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STATIC FRICTION INDETERMINACY PROBLEMS AND MODELING OF STICK-SLIP PHENOMENON IN DISCRETE DYNAMIC SYSTEMS

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The paper presents a new method of modeling of the friction action in discrete dynamic systems in cases of undetermined distribution of static friction forces. This method is based on the Gauss Principle and the piecewise linear luz(...) and tar(...) projections with their original mathematical apparatus. The derived variable-structure model of a two-body system with three frictional contacts describes the stick-slip phenomenon in detail. The model has an analytical form applicable to standard (without iterations) computational procedures.

Key words: static friction, force indeterminacy, stick-slip, multi-body systems, mathematical modeling, Gauss principle, piecewise linear projections

1. Introduction

Modeling and simulation of strongly non-linear dynamic systems with friction is an attractive challenge for researchers. We can encounter numerous publications on sophisticated friction problems in scientific journals dedicated to theoretical and applied mechanics, physics of continuous and granular media, tribology, theory of mechanisms, multi-body and multi-rigid-body systems, finite element method, robotics, automatics, biomedicine, non-linear dynamics, hybrid systems, the numerical methods in simulation, identification and optimization, and even to computer graphics and animation. This interest is comprehensible. The friction problems are very important for life and technology, but a lot of theoretical questions is still without satisfactory solutions. One of them is a singular problem of the static friction forces indeterminacy in context of modeling of the stick-slip phenomenon. The question is: whether simulation of a given multi-body system is possible if the spatial distribution of resultant static friction forces is undetermined? This problem will be discussed in the paper.

1.1. Bibliographical overview on friction indeterminacy problems

Singular type of problems of friction indeterminacy (nonuniqueness of a solution) or inconsistency (nonexistence of the solution) are well known as classic problems of theoretical mechanics. They concern even the simplest models which are grounded on the non-smooth but piecewise linear Coulomb (for kinetic friction) and saturation (for static one) characteristics. Such singularities were firstly noticed by Painlevé in 1895 who analyzed motion of a rigid rod on a frictional surface. He noticed that for some parameter configuration with a large friction force coefficient, the motion was undetermined. Nowadays, similar frictional problems of indeterminacy or inconsistency being observed in many planar rigid-body systems are named as the "Painlevé paradoxes". They were studied, for example by Lötsted (1981), Mason and Wang (1988), Baraff (1991), Genot and Brogliato (1999), Leine *et al.* (2002).

Uniqueness and existence of solutions appear as major problems for universal computational methods elaborated for simulation and contact analysis of multi-rigid-body systems in the 2D or 3D space when load-dependent force frictions are changing, and object's topology is varying. These methods and their algorithms (usually iterative) are intensively used especially for such extremely difficult tasks as path and grasp steering for arm- or finger-mechanisms of robots and surgery manipulators, as physics-based animation for virtual environments (including motion of granular materials), and so on. Usually, they are based on the elementary Coulomb friction model and utilize calculation of friction forces in every computation step. But, if the number of friction forces is larger than the number of degrees of freedom, some of the forces must be unknown! This is especially evident when the stiction states appear. Hence, some special computational tricks and treats must be applied. General description of the method including discussion on the uniqueness and existence is given in state-of-the-art papers by Armstromg-Helouvry *et al.* (1994), Joskowicz *et al.* (1998), Brogliato *et al.* (2002), and in many regular papers. Selected articles are cited below.

Several computational approaches have been perceived: the first one – basing on the "penalty method" enables small penetration between contacting bodies. Because of hard springs added, one has no indeterminacies, but the solutions may be numerically instable (stiffness problems). Schwager and Poshel (2002) described a similar method for granular dynamics studies, basing on artificially composed infinitely small but linear deformations of contacting particles. In their opinion, this algorithm is supposedly stable but such statement seems not to be convincing enough. The second approach, most intensively developed – for example by Glocker and Pfeifer (1993), Baraff (1993, 1994), Stewart and Trinkle (1996), Trinkle *et al.* (1997), Pang and Trinkle (2000), Balkom and Trinkle (2002), treats simulation of the friction multi-rigid-body system as the Linear Complementarity Problem (LCP). The LCP methods (iterative methods, primary designed for frictionless impact systems) are intensively developed because they use very efficient matrix subroutines. Applying the LCP method to friction systems, a special arrangement of constrains must be done with using a linear approximation. These simplifications and some "heuristics" in the numerical code that simplify the problem but make it determined, cause that utility of the LCP method is practically limited to systems with small friction coefficients. The Singular Value Decomposition (SVD) approach by Mirtich (1998) is also based on the LCP-type primary model. But for calculation of some of its components, the model is modified. Using the SVD methods, one obtains a "naturally symmetric" force distribution which is admitted as satisfying (heuristics!). Recently, the "impulse-velocity methods" worked by Mirtich (1998), Moreau (2003), Kaufman *et al.* (2005) have been developed. They also utilize some heuristics to obtain efficient and fast algorithms. For example, only a single pair of contact points is handled at a given time. Such simplifications allow one to obtain a real-time well-realistic animation. Concluding this survey, we can confirm the previous note that to resolve indeterminacy problems in multi-rigid-body simulation, the force-based models are conformably modified with application of heuristic ideas. By the way, physicists working on "molecular dynamics" have treated the friction indeterminacy as a probability problem – see Unger *et al.* (2004). By repetition of simulations with perturbed static friction forces in each step, they obtained and analyzed some statistics of simulation results.

Let us return to our question (now little extended). Whether the simulation of a multi-body system is possible without some supplementary heuristics, if accurate calculation of the resultant static friction forces is impossible? In this paper, we will prove that this is possible for some class of discrete dynamic systems for which the simulation might be handled without calculation of the friction forces distribution in singular states. In our study, the Gauss principle of least constraint will be used for answering this question.

General presentation of the Gauss principle based approach for constrained systems is given in Grzesikiewicz (1990), as well as in the papers by Glocker (1997, 1999), Redon *et al.* (2002), and recently by Fan *et al.* (2005 – a special paper on the 175 anniversary of Gauss' work). According to the Gauss Principle (with its extensions), accelerations of a dynamic multi-body system must fulfill sufficient optimality conditions. The optimization concerns some convex function which expresses the "acceleration energy" of the system. Because of convexity, this problem has a unique solution. So, from the theoretical point of view, the accelerations of the modeled multi-body system should be unique, even some forces can be undetermined. In Fan *et al.* (2005), we can find a special section addressed to the indeterminacy problems in multi-body system dynamics. It is shown that calculation of accelerations in such systems can be supported by a special matrix apparatus (theory of generalized inverses of matrices). However, there are not many papers describing a concrete application of the Gauss principle in simulation studies of multi-body systems. Grzesikiewicz and Wakulicz (1979) described a numerical iterative matrix method for simulation of motion of a train modeled as a multi-body series system with Coulomb dry friction forces in multiple draft gears. This method is very sophisticate and seems to explore the theory of generalized inverses of matrices. Simulation of a braking train seems to be a classical solvable problem with the indeterminacy of static friction forces (static friction forces distribution is unknown but the train stops!). Surprisingly, there are no papers found with explicitly given analytical models for simple dynamic systems with the static friction force indeterminacy. This absence should be filled.

