

## A STEREOLOGICAL UBIQUITIFORMAL SOFTENING MODEL FOR CONCRETE

ZHUO-CHENG OU, GUAN-YING LI, ZHUO-PING DUAN, FENG-LEI HUANG

*State Key Laboratory of Explosion Science and Technology, Beijing Institute of Technology, Beijing, China*  
*e-mail address: zcou@bit.edu.cn*

A stereological ubiquitiformal softening model for describing the softening behavior of concrete under quasi-static uniaxial tensile loadings is presented in this paper. In the model, both the damage evaluation process of fracture cross-sections and their distribution along the specimens axis are taken into account. The numerical results of a certain kind of full grade concrete made of crushed coarse aggregate are found to be in good agreement with the experimental data. Moreover, an experiential relation between the lower bound to the scale invariance of concrete and its tensile strength is also obtained by data fitting of the experimental data, which provides an effective approach to determine the lower bound to scale invariance of concrete.

*Keywords:* ubiquitiform, fractal, concrete, softening curve

### 1. Introduction

Fractals have been widely used as a nonlinear mathematical tool to describe mechanical behavior of heterogeneous materials such as concrete since the pioneer work of Mandelbrot (1982), Mandelbrot *et al.* (1984). It has been found that the internal structure of concrete appears quite a well approximate self-similarity in many aspects over certain ranges of scale. For example, it has been verified experimentally that the fracture surface of concrete can be described by fractals (Saouma and Barton, 1994; Charkaluk *et al.*, 1998). Stroeven has shown that for almost all the aggregate grading in concrete, the distribution of the aggregate particles in various diameters appears the self-similarity feature (Stroeven, 1973, 2000). Moreover, fractals have also been widely used to describe the fracture behavior of concrete (Borodich, 1997; Carpinteri *et al.*, 2002; Khezzzadeh and Mofid, 2006). However, there are still many intrinsic difficulties in fractal applications, especially in the case when the measure of a real geometrical or physical object must be taken into account because kinds of density of fractal parameters defined on the unit fractal measure are not only lacking unambiguous physical meanings but also very difficult to be determined in practice. Recently, Ou *et al.* (2014) demonstrated that such a difficulty was caused by contradiction between the integral dimensional immeasurability of a fractal and the integral dimensional characteristic of a real physical or geometrical object in nature, and proposed a new concept of ubiquitiform. According to Ou *et al.* (2014), a ubiquitiform is defined as a finite order self-similar (or self-affine) physical configuration constructed usually by a finite iterative procedure. It has been shown that a ubiquitiform has a finite integral dimensional measure and must be of integral dimension in Euclidean space, whereas the Hausdorff dimension of a fractal is usually not integral. The Hausdorff dimension of the initial element of a fractal changes abruptly at the point of infinite iteration, which results in divergence of the integral dimensional measure of the fractal and makes the fractal approximation of a real geometrical or physical object to a ubiquitiform unreasonable.

One important phenomenon in tensile failure of concrete is softening, and the most widely used theory is the so-called cohesive crack model (Barenblatt, 1959, 1962). Over the past decades,

several softening curves have been proposed, such as the linear curve (Hillerborg *et al.*, 1976), the bilinear curve (Pettersson, 1981), the nonlinear curve (Reinhardt *et al.*, 1986) and the power-law curve (Gopalaratnam and Shah, 1985; Karihaloo, 1995). Recently, Khezzadeh and Mofid (2006) proposed a quasi-fractal softening curve based on fractal concepts, in which, however, only the damage evaluation process of the fractured cross-section was considered. On the other hand, as demonstrated by Ou *et al.* (2014), a ubiquitous, rather than a fractal, should be used in describing a real geometrical or physical object in the case of the integral dimensional measure of the object.

Therefore, in this study, based on the concept of the ubiquitous, a stereological ubiquitous softening model for concrete, in which both the damage evaluation process of fractured cross-sections and their distribution along the specimens axis are taken into account, and the calculated results of softening curves of concrete are compared with previous experimental data. Moreover, it is interesting to find that there exists a good correspondence between the lower bound to scale invariance and the tensile strength of concrete, and then an exponential formula for the corresponding relationship is obtained.

