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FREE VIBRATIONS AND STABILITY OF DISCRETE SYSTEMS SUBJECTED TO THE SPECIFIC LOAD

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In this paper, we present discrete systems subjected to a generalised specific load by a force directed towards to positive pole and a follower force directed towards the positive pole. The critical load and the course of the natural frequency in relation to the external load are determined. Adequate relationships describing stability of the considered columns are obtained taking into account potential energy of the systems (energetic method) or total mechanical energy (vibration method). Geometrical parameters of heads realising the load and rigidity of rotational springs modelling the finite stiffness of structural constraints of the system are taken into account.

Key words: discrete slender system, divergence instability, rigid body dynamics, natural frequency

1. Introduction

1.1. Theoretical and numerical studies on the stability of discrete systems

The problem of stability of rod systems with a finite degrees of freedom subjected to conservative loads was worked out by Roorda (1981). The critical loads, at which the equilibrium position of system rods becomes unstable, were determined for models with one or two degrees of freedom at the assumed lengths of rigid elements of the systems and stiffnesses of structural constraints. Research results of the influence of internal and external damping on the stability of discrete systems with two rods subjected to non-conservative loads were presented by Ziegler (1956), Herrman and Jong (1965, 1966), Gajewski and Życzkowski (1972) and Gajewski (1972). Ziegler (1956) made investigations on the critical load of a double pendulum with viscoelastic joints subjected to the follower force. He stated that taking into account internal friction in the considered model decreased the critical force in comparison to the identical model without internal damping. Similar contributions were provided by Herrman and Jong (1965, 1966), where Ziegler's model was analysed with consideration given to the internal damping separately for the bottom mounting joint and the upper joint (joint of rods) of the structure. Gajewski and Życzkowski (1972) and Gajewski (1972) analysed stability of a two-rod discrete system (Ziegler's model) taking into account not only internal but also external damping in the considered column. They stated and confirmed their statement through numerical calculations that the external friction stabilizes the system. The above described results were obtained for different lengths of individual elements of the column, distributions of concentrated masses and stiffnesses of joints modelling the finite stiffness of structural constraints of the system.

Tomski and Szmidla (2004b) applied three methods to determine stability of discrete columns with one degree of freedom: energetic method, vibration method and inaccuracy method taking into account the eccentricity of force application as well as pre-deflection of the column. Those methods were presented on the basis of elastic systems restrained to the foundation and loaded by a vertical force.

The problem of stability of the discrete systems with one degree of freedom, in which a pre-imperfection was studied, was considered by Thompson and Hunt (1973), Elishakoff (1980) and Elishakoff *et al.* (1984, 1996). Thompson and Hunt (1973) presented the effect of pre-deflection of the system on stability of an infinitely rigid rod hinged at the base. At the free end, the rod was propped by a horizontal spring with a linear characteristic. The direction of the spring was permanently (parallel to the base), independently of displacement of the system and the free end was loaded by the conservative force. Total influence of pre-deflection of the column and the eccentricity of its loading was presented for the system built as described above, but the direction of the spring action was not parallel to the base.

Elishakoff *et al.* (1996) described the problem of influence of pre-imperfection on the natural frequency of two discrete systems loaded by a vertical force. The physical model of one of the columns stays in good agreement with the system for which the effect of pre-deflection on its stability was considered by Thompson and Hunt (1973). The second system was a modification of the first one, where apart from the modelling of the spring, the ends of the structure were hinged at the props. The props could shift in horizontal and vertical directions.

The model of a three-hinged system of two rigid rods with movement restrained by a spring with a non-linear characteristic was presented by Elishakoff (1980) and Elishakoff *et al.* (1984). The influence of loading, intensity of changes in the pre-deflection and changes in the spring stiffness on the buckling of the system and the range of natural frequency variations were analysed.