1.2. Scope of the studies

Strictly analytical models of single or two-body systems with the Coulomb friction are well known in scientific literature. Most of them are based on the Karnopp (1985) concept. The dry friction structures of such systems are presented below.

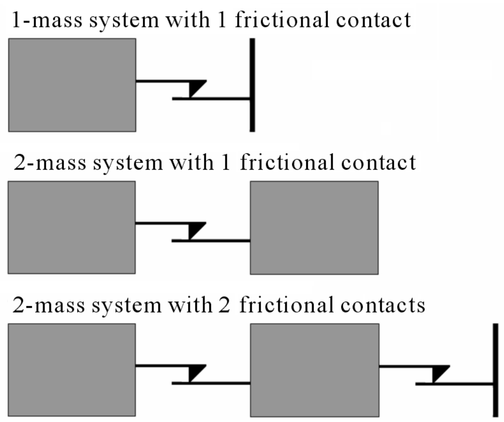

Fig. 1. Three simple friction systems having analytical forms of mathematical models

Mathematical models of systems shown in Fig. 1 have variable-structure forms expressing both kinetic friction (for non-zero velocities) as well as static friction (for zero velocities) actions, so they are applicable to the stick-slip models of mechanisms with friction. Their analytical formulas are ready to use in ODE (Ordinary Differential Equations) computational procedures.

But the mathematical modeling for the next little more complex structure of a friction system (presented on Fig. 2) is noticed to be absent in the literature! This two-body object having three frictional contacts is the simplest system with static friction indeterminacy (for total zero-velocity stiction state, the distribution of static forces is unknown). So, our study on friction indeterminacy problems focuses on such a system.

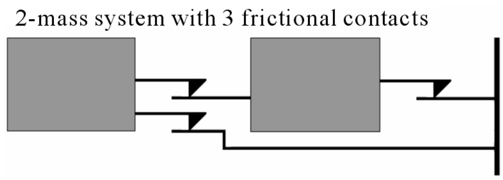

Fig. 2. A simple friction system with static friction indeterminacy

In this study, we use special piecewise linear $\mathrm{luz}(\ldots)$ and $\mathrm{tar}(\ldots)$ projections and their mathematical apparatus. They are very efficient functions for the modeling of non-smooth mechanical systems. Basing on this apparatus, the paper continues an approach presented in many previous authors' publications. Several papers are cited below. The formalism of $\mathrm{luz}(\ldots)$ and $\mathrm{tar}(\ldots)$ projections was described in details and proofs by Żardecki (2001, 2006a). The method of modeling piecewise-linear dynamical systems having freeplays (backlashes, clearances) and frictions (kinetic and static) was presented by Żardecki (2005, 2006b). In the last paper by Żardecki (2006c), all models relating to systems shown in Fig. 1 have been derived with using the $\mathrm{luz}(\ldots)$ and $\mathrm{tar}(\ldots)$ mathematical apparatus and the Gauss principle. In this study, such an approach is continued.

2. Theoretical background for piecewise-linear approach with $\mathrm{luz}(\ldots)$ and $\mathrm{tar}(\ldots)$ projections

Definition

For $x, a \in R$, $a \geqslant 0$

$$
\begin{aligned}
\mathrm{luz}(x,a) &= x + \frac{|x-a| - |x+a|}{2} \\
\mathrm{tar}(x,a) &= x + a\,\mathrm{sgh}(x)
\end{aligned}
$$

where

$$
\mathrm{sgh}(x) = \begin{cases} -1 & \text{if} \quad x<0 \\ s^* \in [-1,1] & \text{if} \quad x=0 \\ 1 & \text{if} \quad x>0 \end{cases}
$$

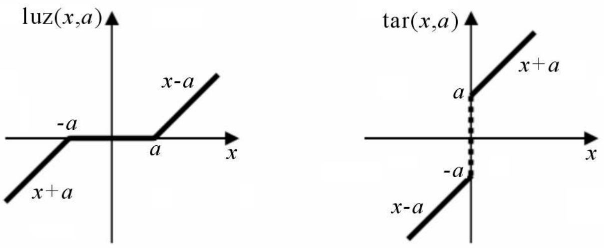

Fig. 3. Geometric interpretations of luz (...) and tar (...) projections

These projections have simple mathematical apparatus containing algebra-like formulas, formulas for some compositions and transformations, theorems on disentanglement of some algebraic equations as well as theorems for differential inclusions and equation transformations – Żardecki (2001, 2006a). We will explore only peculiar formulas and theorems. They will be recalled when necessary.

Below we present some formulas and statements useful for minimization problems with constraints (and for the Gauss principle application in the stick-slip modeling in Sec. 3 and 4). The following "saturation function" (Fig. 4) is used in our studies

$$\overline{x} = \begin{cases} -x_0 & \text{for} \quad x < -x_0 \\ x & \text{for} \quad |x| \leqslant x_0 \\ x_0 & \text{for} \quad x > x_0 \end{cases}$$

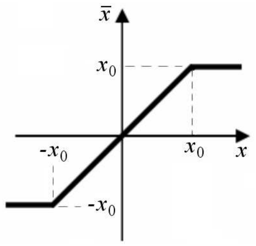

Fig. 4. Piecewise linear saturation characteristics

Remark: In the next points, a simple notation of saturation is applied for variables, e.g.

$$\overline{v}_i \quad \text{means} \quad \overline{v}_i = \overline{v}_i(v_i, v_{0i})$$

Corollary 1

$$\overline{x} = x - \operatorname{luz}(x, x_0)$$

Lemma 1

Let $x, x_0 \in R$, $x_0 \geqslant 0$, $f(x)$ – a convex function.

If $\widetilde{x}$ solves the minimization problem without constraints $\widetilde{x}:\ \min_x f(x)$, then the minimization problem with constraints $\widehat{x}:\ \min_x f(x)\ \wedge\ |x| \leqslant x_0$, has the solution $\widehat{x} = \overline{\overline{\widetilde{x}}} = \widetilde{x} - \operatorname{luz}(\widetilde{x}, x_0)$.

Proof

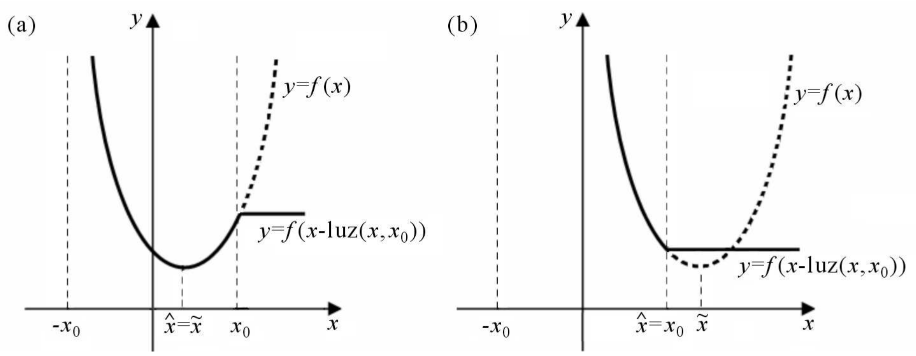

Fig. 5. Minimization of a convex function $y = f(x)$ with limitation $|x| \leqslant x_0$. The pictures are representative: for $|\widetilde{x}| \leqslant x_0$ (a) and for $\widetilde{x} > x_0$ (b)

– If $\widetilde{x} < -x_0$, then $\widehat{x} = \widetilde{x} - (\widetilde{x} + x_0) = -x_0$. Because of the convexity of $f(x)$, for any $\delta > 0$ $\ f(-x_0 + \delta) > f(-x_0)$, so $\widehat{x} = -x_0$.