## 2. Stereological ubiquitous softening model

To describe the damage evaluation process of a concrete specimen, a stereological damage region is assumed in this paper, based on the fracture band theory (Bažant and Oh, 1983). Namely, fracture of a heterogeneous aggregate material such as concrete can be assumed to occur in the form of a blunt smeared crack band. Such a stereological damage region consists of a series of fracture surfaces distributed along the axis of the specimen as a generalized ubiquitous Cantor set, and each of the fracture surface will be described by a generalized ubiquitous Sierpinski carpet having different complexity. The generalized ubiquitous Sierpinski carpet is generated by a series of recursive procedures, i.e. an iteration process from the initial square of unit side length. In each step of the iteration, each remaining square is divided into  $p^2$  identical smaller squares, and the generalized ubiquitous Sierpinski carpet is then obtained by repeatedly removing  $q$  ( $q/p^2 < 1$ ) small squares from the remaining squares. According to Khezzadeh and Mofid (2006), the removing area represents the cracked area of the fractured cross-sections. As has been defined by Ou *et al.* (2014), the complexity  $D$  of such a generalized ubiquitous Sierpinski carpet is

$$D = \frac{\ln(p^2 - q)}{\ln p} \quad (2.1)$$

Therefore, taking different values of  $p$  and  $q$ , the generalized ubiquitous Sierpinski carpet can be used to describe a surface with any complexity. The removed area in the  $n$ -th iteration is

$$\Delta a_n = A_p \frac{q}{p^2} \left( \frac{p^2 - q}{p^2} \right)^{n-1} \quad (2.2)$$

where  $A_p$  is the nominal area of the generalized ubiquitous Sierpinski carpet. The total area of the increased crack surface, when the specimen is failed, is

$$A_1 = \sum_{n=1}^{n_c} \Delta a_n = A_p \left[ 1 - \left( \frac{p^2 - q}{p^2} \right)^{n_c} \right] = A_p \left[ 1 - \left( \frac{1}{p^{n_c}} \right)^{2-D} \right] \quad (2.3)$$

where  $n_c$  represents the critical iteration number of the generalized ubiquitous Sierpinski carpet when the specimen is failed.

According to the fracture band theory (Bažant and Oh, 1983), in this study, a multiple crack surface hypothesis is proposed in the ubiquitous softening model. That is to say, to

describe the damage evolution of the concrete material, besides the main crack surface, there are still several secondary crack surfaces, each of which is described as the above-mentioned generalized ubiquitiformal Sierpinski carpet with different iteration orders. These crack surfaces are assumed to be distributed along the axis of the specimen as a generalized ubiquitiformal Cantor set (Fig. 1). Hereinafter, we denote these crack surfaces as the  $i$ -th order crack surfaces ( $i = 1, 2, 3, \dots, m$ ), and the first order ( $i = 1$ ) one is the main crack surface. According to the structure of the generalized ubiquitiformal Cantor set, the number of the  $i$ -th order crack surface is  $k_i = 2^{i-1}$ . It is also assumed that the iteration number of the  $i$ -th order crack surface is one less than that of the  $(i - 1)$ -th order crack surface. Thus, the increase of the  $i$ -th order crack surface in the  $n$ -th iteration can be calculated by the equation

$$\Delta a_n^i = A_p \frac{q}{p^2} \left( \frac{p^2 - q}{p^2} \right)^{n-i} \quad (2.4)$$

The total increase of the crack surface in the  $n$ -th iteration is

$$\Delta a_n = \sum_{i=1}^m k_i \Delta a_n^i = \sum_{i=1}^m 2^{i-1} A_p \frac{q}{p^2} \left( \frac{p^2 - q}{p^2} \right)^{n-i} = A_p \frac{2^m q p^{2m} (p^2 - q)^{n-m} - q (p^2 - q)^n}{p^{2n} (p^2 + q)} \quad (2.5)$$