1.2. Specific load

Publications discussed in Section 1.1 refer to discrete slender systems subjected to Euler's loads (Timoshenko and Gere, 1961) or Beck's loads (Beck, 1952; Bogacz and Janiszewski, 1986) which are divergence (Timoshenko and Gere, 1961; Gajewski and Życzkowski, 1970; Leipholz, 1974; Ziegler, 1968) or flutter (Beck, 1952; Bogacz and Janiszewski, 1986) systems, adequately. In the scientific literature, there is lack of descriptions of discrete slender systems of the divergence-pseudo-flutter type, subjected to the specific load (Tomski and Szmidla, 2004a). Systems subjected to the above mentioned conservative load generated by the generalised force directed towards the pole (Tomski *et al.*, 1994, 1996, 1998; Tomski and Szmidla, 2004a,c,d) and the follower force directed towards the pole (Tomski *et al.*, 1998, 2004; Tomski and Podgórska-Brzdękiewicz, 2006a,b; Tomski and Szmidla, 2004a,c), while the pole could be positive or negative, are taken into account. The considered cases of the specific load, defined by Tomski and Szmidla (2004a), combine certain elements of the generalised load (Gajewski and Życzkowski, 1988; Kordas, 1963), follower load (Beck, 1952; Bogacz and Janiszewski, 1986) and a load by a force directed towards the pole (Timoshenko and Gere, 1961; Gajewski and Życzkowski, 1970) (see Fig. 1). In the case of loads directed towards the positive pole, the direction of the external force passes through the fixed point located at the in undeflected axis of the column below its free end (Tomski and Szmidla, 2004a). In the case of a load directed towards the negative pole, the discussed point lies above the free end of the system. The analysed loading is realised by loading and receiving heads built of linear elements (Tomski *et al.*, 1994, 1995, 1996, 1998; Tomski and Szmidla, 2004a,c) or elements with a circular outline (constant curvature) (Tomski and Podgórska-Brzdękiewicz, 2006a,b; Tomski and Szmidla, 2004a,c,d; Tomski *et al.*, 2004).

The specific load as the generalised load with a force directed towards the positive pole was first presented by Tomski *et al.* (1994). Then, in the work by Tomski *et al.* (1995), the influence of the generalised load on the stability and vibration of a flat frame built of a vertical column and a horizontal bolt was discussed. The obtained so far results of numerical and experimental computations for slender continuous systems (columns, frames) (Tomski *et al.*, 1995,

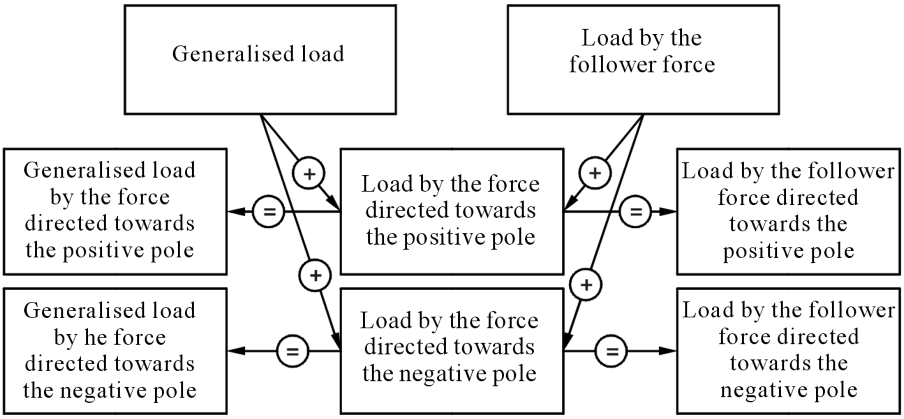

Fig. 1. Diagram showing the origin of nomenclature of the specific load (Tomski and Szmidla, 2004a,d)

1996, 1998, 2004; Tomski and Podgórska-Brzdękiewicz, 2006a,b; Tomski and Szmidla, 2004a,c,d) confirmed the results of theoretical examinations on the determined boundary conditions and solutions to the boundary value problem (taking into account the energetic and vibration methods). An appropriate selection of geometrical and physical parameters of the heads made it possible to qualify the considered systems to one of the two types: divergent or divergence-pseudo-flutter one.

The theoretical investigations presented in this paper are aimed at a mathematical description of discrete systems loaded by the generalised load with a force directed towards the positive pole (A) and with a follower force directed towards the positive pole (B). The problem of stability of the considered systems is solved by applying two methods (see Tomski and Szmidla, 2004b; Ziegler, 1968):

- energetic method (static criterion). It relies on searching for the load at which the total potential energy gets no longer positively determined,
- vibration method (kinematic criterion). It relies on searching for the load at which free movement becomes no longer limited.

The considered columns are discrete systems, where the finite stiffness of structutal constraints, modelled by rotational springs, is taken into account. The load is realised by loading and receiving heads built of elements with a circular outline. The heads are real structures applied in the experimental research on continuous systems.

2. A system loaded by the generalised load with a force directed towards the positive pole (system A)

2.1. Physical model of the system

In Fig. 2, the physical model of the considered system is presented. The system is loaded by the generalised load with a force directed towards the positive pole. The system is composed of three rods with lengths l_1, l_2, l_3, connected by springs with the rotational rigidity c_2, c_3. The elasticity of the system mounting is defined as c_1. The load of the considered structure is realised by a specially designed head (Tomski and Szmidla, 2004a,c,d). The loading head is characterised by a constant radius of curvature R, ①.