– If $|\widetilde{x}| \leqslant x_0$, then $\widehat{x} = \widetilde{x} - \operatorname{luz}(\widetilde{x}, x_0) = \widetilde{x}$, so $\widehat{x} = \widetilde{x}$.

– If $\widetilde{x} > x_0$, then $\widehat{x} = \widetilde{x} - (\widetilde{x} - x_0) = x_0$. Because of the convexity of $f(x)$ for any $\delta > 0$ $\ f(x_0 - \delta) > f(x_0)$, so $\widehat{x} = x_0$.

Hence $\widehat{x} = \overline{\overline{\widetilde{x}}} = \widetilde{x} - \operatorname{luz}(\widetilde{x}, x_0)$.

Corollary 2

Let $x, g_1, g_2, x_0, k_1, k_2, p \in R$, $k_1 + k_2 > 0$, $x_0 \geqslant 0$

$$f(x) = k_1(g_1 - x)^2 + k_2(g_2 - x)^2 + p$$

Because $f(x)$ is convex and has minimum in the point $\widetilde{x} = (k_1 g_1 + k_2 g_2)/(k_1 + k_2)$, so the minimization problem with constraints $\widehat{x}:\ \min_x f(x)\ \wedge\ |x| \leqslant x_0$ has the solution

$$\widehat{x} = \overline{\overline{\widetilde{x}}} = \frac{k_1 g_1 + k_2 g_2}{k_1 + k_2} - \operatorname{luz}\Big(\frac{k_1 g_1 + k_2 g_2}{k_1 + k_2}, x_0\Big)$$

Lemma 2

Let $x_1, x_2, g_1, g_2, x_{01}, x_{02}, k_1, k_2 \in R,\ k_1, k_2 > 0,\ x_{01}, x_{02} \geqslant 0$

$$f(x_1, x_2) = k_1(g_1 - x_1)^2 + k_2(g_2 - x_2)^2$$

The minimization problem $\widehat{x}_1, \widehat{x}_2 :\ \min_{x_1,x_2} f(x_1, x_2) \wedge |x_1| \leqslant x_{01},\ |x_2| \leqslant x_{02}$, has the solution

$$\widehat{x}_i = \overline{\overline{x}}_i = g_i - \operatorname{luz}(g_i, x_{0i}) \qquad i = 1, 2$$

Proof

First, we resolve the problem without constraints

$$\frac{\partial f(x_1, x_2)}{\partial x_i} = -2k_i(g_i - x_i) = 0 \qquad \frac{\partial f^2(x_1, x_2)}{\partial x_i^2} = 2k_i > 0 \qquad i = 1, 2$$

$$\frac{\partial f^2(x_1, x_2)}{\partial x_1 \partial x_2} = \frac{\partial f^2(x_1, x_2)}{\partial x_2 \partial x_1} = 0$$

Because $k_i > 0$ so $f(x_1, x_2)$ is convex for all x_i, g_i $(i = 1, 2)$ and has the minimum: $\widetilde{x}_i = g_i$ $(i = 1, 2)$. Thus

$$\overline{\overline{x}}_i = \widetilde{x}_i - \operatorname{luz}(\widetilde{x}_i, x_{0i}) = g_i - \operatorname{luz}(g_i, x_{0i}) \qquad i = 1, 2$$

Now we check whether $\widehat{x}_i = \overline{\overline{x}}_i$, for $|x_i| \leqslant x_{0i}$ $(i = 1, 2)$. We know that

$$\overline{\overline{x}}_i = \begin{cases} -x_{0i} & \text{if} \quad g_i < -x_{0i} \\ g_i & \text{if} \quad -x_{0i} \leqslant g_i = \widetilde{x}_i \leqslant x_{0i} \\ x_{0i} & \text{if} \quad g_1 > x_{01} \end{cases} \qquad i = 1, 2$$

We solve 6 new simpler optimization tasks for functions with a single variable. Note that for a function of a single variable we can use Lemma 1 and Corollary 2.

– If $g_1 < -x_{01}$, $h_1(x_2) = f(-x_{01}, x_2) = k_1(g_1 + x_{01})^2 + k_2(g_2 - x_2)^2$, so minimization of $h_1(x_2)$ gives $\widehat{x}_2 = \widetilde{x}_2 - \operatorname{luz}(\widetilde{x}_2, x_{02}) = g_2 - \operatorname{luz}(g_2, x_{02})$
– If $|g_1| < x_{01}$, $h_2(x_2) = f(g_1, x_2) = k_1(g_1 - x_2)^2$ – results in the same
– If $g_1 > x_{10}$, $h_3(x_2) = f(x_{01}, x_2) = k_1(g_1 + x_{01})^2 + k_2(g_2 - x_2)^2$ – results in the same
– If $g_2 < -x_{02}$, $h_4(x_1) = f(x_1, -x_{02}) = k_1(g_1 - x_1)^2 + k_2(g_2 + x_{02})^2$, so minimization of $h_4(x_1)$ gives $\widehat{x}_1 = \widetilde{x}_1 - \operatorname{luz}(\widetilde{x}_1, x_{01}) = g_1 - \operatorname{luz}(g_1, x_{01})$
– If $|g_2| < x_{02}$, $h_5(x_1) = f(x_1, g_2) = k_1(g_1 - x_1)^2$ – results in the same

– If $g_2 > x_{02}$, $h_6(x_1) = f(x_1, x_{02}) = k_1(g_1 - x_1)^2 + k_2(g_2 - x_{02})^2$ – results in the same

So $\widehat{x}_i = \overline{\overline{x}}_i = g_i - \operatorname{luz}(g_i, x_{0i})$ $(i = 1, 2)$, indeed.

