams under internal pressure for the static case. Additionally, we analyse in more detail the dynamic stability of beams under internal pressure and demonstrate means to keep an equilibrium.

Key words: stability, elastic beam, dynamic modelling

1. Static cases of loss of stability

Best known examples for the loss of stability under static loads are the Eulerian cases of stability (Leipholz, 1968). For undercritical loads, the equilibrium is determinate, while at critical loads bifurcations in the solutions to state equations occur. Solutions are no more bijective, one load situation may lead to more than one possible geometric configuration of the system. Figure 1 exemplifies this phenomenon in the case of a half-cycle shaped bending beam subject to loading by a single force with constant direction (conservative force) but with a 2D-free floating location of the site of application under load. Two possible trajectories of this point and two realisations of the equilibrium are illustrated. Solutions have been determined numerically, a current application is the design of compliant grasping devices.

Another well-known problem of the loss of stability is buckling. As an example, Fig. 2 shows a valve coherent in the material; due to symmetry only one half of it is drawn. Other examples for statical stability of fully compliant mechanisms are presented in Huba *et al.* (2002).

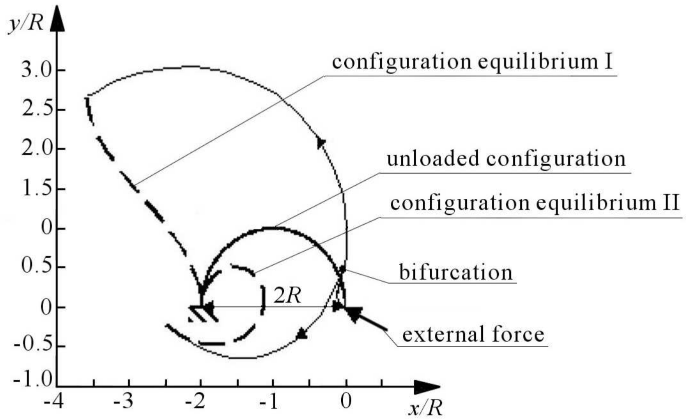

Fig. 1. Equilibrium situations of a half-cycle shaped beam under external load. A force constant in amount and direction traces the free end of the beam

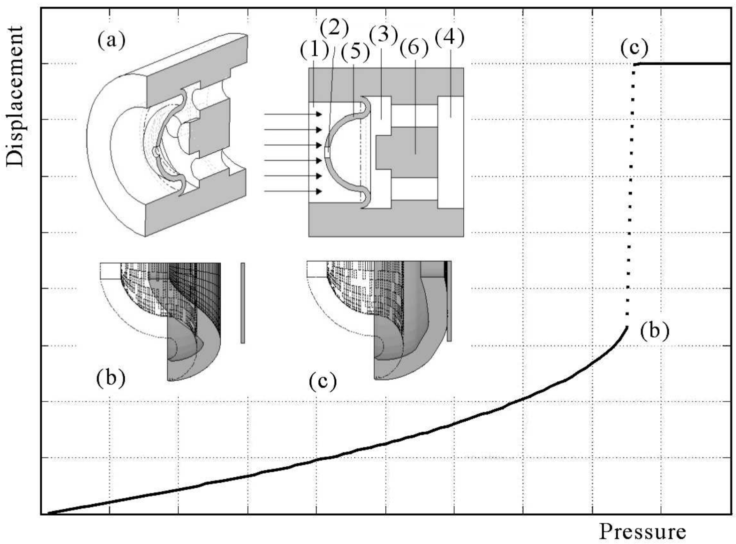

Fig. 2. A valve using compliant mechanisms: pressure-displacement relation; (a) principle of the valve, (b) configuration of the membrane below the lower pressure threshold of the two-point switch mechanism ("open"), (c) configuration of the membrane above the threshold ("closed")

The valve serves to prevent region (4) in Fig. 2 from pressure overload. Due to different diameters and thus different cross–sectional areas in regions (1) to (3), different pressures occur. Depending on pressure differences along the flow line from (1) to (3), elastic membrane (5) is deflected towards plate (6). If the pressure difference between (1) and (3) exceeds a predefined threshold, the valve closes. Calculation of the stress strain relations in the membrane may be based on a quasi-static attempt, if conditions allow to neglect inertial effects of the fluid.