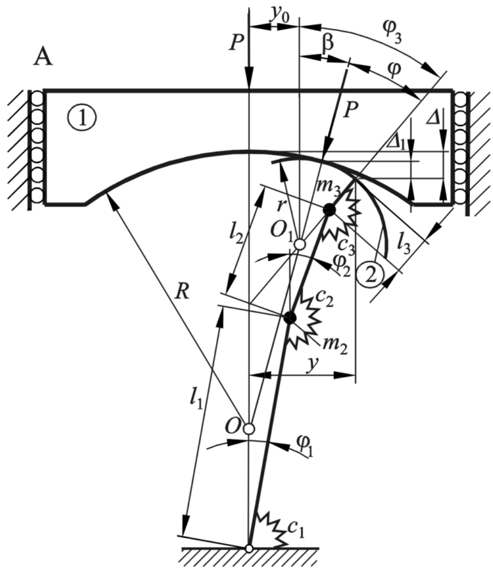

Fig. 2. Physical model of the column loaded by the generalised load with a force directed towards the positive pole

The head receiving the load, ②, is described by a constant radius of curvature r ($r \leqslant R$). In the case of the loading and receiving head, the friction and inertia forces have not been taken into account, on which assumption the direction of the loading force is passing through the constant point O_1 (the centre of curvature of the head receiving the load) and the constant point O (the centre of curvature of the head exerting the load ①), placed at the distance R from the free end of the structure.

Head ② is rigidly connected to the rod with l_3 in length. Concentrated masses m_2 and m_3 are the reduced masses of the rods with lengths l_1, l_2.

The loading and receiving elements of heads as well as rods of the system are assumed to be infinitely rigid.

2.2. Mechanical energy of the system

The relationships following from the geometry of the loading and receiving heads and from the direction of the external loading force are specified to determine the potential energy. The system is described by three variables $\varphi_i(t)$ $(i = 1, 2, 3)$. On the basis of Fig. 2 one can write:
— for the loading head

$$y_0 = (R - r)\sin\beta \qquad y = y_0 + r\sin\varphi_3 \qquad \beta + \varphi = \varphi_3 \tag{2.1}$$

— for the rods of the system, due to equality of longitudinal displacements

$$y = l_1\sin\varphi_1 + l_2\sin\varphi_2 + l_3\sin\varphi_3 \tag{2.2}$$

hence

$$y_0 = l_1\sin\varphi_1 + l_2\sin\varphi_2 + (l_3 - r)\sin\varphi_3 \tag{2.3}$$

The longitudinal displacement of the free end of the column $\varDelta$ and the displacement $\varDelta_1$ resulting from a change of the point of application of the external force P towards the free end of the structure, are also determined

$$\begin{aligned} \varDelta &= l_1(1 - \cos\varphi_1) + l_2(1 - \cos\varphi_2) + l_3(1 - \cos\varphi_3) \\ \varDelta_1 &= r(\cos\beta - \cos\varphi_3) \end{aligned} \tag{2.4}$$

On the basis of the above conditions, the components of potential energy are determined as follows:
— energy of elastic strain

$$V_1 = \frac{1}{2}c_1\varphi_1^2 + \frac{1}{2}c_2(\varphi_2 - \varphi_1)^2 + \frac{1}{2}c_3(\varphi_3 - \varphi_2)^2 \tag{2.5}$$

— potential energy of a component of the vertical force P

$$\begin{aligned} V_2 &= -P(\varDelta - \varDelta_1) = \\ &= -P[l_1(1 - \cos\varphi_1) + l_2(1 - \cos\varphi_2) + l_3(1 - \cos\varphi_3) - r(\cos\beta - \cos\varphi_3)] \end{aligned} \tag{2.6}$$

— potential energy of a component of the horizontal force P

$$V_3 = \frac{P}{2}\sin\beta(y_0 + r\sin\beta) \tag{2.7}$$

The total kinetic energy T is a sum of the kinetic energy of the m_2 and m_3 masses

$$T = T_1 + T_2 = \frac{m_2}{2} l_1^2 \Big(\frac{\partial \varphi_1}{\partial t}\Big)^2 + \frac{m_3}{2}\Big(l_1 \frac{\partial \varphi_1}{\partial t} + l_2 \frac{\partial \varphi_2}{\partial t}\Big)^2 \tag{2.8}$$

where $\varphi_i = \varphi_i(t)$.

The relationships which determine the parameters describing stability of the considered structure are found on the basis of the presented energy description.