Lemma 3

Let $x_1, x_2, g_1, g_2, w, x_{01}, x_{02}, w_0, k_1, k_2 \in R$, $k_1, k_2 > 0$, $x_{01}, x_{02}, w_0 \geqslant 0$

$$f(x_1, x_2, w) = k_1(g_1 - (x_1 + w))^2 + k_2(g_2 - (x_2 - w))^2$$

The solutions $\widehat{x}_1, \widehat{x}_2, \widehat{w}$ to the minimization problem

$$\widehat{x}_1, \widehat{x}_2, \widehat{w} : \min_{x_1, x_2, w_1} f(x_1, x_2, w) \ \wedge \ |x_1| \leqslant x_{01}, \quad |x_2| \leqslant x_{02}, \quad |w| \leqslant w_0 \quad \text{fulfill}$$

$$\widehat{x}_1 + \widehat{w} = g_1 - \operatorname{luz}(g_1, x_{01} + w_0) \qquad \widehat{x}_2 - \widehat{w} = g_2 - \operatorname{luz}(g_2, x_{02} + w_0)$$

<u>Proof</u>

Note that direct resolution of the task without limitation is impossible

$$\frac{\partial f(\dots)}{\partial x_1} = -2k_1(g_1 - x_1 - w) = 0 \qquad \frac{\partial f(\dots)}{\partial x_2} = -2k_2(g_2 - x_2 + w) = 0$$

$$\frac{\partial f(\dots)}{\partial w} = -2k_1(g_1 - x_1 - w) + 2k_2(g_2 - x_2 + w) = 0$$

x_1, x_2, w are linearly dependent (indeterminacy problem!).

Setting new variables $v_1 = x_1 + w$, $v_2 = x_2 - w$, we can redefine the problem. Now $f(v_1, v_2) = k_1(g_1 - v_1)^2 + k_2(g_2 - v_2)^2$. The constraints fulfill the relations $|v_1| = |x_1 + w| \leqslant |x_i| + |w| \leqslant x_{01} + w_0$, $|v_2| = |x_2 - w| \leqslant |x_2| + |w| \leqslant x_{02} + w_0$.

We resolve the new problem with constraints applying Lemma 2

$$\widehat{v}_1, \widehat{v}_2 : \quad \min_{v_1, v_2} f(v_1, v_2) \ \wedge \ |\widehat{v}_1| \leqslant x_{01} + w_0, \quad |\widehat{v}_2| \leqslant x_{02} + w_0$$

Because the solution to the problem without constraints is $\widetilde{v}_i = g_i$ $(i = 1, 2)$. So $\widehat{v}_i = \overline{\overline{v}}_i = g_i - \operatorname{luz}(g_i, x_{0i} + w_0)$ and finally $\widehat{x}_1 + \widehat{w} = g_1 - \operatorname{luz}(g_1, x_{01} + w_0)$, $\widehat{x}_2 - \widehat{w} = g_2 - \operatorname{luz}(g_2, x_{02} + w_0)$ ■

Note that this lemma does not give solutions (they are indeterminate) but some relations between them.

3. A method of modeling of friction forces and stick-slip phenomena

The $\operatorname{luz}(\dots)$ and $\operatorname{tar}(\dots)$ projections and their mathematical apparatus simplify a synthesis and analysis of stick-slip phenomena in multi-body systems with friction(s) expressed by piecewise linear characteristics. It means that the range of method usability is limited to objects which have constant friction force topology and friction forces not load-dependent. The method is commented below for the simplest one-mass system with friction.

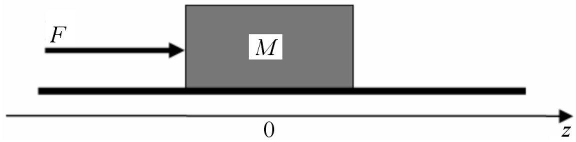

Fig. 6. One-mass system with friction; M – mass, F – external force, which expresses conjunctions with other elements of the multi-body system

The synthesis of the model is done in several steps.

➢ Firstly, friction force characteristics are assumed. Typical friction force characteristics $F_T(V)$ (Fig. 7) are presented in an extended form (with "hidden" but limited static friction force for $V = 0$). Such characteristics can be described directly or piecewise-linear approximated by the $\operatorname{luz}(\ldots)$ and $\operatorname{tar}(\ldots)$ projections.

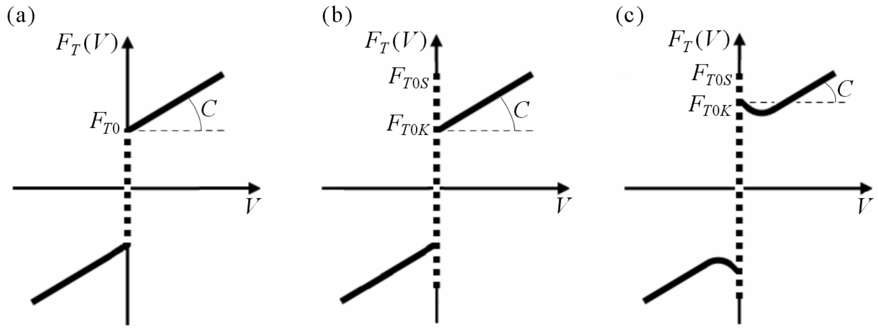

Fig. 7. Typical friction force characteristics: (a) exactly Coulomb's, (b) Coulomb's + static friction augmented, (c) Coulomb's + static friction augmented + Stribeck's effect; area $V = 0$ for static friction action denoted by double line; F_T – friction force, V – relative velocity of elements, F_{T0K} – kinetic dry friction force, F_{T0S} – maximum static friction force, F_{T0} – maximum dry friction force (for Coulomb's characteristics $F_{T0} = F_{T0K} = F_{T0S}$), C – damping factor

In our studies, we use the Coulomb extended characteristic which is usually treated as basic for friction problems. Such a characteristic is written directly as $F_T(V) = C\operatorname{tar}(V, F_{T0}/C)$ (Żardecki. 2006b,c). For $V \neq 0$, they express the kinetic friction force F_{TK}. For $V = 0$, $F_T(0) = F_{TS} = F_{T0}s^*$ ($s^* \in [-1, 1]$), so the static friction force F_{TS} should be additionally determined by the resultant force F_W (in one-mass system $F_W = F$). Generally, $F_{TS}(F_W)$ are like saturation characteristics, but the forces F_W may have complex forms or be even undeterminable.

Having assumed a type of friction force extended characteristics, their parameters must be given. Sometimes (for example when contact surfaces have heterogeneous properties), calculation of friction force parameters can require some additional assumptions (even heuristics!). In our studies, we assume that the friction force parameters are known.

➢ In the second step, the primary inclusion model is written. In our case, this is

$$M\ddot{z}(t) \in F(t) - C\,\mathrm{tar}\Big(\dot{z}(t), \frac{F_{T0}}{C}\Big)$$

The inclusion model must be translated to the ODE form. The problem concerns only the state $\dot{z}(t) = 0$, because for $\dot{z}(t) \neq 0$ the $\mathrm{tar}(\dots)$ describes friction characteristics one to one. So:

— if $\dot{z}(t) \neq 0$

$$M\ddot{z}(t) = F(t) - C\,\mathrm{tar}\Big(\dot{z}(t), \frac{F_{T0}}{C}\Big)$$

— if $\dot{z}(t) = 0$

$$M\ddot{z}(t) \in F(t) - s^* F_{T0} \qquad s^* \in [-1, 1]$$

➢ The inclusion model is analyzed for the state $\dot{z}(t) = 0$. The static friction force $F_{TS} = s^* F_{T0}$ is unknown but limited $(F_{TS} \in [-F_{T0}, F_{T0}])$. For application of the Gauss principle, the acceleration energy Q is defined. Here

$$Q = M\ddot{z}^2 = \frac{(F - F_{TS})^2}{M}$$

According to the Gauss principle, the function $Q(\dots)$ (here $Q(F_{TS})$) is minimized. For the one-mass system, the optimization tasks has a form

$$F_{TS}: \qquad \min_{F_{TS}} \frac{(F - F_{TS})^2}{M} \;\wedge\; |F_{TS}| \leqslant F_{T0}$$