High deflections provoke only small stresses and strains, functions and shape constancy are guaranteed for a large number of cycles, providing short and constant switching times with constant diameters of the valve in all "open" states. Figures 2b and 2c compare the unswitched and switched states as far as the geometry is concerned. Additionally, an example of this problem is described in Huba *et al.* (2002).

2. Dynamic cases of loss of stability

A number of systems, especially non-conservative ones, may only insufficiently be described by static approaches (Djanelidse, 1958). For example, if a straight bending beam is subject to a load like illustrated in Fig. 3, calculations based on static approaches do not identify unstable situations, which, nevertheless, may be found when using dynamic methods. Below the threshold of the critical load, one stable equilibrium exists. The critical load induces mo-

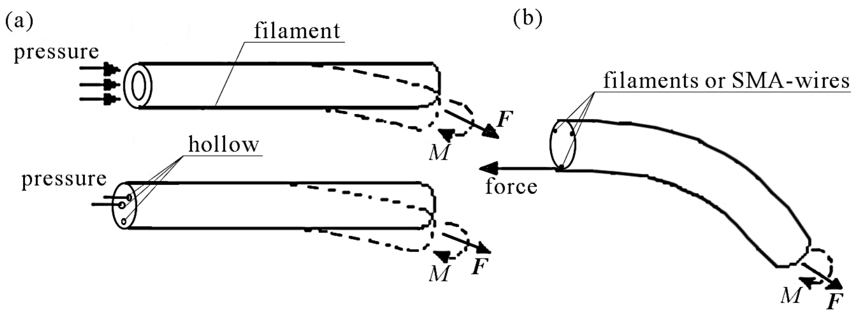

Fig. 3. Deformation of compliant beam structures loaded by external forces and/or bending moment applied to the free-moving end of the beam. Resistance is provided by elasticity of the beam in combination with internal pressure and/or influence of filaments

tion, which may lead to stable situations as well as to unstable ones. Methods to quantitatively identify these situations and to derive related stability criteria are mostly lacking in literature, thus development of methodology in this field is an important task for the future. Our contribution to this duty will be the dynamical analysis of a hollow beam under the influence of internal pressure and of non-conservative forces. The technical application we aim at is a pneumatic "finger" for grasping, especially for manipulation. Due to the underlying technology, we would like to know the influence of internal pressure or length-constant filaments in the wall of the structure on the stability behaviour of the structure. The effects of internal pressure on deformation may be represented by one force and a bending moment acting on the moving end of the beam.

The deformation of the beam is expressed as a function of time and spacial coordinate x (Fig. 4). The external force is provoked by a point-like mass m attached to the moving end of the beam, which performs small movements v parallel to the y-axis. Thus, redistribution of the mass reduces the kinetic energy balance to the analysis of states of the point-like mass, without neglecting the elasticity of the beam. As a consequence, the reduced model provides higher amplitudes and smaller critical loads than a more complex model based on continuum mechanics. Thus, the results of the calculations assure additional security, especially as far as critical loads are concerned.

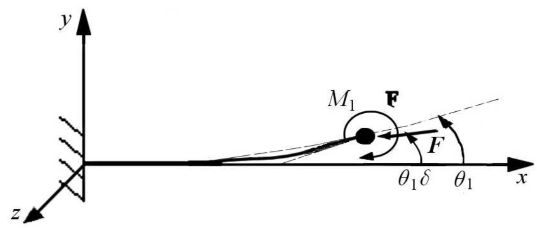

Fig. 4. A dynamic case of loading of the beam by non-conservative forces and the bending moment

The force acting on the beam generated by the point-like mass is

$$F_k = -m\ddot{v}_l \tag{2.1}$$

The bending moment of the beam $M = EI_z v''$ may be expressed as

$$EI_z v'' = F(v_l - v) + (F_k - F\theta_l\delta)(l - x) + M_l \tag{2.2}$$

In further considerations a dynamic model of the material will be used.