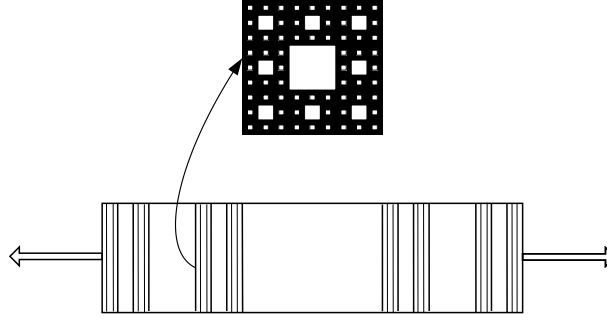


Fig. 1. Stereological ubiquitiformal softening model

According to the iteration law of the generalized ubiquitiformal Sierpinski carpet, the ratio of the area of the  $(i + 1)$ -th crack surface to that of the  $i$ -th crack surface is

$$A_{i+1} = \frac{p^2 - q}{p^2} A_i \quad (2.6)$$

where  $i = 1, 2, 3, \dots, m$ , and, from Eq. (2.6), we have

$$A_i = \left( \frac{p^2 - q}{p^2} \right)^{i-1} A_1 \quad (2.7)$$

Then, the total crack surface increased in the fracture process is

$$A = \sum_{i=1}^m k_i A_i = A_1 \sum_{i=1}^m 2^{i-1} \left( \frac{p^2 - q}{p^2} \right)^{i-1} = A_1 \frac{p^{2m} - 2^m (p^2 - q)^m}{(2q - p^2) p^{2m-2}} \quad (2.8)$$

For convenience, here, the homogeneous deformation along the axis of the specimen is assumed, and then the elongations generated in each iteration  $\Delta w$  are the same, which can be written as

$$\Delta w = \frac{w_c}{n_c} \quad (2.9)$$

where  $w_c$  is the critical elongation of the specimen.

In general, on the one hand, the energy consumed in each iteration is proportional to the increase of the area of the crack surface, that is

$$\Delta U_n = G_f \Delta a_n \quad (2.10)$$

where  $G_f$  is the fracture energy. On the other hand, the required energy to generate new cracks equal to the area under the softening curve in a interval of length  $\Delta w$  implies that

$$\Delta W_n = A \sigma_n \Delta w \quad (2.11)$$

Thus, from Eqs.(2.10) and (2.11), there is

$$G_f \frac{\Delta a_n}{A} = \sigma_n \Delta w \quad (2.12)$$

The relationship between the stress and the elongation in each iteration can be obtained from Eqs. (2.5), (2.8), (2.9) and (2.12), as

$$\begin{aligned} \sigma_n &= \frac{G_f \Delta a_n}{A \Delta w} = \frac{G_f A_p n_c p^{2m-2} (2q - p^2) 2^m q p^{2m} (p^2 - q)^{n-m}}{A_1 w_c p^{2n} (p^2 + q) [p^{2m} - 2^m (p^2 - q)^m]} \\ &\quad - \frac{G_f A_p n_c p^{2m-2} (2q - p^2) q (p^2 - q)^n}{A_1 w_c p^{2n} (p^2 + q) [p^{2m} - 2^m (p^2 - q)^m]} \quad 1 \leq n \leq n_c \end{aligned} \quad (2.13)$$

It should be noticed that the values of both the stress and the elongation in Eq. (2.13) are discrete, starting from  $n = 1$ . In order to obtain a continuous stress-elongation curve in the interval of  $[0, w_c]$ , the Khezzzadeh and Mofid modification (Khezzzadeh and Mofid, 2006) is used, which is described briefly below. Firstly, it is assumed that the value of the softening function in  $w = 0$  is equal to the tensile strength of the specimen, i.e.,  $\sigma(0) = f_t$ , and that the stress-elongation curve is linear in the interval of  $[0, \Delta w]$ . Next, an energy modification factor  $\mu$  is then introduced to make sure that the area under the softening curve is equal to  $G_f$ , namely,

$$(1 - \mu)G_f = \left[ f_t + \sigma \left( \frac{\Delta w}{2} \right) \right] \frac{\Delta w}{2} \quad (2.14)$$