According to Corollary 2, the optimal solution is

$$F_{TS} = s^* F_{T0} = F - \mathrm{luz}\,(F, F_{T0})$$

➢ Finally, the inclusion model is translated to the ODE form. Here one obtains

$$M\ddot{z}(t) = \begin{cases} F(t) - C\,\mathrm{tar}\Big(\dot{z}(t), \dfrac{F_{T0}}{C}\Big) & \text{if } \dot{z}(t) \neq 0 \\ \mathrm{luz}\,(F(t), F_{T0}) & \text{if } \dot{z}(t) = 0 \end{cases}$$

This formula strictly corresponds to the one-mass Karnopp model (Karnopp, 1985) and clearly describes the stick-slip phenomenon. Note, when $\dot{z}(t) = 0$

and $|F(t)| \leqslant F_{T0}$, then $\operatorname{luz}(F(t), F_{T0}) = 0$, then $\ddot{z}(t) = 0$. This means stiction. When $|F(t)| > F_{T0}$, we have $\operatorname{luz}(F(t), F_{T0}) \neq 0$ and $\ddot{z}(t) \neq 0$ – the state $\dot{z}(t) = 0$ is temporary.

Advantages of using the $\operatorname{luz}(\dots)$ and $\operatorname{tar}(\dots)$ projections concern not only brief analytic forms of the friction characteristics and clear the stick-slip description. Using their mathematical apparatus, we can transform the stick-slip models by parametric operations, and this seems to be an important benefit too (more details in the paper by Żardecki (2006c)).

4. A model of the two-mass system with three frictional contacts – the simplest indeterminacy problem for static friction forces

The two-mass system with three friction sources, which is shown in Fig. 2, is representative for several physical object configurations. In such a case, the mass blocks rub with each other as well as with a motionless base surface (or casing).

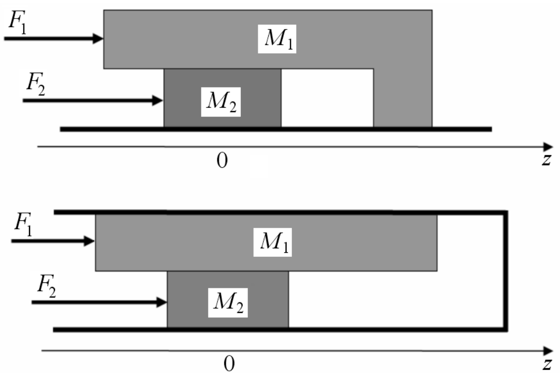

Fig. 8. Exemplary physical configurations of the two-mass system with three frictional contacts

➢ One assumes that all kinetic friction forces fulfill the Coulomb characteristics. The following denote: M_1, M_2 – masses of bocks, F_{T012}, C_{12} – coefficients of the Coulomb characteristics for friction existing between the blocks, F_{T010}, C_{10} – coefficients for friction between the top block and base surface, F_{T020}, C_{20} – coefficients for friction between the bottom block and base surface, F_1, F_2 – external forces.

➢ The primary inclusion model is

$$M_1\ddot{z}_1 \in F_1 - C_{12}\,\mathrm{tar}\Big(\dot{z}_1-\dot{z}_2, \frac{F_{T012}}{C_{12}}\Big) - C_{10}\,\mathrm{tar}\Big(\dot{z}_1, \frac{F_{T010}}{C_{10}}\Big)$$
$$M_2\ddot{z}_2 \in F_2 + C_{12}\,\mathrm{tar}\Big(\dot{z}_1-\dot{z}_2, \frac{F_{T012}}{C_{12}}\Big) - C_{20}\,\mathrm{tar}\Big(\dot{z}_2, \frac{F_{T020}}{C_{20}}\Big)$$

where $s_{12}^*, s_{10}^*, s_{20}^* \in [-1,1]$ (see definition of the $\mathrm{tar}\,(\dots)$).

This model can be rewritten as:

— if $\dot{z}_1 \neq 0$, $\dot{z}_2 \neq 0$, $\dot{z}_1 \neq \dot{z}_2$

$$M_1\ddot{z}_1 = F_1 - C_{12}\,\mathrm{tar}\Big(\dot{z}_1-\dot{z}_2, \frac{F_{T012}}{C_{12}}\Big) - C_{10}\,\mathrm{tar}\Big(\dot{z}_1, \frac{F_{T010}}{C_{10}}\Big)$$
$$M_2\ddot{z}_2 = F_2 + C_{12}\,\mathrm{tar}\Big(\dot{z}_1-\dot{z}_2, \frac{F_{T012}}{C_{12}}\Big) - C_{20}\,\mathrm{tar}\Big(\dot{z}_2, \frac{F_{T020}}{C_{20}}\Big)$$

— if $\dot{z}_1 = \dot{z}_2 \neq 0$

$$M_1\ddot{z}_1 \in F_1 - F_{T012}s_{12}^* - C_{10}\,\mathrm{tar}\Big(\dot{z}_1, \frac{F_{T010}}{C_{10}}\Big)$$
$$M_2\ddot{z}_2 \in F_2 + F_{T012}s_{12}^* - C_{20}\,\mathrm{tar}\Big(\dot{z}_2, \frac{F_{T020}}{C_{20}}\Big)$$

— if $\dot{z}_1 = 0$, $\dot{z}_2 \neq 0$

$$M_1\ddot{z}_1 \in F_1 + C_{12}\,\mathrm{tar}\Big(\dot{z}_2, \frac{F_{T012}}{C_{12}}\Big) - F_{T010}s_{10}^*$$
$$M_2\ddot{z}_2 = F_2 - (C_{12}+C_{20})\,\mathrm{tar}\Big(\dot{z}_2, \frac{F_{T012}+F_{T020}}{C_{12}+C_{20}}\Big)$$

— if $\dot{z}_1 \neq 0$, $\dot{z}_2 = 0$

$$M_1\ddot{z}_1 = F_1 - (C_{12}+C_{10})\,\mathrm{tar}\Big(\dot{z}_1, \frac{F_{T012}+F_{T010}}{C_{12}+C_{10}}\Big)$$
$$M_2\ddot{z}_2 \in F_2 + C_{12}\,\mathrm{tar}\Big(\dot{z}_1, \frac{F_{T012}}{C_{12}}\Big) - F_{T020}s_{12}^*$$

— if $\dot{z}_1 = 0$, $\dot{z}_2 = 0$

$$M_1\ddot{z}_1 \in F_1 - F_{T012}s_{12}^* - F_{T010}s_{10}^*$$
$$M_2\ddot{z}_2 \in F_2 + F_{T012}s_{12}^* - F_{T020}s_{20}^*$$

For the state $\dot{z}_1 = 0$, $\dot{z}_2 \neq 0$ as well as for $\dot{z}_1 \neq 0$, $\dot{z}_2 = 0$ the equations and inclusions have been little compressed. The formulas

$$\mathrm{tar}\,(-x,a) = -\,\mathrm{tar}\,(x,a)$$
$$k_1\,\mathrm{tar}\,(x,a_1) + k_2\,\mathrm{tar}\,(x,a_2) = (k_1+k_2)\,\mathrm{tar}\Big(x, \frac{k_1a_1+k_2a_2}{k_1+k_2}\Big)$$

for $k_1, k_2 \geqslant 0$ were used.