Suppose that F_k, v_l and θ_l are independent of x, then the solution of the previous equation is

$$v(x) = A\sin\lambda x + B\cos\lambda x + \Big(\theta_l\delta - \frac{F_k}{\lambda^2 EI_z}\Big)x + (v_l - \theta_l\delta l) - \frac{F_k l + M_l}{\lambda^2 EI_z} \tag{2.3}$$

with

$$\lambda^2 = \frac{F}{EI_z}$$

The bending moment M_l is supposed to be proportional to the angle θ_l by a factor k

$$M_l = -k\theta_l \tag{2.4}$$

In combination with boundary conditions $v(0) = 0$ and $v'(0) = 0$, these equations yield

$$v(x) = \Big(\theta_l\delta - \frac{F_k}{\lambda^2 EI_z}\Big)\Big(x - \frac{\sin\lambda x}{\lambda}\Big) + \Big(v_l - \theta_l\delta l - \frac{F_k l + M_l}{\lambda^2 EI_z}\Big)(1 - \cos\lambda x) \tag{2.5}$$

Taking into account the two additional boundary conditions $v(l) = v_l$ and $v'(l) = \theta_l$ allows the derivation of the following equations

$$\begin{aligned} v_l &= \Big(\theta_l\delta - \frac{F_k}{\lambda^2 EI_z}\Big)\Big(l - \frac{\sin\lambda l}{\lambda}\Big) + \Big(v_l - \theta_l\delta l - \frac{F_k l + M_l}{\lambda^2 EI_z}\Big)(1 - \cos\lambda l) \\ \theta_l &= \Big(\theta_l\delta - \frac{F_k}{\lambda^2 EI_z}\Big)(1 - \sin\lambda l) + \Big(v_l - \theta_l\delta l - \frac{F_k l + M_l}{\lambda^2 EI_z}\Big)\lambda\sin\lambda l \end{aligned} \tag{2.6}$$

Knowing conditions (2.4) for the moment and force F_k, we eliminate θ_l and get the equation of motion for the point-like mass

$$\ddot{v}_l + \omega^2 v_l = 0 \tag{2.7}$$

with the natural frequency ω

$$\omega^2 = \frac{\lambda^3 EI_z\Big(\lambda(\delta - 1)\cos\lambda l - \frac{k}{EI_z}\sin\lambda l - \delta\lambda\Big)}{mBigl[\Big(2\frac{k}{EI_z} + \lambda^2 l\Big)\cos\lambda l - \lambda\Big(1 - \frac{k}{EI_z}l\Big)\sin\lambda l - 2\frac{k}{EI_z}\Big]} \tag{2.8}$$

The sign of $\omega^2 \neq 0$ decides the solution to equation (2.7)

$$v_l = \begin{cases} K_1\sin\omega t + K_2\cos\omega t & \text{for} \quad \omega^2 > 0 \\ K_3\sinh\omega t + K_4\cosh\omega t & \text{for} \quad \omega^2 < 0 \end{cases} \tag{2.9}$$

In the case of $\omega^2 > 0$, the amplitude is constant. For $\omega^2 < 0$ the solution tends to infinity. The sign of ω^2 may change if the sign of the denominator or of the numerator in (2.8) changes. These events occur if

$$\lambda(\delta - 1)\cos\lambda l - \frac{k}{EI_z}\sin\lambda l - \delta\lambda = 0 \tag{2.10}$$

or if

$$\Big(2\frac{k}{EI_z} + \lambda^2 l\Big)\cos\lambda l - \lambda\Big(1 - \frac{k}{EI_z}l\Big)\sin\lambda l - 2\frac{k}{EI_z} = 0 \tag{2.11}$$