Thus one has

$$\begin{aligned} \sigma &= \frac{\mu A_p G_f n_c p^{2m-2} (2q - p^2) 2^m q p^{2m} (p^2 - q)^{\frac{n_c}{w_c} w - m}}{A_1 w_c p^{2 \frac{n_c}{w_c} w} (p^2 + q) [p^{2m} - 2^m (p^2 - q)^m]} \\ &\quad - \frac{\mu A_p G_f n_c p^{2m-2} (2q - p^2) q (p^2 - q)^{\frac{n_c}{w_c} w}}{A_1 w_c p^{2 \frac{n_c}{w_c} w} (p^2 + q) [p^{2m} - 2^m (p^2 - q)^m]} \quad \frac{w_c}{n_c} \leq w \leq w_c \end{aligned} \quad (2.15)$$

For convenience, we assume that  $q = 1$  in the ubiquitous softening model, then Eq. (2.15) can be rewritten as

$$\begin{aligned} \sigma &= \frac{\mu A_p G_f n_c p^{2m-2} (2 - p^2) 2^m p^{2m} (p^2 - 1)^{\frac{n_c}{w_c} w - m}}{A_1 w_c p^{2 \frac{n_c}{w_c} w} (p^2 + 1) [p^{2m} - 2^m (p^2 - 1)^m]} \\ &\quad - \frac{\mu A_p G_f n_c p^{2m-2} (2 - p^2) (p^2 - 1)^{\frac{n_c}{w_c} w}}{A_1 w_c p^{2 \frac{n_c}{w_c} w} (p^2 + 1) [p^{2m} - 2^m (p^2 - 1)^m]} \quad \frac{w_c}{n_c} \leq w \leq w_c \end{aligned} \quad (2.16)$$

For  $w = \Delta w$ , we have

$$\begin{aligned} \sigma(\Delta w) &= \frac{\mu A_p G_f n_c p^{2m-4} (2 - p^2) 2^m p^{2m} (p^2 - 1)^{1-m}}{A_1 w_c (p^2 + 1) [p^{2m} - 2^m (p^2 - 1)^m]} \\ &\quad - \frac{\mu A_p G_f n_c p^{2m-4} (2 - p^2) (p^2 - 1)}{A_1 w_c (p^2 + 1) [p^{2m} - 2^m (p^2 - 1)^m]} \end{aligned} \quad (2.17)$$

We can obtain the slope of the softening curve in the interval  $[0, \Delta w]$

$$C = \frac{\sigma(\Delta w) - f_t}{\Delta w} = \frac{\mu A_p G_f n_c^2 p^{2m-4} (2-p^2) 2^m p^{2m} (p^2-1)^{1-m}}{A_1 w_c^2 (p^2+1) [p^{2m} - 2^m (p^2-1)^m]} - \frac{\mu A_p G_f n_c^2 p^{2m-4} (2-p^2) (p^2-1)}{A_1 w_c^2 (p^2+1) [p^{2m} - 2^m (p^2-1)^m]} - \frac{n_c}{w_c} f_t \quad (2.18)$$

Then we have

$$\sigma = \frac{\mu A_p G_f n_c^2 p^{2m-4} (2-p^2) 2^m p^{2m} (p^2-1)^{1-m}}{A_1 w_c^2 (p^2+1) [p^{2m} - 2^m (p^2-1)^m]} w - \frac{\mu A_p G_f n_c^2 p^{2m-4} (2-p^2) (p^2-1)}{A_1 w_c^2 (p^2+1) [p^{2m} - 2^m (p^2-1)^m]} w - \frac{n_c}{w_c} f_t w + f_t \quad 0 \leq w \leq \frac{w_c}{n_c} \quad (2.19)$$

From Eq. (2.14) and Eq. (2.19) one can obtain the energy modification factor  $\mu$  as

$$\mu = 1 - \frac{A_p p^{2m-4} (2-p^2) [2^m p^{2m} (p^2-1)^{1-m} - (p^2-1)]}{4A_1 (p^2+1) [p^{2m} - 2^m (p^2-1)^m]} - \frac{3f_t w_c}{4G_f n_c} = \frac{4G_f n_c - 3f_t w_c}{4G_f n_c} \cdot \frac{4A_1 (p^2+1) [p^{2m} - 2^m (p^2-1)^m]}{A_p p^{2m-4} (2-p^2) [2^m p^{2m} (p^2-1)^{1-m} - (p^2-1)] + 4A_1 (p^2+1) [p^{2m} - 2^m (p^2-1)^m]} \quad (2.20)$$