➢ Now we analyze the inclusion forms. They concern four velocity conditions:

1) When $\dot{z}_1 = \dot{z}_2 \neq 0$ (then $\dot{z}_1 - \dot{z}_2 = 0$)
   – problem of F_{TS12} $(F_{TS12} = F_{T012} s_{12}^*)$
2) When $\dot{z}_1 = 0,\ \dot{z}_2 \neq 0$ (then $\dot{z}_1 - \dot{z}_2 \neq 0$)
   – problem of F_{TS10} $(F_{TS10} = F_{T010} s_{10}^*)$
3) When $\dot{z}_1 \neq 0,\ \dot{z}_2 = 0$ (then $\dot{z}_1 - \dot{z}_2 \neq 0$)
   – problem of F_{FS20} $(F_{FS20} = F_{T020} s_{20}^*)$
4) When $\dot{z}_1 = \dot{z}_2 = 0$ (then $\dot{z}_1 - \dot{z}_2 = 0$)
   – problem of F_{TS12}, F_{TS10}, F_{TS20}

Note, there is no problem of double singularities, for example a pair of F_{TS12}, F_{TS10}. The problem concerns of the F_{TS12}, F_{TS10}, F_{TS20} triplet at once.

In each case, the acceleration energy $Q = M_1\ddot{z}_1^2 + M_2\ddot{z}_2^2$ is defined and an appropriate minimization task is resolved. Calculations of every static friction force (cases 1, 2, 3) can be realised in a standard way. Analysis of the triplet F_{TS12}, F_{TS10}, F_{TS20} will be a task with indeterminacy!

Case 1 $(\dot{z}_1 = \dot{z}_2 \neq 0)$

$$M_1\ddot{z}_1 \in F_{W1} - F_{TS12} \qquad \text{where} \qquad F_{W1} = F_1 - C_{10}\,\mathrm{tar}\Big(\dot{z}_1, \frac{F_{T010}}{C_{10}}\Big)$$
$$M_2\ddot{z}_2 \in F_{W2} + F_{TS12} \qquad \text{where} \qquad F_{W2} = F_2 - C_{20}\,\mathrm{tar}\Big(\dot{z}_2, \frac{F_{T020}}{C_{20}}\Big)$$

The acceleration energy Q as function of F_{TS12} is

$$\begin{aligned} Q(F_{TS12}) &= \frac{(F_{W1} - F_{TS12})^2}{M_1} + \frac{(F_{W2} + F_{TS12})^2}{M_2} = \\ &= \frac{(F_{W1} - F_{TS12})^2}{M_1} + \frac{(-F_{W2} - F_{TS12})^2}{M_2} \end{aligned}$$

So the optimization problem $F_{TS12} :\ \min_{F_{TS12}} Q(F_{TS12}) \wedge |F_{TS12}| \leqslant F_{T012}$ is compatible to the task in Corollary 2. Note, in our case $k_1 = 1/M_1,\ g_1 = F_{W1}$, $k_2 = 1/M_2,\ g_2 = -F_{W2},\ p = 0$, and

$$\frac{k_1 g_1 + k_2 g_2}{k_1 + k_2} = \frac{\frac{F_{W1}}{M_1} - \frac{F_{W2}}{M_2}}{\frac{1}{M_1} + \frac{1}{M_2}} = \frac{M_2F_{W1} - M_1F_{W2}}{M_1 + M_2} = \widetilde{F}_{TS12}$$

By application of Corollary 2, one finally obtains

$$F_{TS12} = \frac{M_2F_{W1} - M_1F_{W2}}{M_1 + M_2} - \mathrm{luz}\Big(\frac{M_2F_{W1} - M_1F_{W2}}{M_1 + M_2}, F_{T012}\Big)$$

Note that

$$F_{W1} - F_{TS12} = \frac{M_1(F_{W1}+F_{W2})}{M_1+M_2} + \operatorname{luz}\Big(\frac{M_2F_{W1}-M_1F_{W2}}{M_1+M_2}, F_{T012}\Big)$$
$$F_{W2} + F_{TS12} = \frac{M_2(F_{W1}+F_{W2})}{M_1+M_2} - \operatorname{luz}\Big(\frac{M_2F_{W1}-M_1F_{W2}}{M_1+M_2}, F_{T012}\Big)$$

<u>Case 2</u> ($\dot{z}_1 = 0,\ \dot{z}_2 \neq 0$)

$$M_1\ddot{z}_1 \in F_{W1} - F_{TS10} \qquad \text{where} \qquad F_{W1} = F_1 + C_{12}\operatorname{tar}\Big(\dot{z}_2, \frac{F_{T012}}{C_{12}}\Big)$$
$$M_2\ddot{z}_2 = F_{W2} \qquad \text{where} \qquad F_{W2} = F_2 - (C_{12}+C_{20})\operatorname{tar}\Big(\dot{z}_2, \frac{F_{T012}+F_{T020}}{C_{12}+C_{20}}\Big)$$

The acceleration energy Q as function of F_{TS10} is

$$Q(F_{TS10}) = \frac{(F_{W1}-F_{TS10})^2}{M_1} + \frac{F_{W2}^2}{M_2}$$

So the optimization problem $F_{TS10}:\ \min_{F_{TS10}} Q(F_{TS10})\ \wedge\ |F_{TS10}| \leqslant F_{T010}$ is compatible to the task in Corollary 2. In this case $k_1 = 1/M_1,\ g_1 = F_{W1}$, $k_2 = 0,\ g_2 = 0,\ p = F_{W2}^2/M_2$, so

$$\frac{k_1g_1 + k_2g_2}{k_1+k_2} = F_{W1} = \widetilde{F}_{TS10}$$

According to Corollary 2, we have $F_{TS10} = F_{W1} - \operatorname{luz}(F_{W1}, F_{T010})$.

Note that $F_{W1} - F_{TS10} = \operatorname{luz}(F_{W1}, F_{T010})$.