Introduction of dimensionless parameters $\widetilde{F} = \frac{Fl^2}{EI_z}$ and $\widetilde{k} = kl/(EI_z)$ facilitates the further procedure. We transform λ and k to dimensionless parameters

$$\lambda = \frac{\sqrt{\widetilde{F}}}{l} \qquad\qquad k = \frac{\widetilde{k}EI_z}{l}$$

Using (2.10) and (2.11), we get relationships for $\widetilde{k}(\widetilde{F})$ (Fig. 5)

$$\begin{aligned}
\widetilde{k} &= -\frac{\sqrt{\widetilde{F}}}{\sin\sqrt{\widetilde{F}}}[\cos\sqrt{\widetilde{F}}(1-\delta)+\delta] \\
\widetilde{k} &= \frac{\sin\sqrt{\widetilde{F}}-\cos\sqrt{\widetilde{F}}}{\sqrt{\widetilde{F}}\sin\sqrt{\widetilde{F}}+2\cos\sqrt{\widetilde{F}}-2}
\end{aligned} \tag{2.12}$$

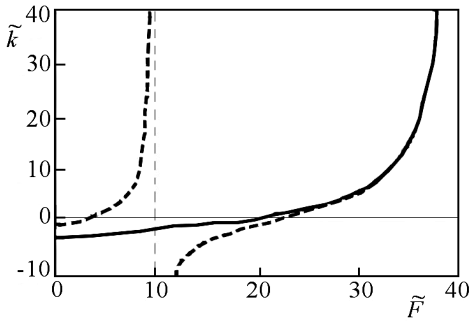

Fig. 5. Critical loads from equation $(2.12)_1$ (dashed line) and $(2.12)_2$ (solid line) in the $(\widetilde{k},\widetilde{F})$ plane

The asymptote of the graph of equation $(2.12)_2$ coincides with the second asymptote described by equation $(2.12)_1$ and is determined from the formula

$$\sqrt{\widetilde{F}}\sin\sqrt{\widetilde{F}}+2\cos\sqrt{\widetilde{F}}=2\widetilde{k}$$

and the bending moment are indeterminate for $\widetilde{F}=0$. In Fig. 5, the straight line of $(2.12)_2$ intersects the function derived in equation $(2.12)_1$. Consecutively, an *a priori* exclusion of the critical loads is impossible. In the case of $\widetilde{k}=0$, a simple change of the force direction increases the critical force by a factor of ten. Dashed line I in Fig. 6 illustrates the strategy to avoid instability. Line II shows the positive effect of an additional external moment on the avoidance of the critical loads.

Graphic representations of $\widetilde{k}(\widetilde{F},\delta)$ allow the derivation of strategies to avoid the critical forces by guiding the processes on "uncritical pathes". One of those solutions is given by the dashed line in Fig. 7. The price to be paid for this avoidance is control: preemptive (planning) or current (Zentner, 2003).

Introduction of a bending moment acting as an extended load not only provides more general solutions to stability problems, but also offers access to new strategies for the avoidance of instable phases in the processes. Superposition of loads helps in "shifting" the critical loads into uncritical zones of the

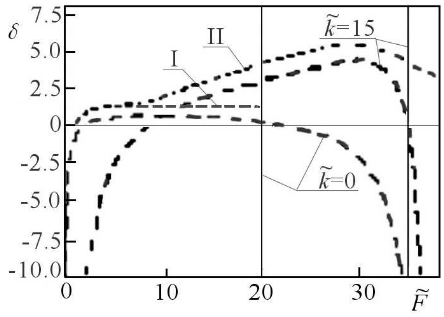

Fig. 6. Critical loads from equation $(2.12)_1$ (dashed line) and $(2.12)_2$ (solid line) in plane $(\delta, \widetilde{F})$ for $\widetilde{k} = 0$ (line I) and $\widetilde{k} = 15$ (line II). Adaptive change of loads allows for avoidance of critical loads