The ubiquitousformal softening curve of concrete is then

$$\sigma = \begin{cases} \frac{\mu A_p G_f n_c p^{2m-2} (2-p^2) [2^m p^{2m} (p^2-1)^{-m} - 1]}{A_1 w_c (p^2+1) [p^{2m} - 2^m (p^2-1)^m]} \left(1 - \frac{1}{p^2}\right) \frac{n_c}{w_c} w & \frac{w_c}{n_c} \leq w \leq w_c \\ \left( \frac{\mu A_p G_f n_c^2 p^{2m-4} (2-p^2) [2^m p^{2m} (p^2-1)^{1-m} - (p^2-1)]}{A_1 w_c^2 (p^2+1) [p^{2m} - 2^m (p^2-1)^m]} - \frac{n_c}{w_c} f \right) w + f_t & 0 \leq w \leq \frac{w_c}{n_c} \end{cases} \quad (2.21)$$

In the ubiquitousformal softening model, the iteration number is calculated by the relation

$$\left(\frac{1}{p}\right)^N = \frac{\delta_{min}}{l} \quad (2.22)$$

where  $\delta_{min}$  and  $l$  are the minimum and maximum scales of the concrete respectively, which are related to the micro and macro structure of the concrete. However, the iteration number calculated from Eq. (2.22) is always not an integer, whereas the iteration number of the generalized ubiquitousformal Cantor set should be an integer. Thus we assume that  $n = [N]$  in this paper, where the square brackets represents the maximum integer no larger than the argument.

### 3. Numerical results of full grade concrete

To confirm the availability of the ubiquitousformal softening model, the model is used to calculate the softening curve of a full grade concrete specimen made of crushed coarse aggregate, and the numerical results are compared with the experimental result (Deng *et al.*, 2005). In the experiment, the uniaxial tension-compression behavior of the full grade concrete specimens made of crushed coarse aggregate was studied on an INSTRON8506 material testing machine under constant-displacement loading, with the maximum load of 3000 kN. Four displacement extensometers were set around the test specimen, and the data collection and the loading control were completed by using a computer. The composition of the concrete and the experimental data are listed in Tables 1 and 2, respectively.

**Table 1.** Concrete mix of the concrete [kg/m<sup>3</sup>]

Water	Cement	Ash	Artificial sand	Artificial stone [mm]				Superplasticizerits	
				5-20	20-40	40-80	80-150	JGB	DH9
87	131	44	585	328	328	492	492	10.5	1.23

**Table 2.** Experimental data of the concrete specimen

Curing period [day]	Tensile strength $f_t$ [MPa]	Elastic modulus $E_t$ [GPa]	Critical elongation $w_c$ [mm]	Fracture energy $G_f$ [N/m]
110	1.908	40.0	1.390	497.220
55	1.508	37.0	1.355	448.401
46	1.310	35.0	1.199	422.878
16	1.180	31.1	1.680	445.738
15	1.044	28.9	1.289	369.463
11	0.804	22.0	1.193	273.233

In the ubiquitous softening model, the parameter is:  $p = 2.07$ , which is the same as in Khezzadeh and Mofid (2006), and the adaptive result for  $m$  is  $m = 2$ .

For a certain concrete, the parameters  $p$ ,  $m$ ,  $A_p$  and  $n_c$  in the softening model are determined, and the material parameters  $G_f$ ,  $w_c$  and  $f_t$  are also known. Thus the parameter  $A_1$  and  $\mu$  can be regarded as constants. Therefore, for convenience, we rewrite Eq. (2.21) as

$$\sigma = \begin{cases} C_1 C_2^w & \frac{w_c}{n_c} \leq w \leq w_c \\ f_t - C_3 w & 0 \leq w \leq \frac{w_c}{n_c} \end{cases} \quad (3.1)$$

where  $C_1$ ,  $C_2$  and  $C_3$  are constant. The values of these parameters for concrete specimens with different curing periods as well as the experimental data are all listed in Table 3.