<u>Case 3</u> ($\dot{z}_1 \neq 0,\ \dot{z}_2 = 0$)

$$M_1\ddot{z}_1 = F_{W1} \qquad \text{where} \qquad F_{W1} = F_1 - (C_{12}+C_{10})\operatorname{tar}\Big(\dot{z}_1, \frac{F_{T012}+F_{T010}}{C_{12}+C_{10}}\Big)$$
$$M_2\ddot{z}_2 \in F_{W2} - F_{TS20} \qquad \text{where} \qquad F_{W2} = F_2 + C_{12}\operatorname{tar}\Big(\dot{z}_1, \frac{F_{T012}}{C_{12}}\Big)$$

The acceleration energy Q as function of F_{TS20} is

$$Q(F_{TS20}) = \frac{F_{W1}^2}{M_1} + \frac{(F_{W2}-F_{TS20})^2}{M_2}$$

So the optimization problem $F_{TS20}:\ \min_{F_{TS20}} Q(F_{TS20})\ \wedge\ |F_{TS20}| \leqslant F_{T020}$ is compatible to the task in Corollary 2. In this case $k_1 = 1/M_2,\ g_1 = F_{W2}$, $k_2 = 0,\ g_2 = 0,\ p = F_{W1}^2/M_1$, so

$$\frac{k_1g_1 + k_2g_2}{k_1+k_2} = F_{W2} = \widetilde{F}_{TS20}$$

According to Corollary 2, one finally obtains the static friction force $F_{TS20} = F_{W2} - \operatorname{luz}(F_{W2}, F_{T020})$.

Note that $F_{W2} - F_{TS20} = \operatorname{luz}(F_{W2}, F_{T020})$.

<u>Case 4</u> $(\dot{z}_1 = \dot{z}_2 = 0)$

$$M_1\ddot{z}_1 \in F_1 - (F_{TS10} + F_{TS12})$$
$$M_2\ddot{z}_2 \in F_2 - (F_{TS20} - F_{TS12})$$

The acceleration energy Q as function of F_{TS10}, F_{TS20}, F_{TS12} is

$$Q(F_{TS10}, F_{TS20}, F_{TS12}) = \frac{[F_1 - (F_{TS10} + F_{TS12})]^2}{M_1} + \frac{[F_2 - (F_{TS20} - F_{TS12})]^2}{M_2}$$

The optimization problem

$$\begin{aligned} & F_{TS10}, F_{TS20}, F_{TS12} : \\ & \quad \min_{F_{TS10}, F_{TS20}, F_{TS12}} Q(F_{TS10}, F_{TS20}, F_{TS12}) \ \wedge \ |F_{TS10}| \leqslant F_{T010}, \\ & \quad |F_{TS20}| \leqslant F_{T020}, \quad |F_{TS12}| \leqslant F_{T012} \end{aligned}$$

is compatible to the task in Lemma 3 (appropriate for the indeterminacy problem).

Here $k_1 = 1/M_1$, $g_1 = F_1$, $k_2 = 1/M_2$, $g_2 = F_2$.

By application of Lemma 3, we know that the solutions fulfill

$$F_{TS10} + F_{TS12} = F_1 - \operatorname{luz}(F_1, F_{T010} + F_{T012})$$
$$F_{TS20} - F_{TS12} = F_2 - \operatorname{luz}(F_2, F_{T020} + F_{T012})$$

We have not calculated the static friction forces (they are undetermined), but their necessary combinations have been found. Note that

$$F_1 - (F_{TS10} + F_{TS12}) = \operatorname{luz}(F_1, F_{T010} + F_{T012})$$
$$F_2 - (F_{TS20} - F_{TS12}) = \operatorname{luz}(F_2, F_{T020} + F_{T012})$$

➢ Finally, the inclusion model is translated to the variable structure ODE form. Such a model is convenient for analysis of the stick-slip phenomena. For the two-mass system with three frictional contacts, we obtain:

— When $\dot{z}_1 \neq 0$, $\dot{z}_2 \neq 0$, $\dot{z}_1 \neq \dot{z}_2$

$$M_1\ddot{z}_1 = F_1 - C_{12}\operatorname{tar}\Big((\dot{z}_1 - \dot{z}_2), \frac{F_{T012}}{C_{12}}\Big) - C_{10}\operatorname{tar}\Big(\dot{z}_1, \frac{F_{T010}}{C_{10}}\Big) \tag{4.1}$$

$$M_2\ddot{z}_2 = F_2 + C_{12}\operatorname{tar}\Big(\dot{z}_1 - \dot{z}_2, \frac{F_{T012}}{C_{12}}\Big) - C_{20}\operatorname{tar}\Big(\dot{z}_2, \frac{F_{T020}}{C_{20}}\Big) \tag{4.2}$$

No stiction states, only slipping
— when $\dot{z}_1 = \dot{z}_2 \neq 0$

$$
\begin{aligned}
M_1\ddot{z}_1 = {} & \frac{M_1}{M_1+M_2}\Big[F_1 - C_{10}\,\mathrm{tar}\Big(\dot{z}_1,\frac{F_{T10}}{C_{10}}\Big) + F_2 - C_{20}\,\mathrm{tar}\Big(\dot{z}_2,\frac{F_{T20}}{C_{20}}\Big)\Big] + \\
& + \mathrm{luz}\left(\frac{M_2\Big[F_1 - C_{10}\,\mathrm{tar}\Big(\dot{z}_1,\frac{F_{T10}}{C_{10}}\Big)\Big] - M_1\Big[F_1 - C_{20}\,\mathrm{tar}\Big(\dot{z}_2,\frac{F_{T20}}{C_{20}}\Big)\Big]}{M_1+M_2}, F_{T012}\right)
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
M_2\ddot{z}_2 = {} & \frac{M_2}{M_1+M_2}\Big[F_1 - C_{10}\,\mathrm{tar}\Big(\dot{z}_1,\frac{F_{T10}}{C_{10}}\Big) + F_2 - C_{20}\,\mathrm{tar}\Big(\dot{z}_2,\frac{F_{T20}}{C_{20}}\Big)\Big] + \\
& - \mathrm{luz}\left(\frac{M_2\Big[F_1 - C_{10}\,\mathrm{tar}\Big(\dot{z}_1,\frac{F_{T10}}{C_{10}}\Big)\Big] - M_1\Big[F_1 - C_{20}\,\mathrm{tar}\Big(\dot{z}_2,\frac{F_{T20}}{C_{20}}\Big)\Big]}{M_1+M_2}, F_{T012}\right)
\end{aligned}
\tag{4.4}
$$

If

$$
\left|\frac{M_2\Big[F_1 - C_{10}\,\mathrm{tar}\Big(\dot{z}_1,\frac{F_{T10}}{C_{10}}\Big)\Big] - M_1\Big[F_1 - C_{20}\,\mathrm{tar}\Big(\dot{z}_2,\frac{F_{T20}}{C_{20}}\Big)\Big]}{M_1+M_2}\right| \leqslant F_{T012}
$$

then $luz(\dots) = 0$, and the equations for $\ddot{z}_1$, $\ddot{z}_2$ have identical forms. Since $\ddot{z}_1 - \ddot{z}_2 = 0$ and $\dot{z}_1 - \dot{z}_2 = 0$, it means that the blocks are stuck. In other cases, the state $\dot{z}_1 - \dot{z}_2 = 0$ is temporary (without stiction).
— When $\dot{z}_1 = 0$, $\dot{z}_2 \neq 0$

$$
M_1\ddot{z}_1 = \mathrm{luz}\Big(F_1 + C_{12}\,\mathrm{tar}\Big(\dot{z}_2,\frac{F_{T012}}{C_{12}}\Big), F_{T010}\Big) \tag{4.5}
$$

$$
M_2\ddot{z}_2 = F_2 - (C_{12}+C_{20})\,\mathrm{tar}\Big(\dot{z}_2,\frac{F_{T012}+F_{T020}}{C_{12}+C_{20}}\Big) \tag{4.6}
$$