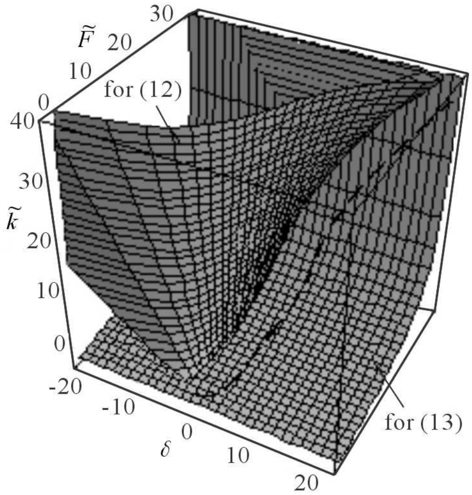

Fig. 7. Critical loads, determined by equations (2.12); dashed line: one possible strategy for the guidance of the loading process avoiding critical situations

load condition, as illustrated in Fig. 6. Physical realisation of such additional loads enable one to use embedded filaments in the basis structure (Fig. 3), which provoke moments under an external force. Thus, the moment is realised under dangerous load conditions, without the use of additional sensors.

Our results cover those of Djanelidse (1958) for a static case without an external bending moment ($\widetilde{k} = 0$). For $\widetilde{k} = 0$ and $\delta = 0$, equation $(2.12)_1$ provides a formula $\cos\sqrt{\widetilde{F}} = 0$, where the eigenvalues equal the critical force in the Euleian sense.

3. Conclusions

Our attempts to model the behaviour of compliant mechanisms, like pneumatic manipulator "fingers", mechanically identified the needs to extend methods for the determination of critical loads. For this purpose, we analysed beams under a load by internal pressure in combination with external forces and/or bending moments. In some loading situations of the pneumatic "finger", the analysis of the derived equations allowed a multifold increase of the critical loads. Guidance of the loading process was required to trace uncritical regions of parameter combinations describing the equilibrium situation.

Of special impact on the increase of the critical loads is the controlled overlay of bending moments onto the forces applied. In situations with given external loads, this overlay allows the minimisation of structural weight.

References

1. Djanelidse G., 1958, Ob ustojchiwosti sterschnja pri dejstwii sledjatschej sily, *Trudy Leningradskogo Politechnicheskogo Instituta*, **192**

2. Huba A., Molnár L., Takács Á., 2002, Hohlraumsonde mit wurmförmiger Bewegung, *ISOM 2002. Advanced Driving Systems, Ist. Internat. Symp. on Mechatronics*, Chemnitz, 466-470

3. Huba A., Molnár L., 2002, Dynamic models of silicone rubbers based on the synthesis method, *Materials Science, Testing and Informatics*, Trans. Tech. Publications Uetikon/Zrich, 95-100

4. Leipholz H., 1968, *Stabilitätstheorie*, B.G. Teubner, Stuttgart

5. Zentner L., 2003, *Untersuchung und Entwicklung nachgiebiger Strukturen basierend auf innendruckbelasteten Röhren mit stoffschlüssigen Gelenken*, Ilmenau ISLE Verlag, ISBN 3-932633-77-6

O stateczności podatnych belek poddanych obciążeniu ciśnieniem wewnętrznym

Streszczenie

Naukowe podejście do kwestii stateczności cechuje niezmienna adaptacja tego terminu do różnych dziedzin zastosowań. Niemniej, wyznaczanie obciążeń krytycznych wciąż pozostaje jednym z głównych zadań teorii stateczności. W pracy zaprezentowano kilka przykładów różnych typów utraty stateczności w belkach poddanych obciążeniu ciśnieniem wewnętrznym w przypadku statycznym. Dodatkowo, szczegółowo przeanalizowano zagadnienie stateczności dynamicznej takich układów i pokazano sposoby utrzymywania ich w równowadze.

Manuscript received August 3, 2006; accepted for print August 18, 2006