**Table 3.** Parameters of the ubiquitous softening model for concrete specimen

Curing period [day]	Tensile strength $f_t$ [MPa]	Crit. elongation $w_c$ [mm]	Iteration number $n_c$	$\delta_{min}$ [ $\mu\text{m}$ ]	$C_1$	$C_2$	$C_3$
110	1.908	1.390	12	24	1.145	0.1008	8.894
55	1.508	1.355	11	50	1.0047	0.1156	5.9895
46	1.310	1.199	10	104	1.0188	0.1090	4.4188
16	1.180	1.680	10	104	0.7080	0.2055	3.7931
15	1.044	1.289	9	215	0.7194	0.1563	3.4388
11	0.804	1.193	8	445	0.4916	0.1682	2.8644

The comparison between the softening curves calculated by using the ubiquitous softening model and the experimental results are shown in Fig. 2. It can be seen that the ubiquitous softening model is in good agreement with the experimental data. It should be pointed out that the difference of the stress between the softening curve in the interval  $[0, \Delta w]$  increases with the tensile strength of the specimen, except for the specimen with a curing period of 16 days. However, it can also be seen that the experimental data for this specimen, especially the critical elongation, is abnormal. The difference of the softening curve of this specimen is caused mainly by abnormality of the experimental data.

As has been mentioned by Ou *et al.* (2014), the lower bound to the scale invariance  $\delta_{min}$ , namely, the minimum scale of concrete, is a crucial parameter for a ubiquitous, and it can be