The stiction state between the mass M_1 and the base surface appears when

$$
\Big|F_1 + C_{12}\,\mathrm{tar}\Big(\dot{z}_2,\frac{F_{T012}}{C_{12}}\Big)\Big| \leqslant F_{T010}
$$

In other cases, the state $\dot{z}_1 = 0$ is temporary.
— When $\dot{z}_1 \neq 0$, $\dot{z}_2 = 0$

$$
M_1\ddot{z}_1 = F_1 - (C_{12}+C_{10})\,\mathrm{tar}\Big(\dot{z}_1,\frac{F_{T012}+F_{T010}}{C_{12}+C_{10}}\Big) \tag{4.7}
$$

$$
M_2\ddot{z}_2 = \mathrm{luz}\Big(F_2 + C_{12}\,\mathrm{tar}\Big(\dot{z}_1,\frac{F_{T012}}{C_{12}}\Big), F_{T020}\Big) \tag{4.8}
$$

The stiction state between the mass M_2 and the base surface appears when

$$\Big|F_2 + C_{12}\,\mathrm{tar}\Big(\dot{z}_1, \frac{F_{T012}}{C_{12}}\Big)\Big| \leqslant F_{T020}$$

In other cases, the state $\dot{z}_2 = 0$ is temporary.

— When $\dot{z}_1 = 0$, $\dot{z}_2 = 0$

$$M_1\ddot{z}_1 = \mathrm{luz}\,(F_1, F_{T012} + F_{T010}) \tag{4.9}$$

$$M_2\ddot{z}_2 = \mathrm{luz}\,(F_2, F_{T012} + F_{T020}) \tag{4.10}$$

The total stiction state appears when $|F_1| \leqslant F_{T012} + F_{T010}$ and $|F_2| \leqslant F_{T012} + F_{T020}$. In this case $\ddot{z}_1 = \ddot{z}_2 = 0$ and $\dot{z}_1 = \dot{z}_2 = 0$. When $|F_1| > F_{T012} + F_{T010}$ but $|F_2| \leqslant F_{T012} + F_{T020}$ we have $\ddot{z}_1 \neq \ddot{z}_2 = 0$, but $\dot{z}_1 = \dot{z}_2 = 0$, so the block M_2 is stuck with the base surface but the state $\dot{z}_1 = 0$ of the block M_1 and the state $\dot{z}_1 - \dot{z}_2 = 0$ of the blocks M_1, M_2 are temporary. An analogical situation is when $|F_1| \leqslant F_{T012} + F_{T010}$ but $|F_2| > F_{T012} + F_{T020}$. When both $|F_1| > F_{T012} + F_{T010}$ and $|F_2| > F_{T012} + F_{T020}$, the state $\dot{z}_1 = \dot{z}_2 = 0$ is without any stiction.

The final variable structure model of the two-mass system with three friction sources in the form ready to use with standard ODE (without iterations) procedures is presented below

$$M_1\ddot{z}_1 = \begin{cases} \text{Eq. (4.1)} & \text{for} \quad \dot{z}_1 \neq 0, \quad \dot{z}_2 \neq 0, \quad \dot{z}_1 \neq \dot{z}_2 \\ \text{Eq. (4.3)} & \text{for} \quad \dot{z}_1 = \dot{z}_2 \neq 0 \\ \text{Eq. (4.5)} & \text{for} \quad \dot{z}_1 = 0, \quad \dot{z}_2 \neq 0 \\ \text{Eq. (4.7)} & \text{for} \quad \dot{z}_1 \neq 0, \quad \dot{z}_2 = 0 \\ \text{Eq. (4.9)} & \text{for} \quad \dot{z}_1 = 0, \quad \dot{z}_2 = 0 \end{cases}$$

$$M_2\ddot{z}_2 = \begin{cases} \text{Eq. (4.2)} & \text{for} \quad \dot{z}_1 \neq 0, \quad \dot{z}_2 \neq 0, \quad \dot{z}_1 \neq \dot{z}_2 \\ \text{Eq. (4.4)} & \text{for} \quad \dot{z}_1 = \dot{z}_2 \neq 0 \\ \text{Eq. (4.6)} & \text{for} \quad \dot{z}_1 = 0, \quad \dot{z}_2 \neq 0 \\ \text{Eq. (4.8)} & \text{for} \quad \dot{z}_1 \neq 0, \quad \dot{z}_2 = 0 \\ \text{Eq. (4.10)} & \text{for} \quad \dot{z}_1 = 0, \quad \dot{z}_2 = 0 \end{cases}$$

5. Final remarks

A new method for modeling of friction actions and stick-slip phenomena in discrete dynamic systems including the static friction distribution indeterminacy has been presented in the paper. The method is based on the Gauss least constraints principle and the piecewise linear $\mathrm{luz}\,(\dots)$ and $\mathrm{tar}\,(\dots)$ projections with their original mathematical apparatus. Details of model derivations have been shown on an example of a two-mass system with three friction sources (mathematical model of such a system has not been noticed in others publications). Thanks to the $\mathrm{luz}(\ldots)$ and $\mathrm{tar}(\ldots)$ projections, the model has been given a clear analytical form ready to use with standard ordinary differential equations procedures (without iteration). It can be useful even for real time processing (steering). The idea presented here on the exemplary system can be applied for more complex systems having a bigger number of friction forces than the number of degrees of freedom.

The presented method has been discussed in the case of the Coulomb friction. But the piecewise linear approximation basing on the $\mathrm{luz}(\ldots)$ and $\mathrm{tar}(\ldots)$ projections is applicable also to more sophisticated friction characteristics (expressing the Stribeck effect, non-symmetry and so on). Even though the stick-slip models have been derived here for simple Coulomb's friction characteristics, their final forms can be easily adapted to other, more complex ones. For example, when the magnitudes of kinetic and static dry friction forces are not identical, two different parameters F_{T0K} and F_{T0S} can be applied in the variable-structure model. The essence of the model is not changing.

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Problemy nieokreśloności tarcia statycznego i modelowanie zjawiska ***stick-slip*** w układach dyskretnych

Streszczenie

W artykule przedstawia się nową metodę modelowania procesów *stick-slip* w dyskretnych układach dynamicznych z tarciami dopuszczającą nieokreśloność rozkładu sił tarcia statycznego. Metoda opiera się na zasadzie Gaussa oraz wykorzystaniu specjalnych przedziałami liniowych odwzorowań luz (...) i tar (...) z ich oryginalnym aparatem matematycznym. W pracy prezentowane jest szczegółowe wyprowadzenie modelu opisującego *stick-slip* w układzie 2 masowym z 3 miejscami tarcia. Dzięki zastosowaniu odwzorowań $\operatorname{luz}(\ldots)$ i $\operatorname{tar}(\ldots)$ modele układów z tarciem mają analityczne formy przystosowane do standardowych procedur symulacyjnych.

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