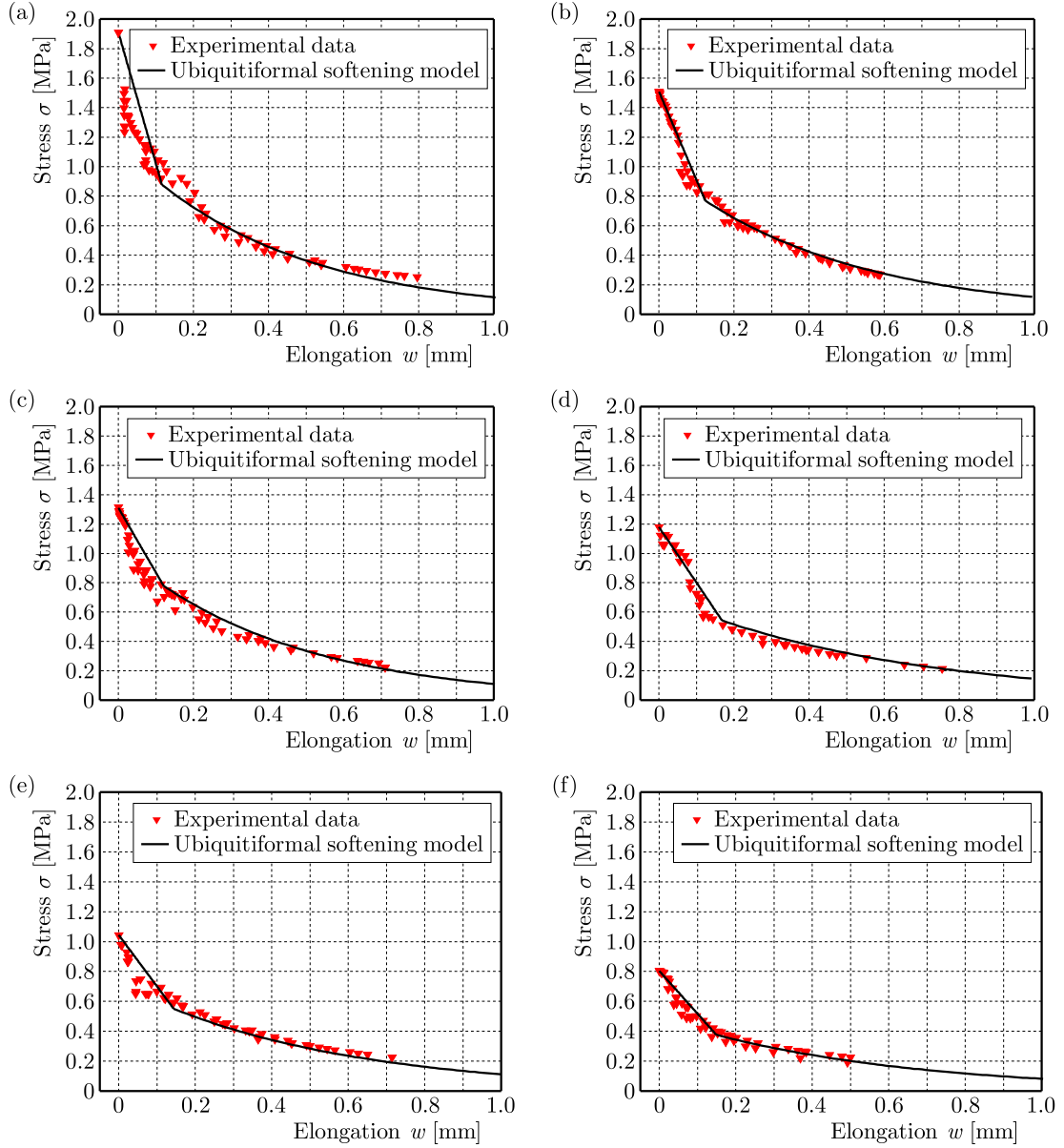


Fig. 2. The ubiquitousformal softening curve: (a) specimen of 1.908 MPa, (b) specimen of 1.508 MPa, (c) specimen of 1.310 MPa, (d) specimen of 1.180 MPa, (e) specimen of 1.044 MPa, (f) specimen of 0.804 MPa

seen that this crucial parameter is related with the tensile strength of the concrete specimen with different curing periods. The lower bound to the scale invariance  $\delta_{min}$  for the specimen with different tensile strength is shown in Fig. 3. By fitting the data with a power function, the relation between the lower bound to the scale invariance  $\delta_{min}$  and the tensile strength can be obtained as

$$\delta_{min} = 221.28 \cdot f_t^{-3.24} \quad (3.2)$$

where the units of  $\delta_{min}$  and  $f_t$  are  $\mu\text{m}$  and MPa, respectively. This relationship provides a reasonable approach to determine the lower bound to the scale invariance of concrete. Furthermore, by analysing the influencing factors of the concrete tensile strength, the approach to determine the lower bound to the scale invariance of concrete by other physical parameters may be obtained.



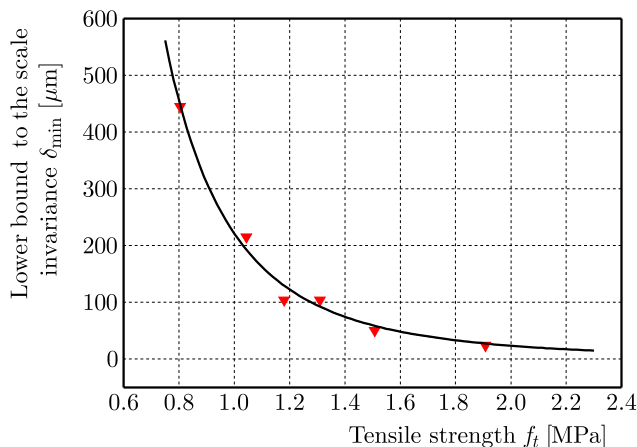


Fig. 3. The relation between the lower bound to the scale invariance  $\delta_{min}$  and tensile strength of concrete

It should be mentioned that, although such an ubiquitous softening model for concrete seems to be similar to the fractal one (Khezzadeh and Mofid, 2006), it has more definite physical meanings. The relation between the lower bound to the scale invariance and tensile strength of concrete is obtained numerically, which may provide a reasonable approach to determine the lower bound to the scale invariance of concrete.

#### 4. Conclusion

A stereological type of ubiquitous softening model that can well describe the softening behavior of concrete under quasi-static tensile loadings is proposed in this paper. Both the damage evaluation process of fracture cross-sections and their distribution along the specimens axis are considered. Moreover, by fitting the experimental data, a relation between the lower bound to the scale invariance and the tensile strength of concrete is obtained, which provides a reasonable approach to determine the lower bound to the scale invariance of concrete.

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