2.3. Stability and free vibrations of the column

Appropriate expressions for stability and free vibrations of the considered column subjected to the generalised load by the force directed towards the positive pole are presented with small displacements of the structures assumed, that is when $\sin\varphi_i = \varphi_i$, $\sin\beta = \beta$. The cosine function is expanded in power series taking into account only its first two terms

$$\cos\varphi_i = 1 - \frac{\varphi_i^2}{2} \qquad\qquad \cos\beta = 1 - \frac{\beta^2}{2} \tag{2.9}$$

The dimensionless quantities are additionally considered in dependences (2.5)-(2.7) describing the components of potential energy

$$\begin{aligned} l_i^* &= \frac{l_i}{l} \quad (i = 1,2,3) \qquad & P^* &= \frac{Pl}{c_2} \qquad & c_1^* &= \frac{c_1}{c_2} \\ c_3^* &= \frac{c_3}{c_2} & R^* &= \frac{R}{l} & r^* &= \frac{r}{l} \end{aligned} \tag{2.10}$$

where l is the total length of the system.

For the given assumptions, the potential energy takes the form

$$\begin{aligned} V(\varphi_i) = V_1 + V_2 + V_3 = \frac{1}{2}c_2\Big\{c_1^*\varphi_1^2 + (\varphi_2 - \varphi_1)^2 + c_3^*(\varphi_3 - \varphi_2)^2 + \\ -2P^*\Big[l_1^*\frac{\varphi_1^2}{2} + l_2^*\frac{\varphi_2^2}{2} + (l_3^* - r^*)\frac{\varphi_3^2}{2}\Big] + 2P^*r^*(\varphi_3^2 - \beta^2) + P^*R^*\beta^2\Big\} \end{aligned} \tag{2.11}$$

2.3.1. Stability of the system

Taking into account the condition describing the equality of longitudinal displacements and arising from Eqs. (2.1) and (2.2), one can write

$$\beta = \frac{l_1^*\varphi_1 + l_2^*\varphi_2 + (l_3^* - R^*)\varphi_3}{R^* - r^*} \tag{2.12}$$

Considering the above equation, potential energy (2.11) takes the form

$$\begin{aligned}V(\varphi_i) &= \frac{1}{2}c_2\Big\{c_1^*\varphi_1^2 + (\varphi_2-\varphi_1)^2 + c_3^*(\varphi_3-\varphi_2)^2 + \\ &-2P^*\Big[l_1^*\frac{\varphi_1^2}{2} + l_2^*\frac{\varphi_2^2}{2} + (l_3^*-r^*)\frac{\varphi_3^2}{2}\Big] + \frac{P^*}{R^*-r^*}[l_1^*\varphi_1 + l_2^*\varphi_2 + (l_3^*-R^*)\varphi_3]^2\Big\}\end{aligned} \tag{2.13}$$

To determine the parameters corresponding with stability of the considered system, essential relationships are specified taking into account the necessary condition for the existence of minimum of the total potential energy

$$\frac{\partial V(\varphi_i)}{\partial \varphi_i} = 0 \tag{2.14}$$

Taking into account relationships (2.13) and (2.14), the following system of equations is obtained

$$[d_{in}][\varphi_i] = 0 \qquad\qquad n = 1,2,3 \tag{2.15}$$

while

$$\begin{aligned}
d_{11} &= 1 + c_1^* - P^*l_1^* + \frac{P^*l_1^{*2}}{R^*-r^*} & d_{12} &= d_{21} = \frac{P^*l_1^*l_2^*}{R^*-r^*} - 1\\
d_{22} &= 1 + c_3^* - P^*l_2^* + \frac{P^*l_2^{*2}}{R^*-r^*} & d_{13} &= d_{31} = \frac{P^*l_1^*(l_3^*-r^*)}{R^*-r^*}\\
d_{23} &= \frac{P^*l_2^*(l_3^*-r^*)}{R^*-r^*} & d_{33} &= \frac{P^*l_2^*(l_3^*-r^*)}{R^*-r^*} - c_3^*\\
d_{33} &= c_3^* - P^*(l_3^*-r^*) + \frac{P^*(l_3^*-r^*)^2}{R^*-r^*}
\end{aligned} \tag{2.16}$$

Due to homogeneity of the system of Eq. (2.15), the critical parameter of the load is determined considering that the matrix determinant d_{in} is set to zero. The results of numerical computations of stability of the considered system are presented in Fig. 3. They are limited to chosen parameters of the system related to geometry of the loading and receiving heads, length of the column rods, chosen rigidity of the springs connected to the rods and rigidity of the prop.

The change of the critical parameter of the load P_c^* was determined for four values of the parameter Δr^* ($\Delta r^* = R^* - r^*$) with a change in the radius R^* of the loading head in the range $R^* \in (R^*_{(j')}, 1)$, $j = 1,\ldots,4$. Each curve is characterised by the presence of the maximum critical load at the considered values of the parameter R^*. In a particular case, when

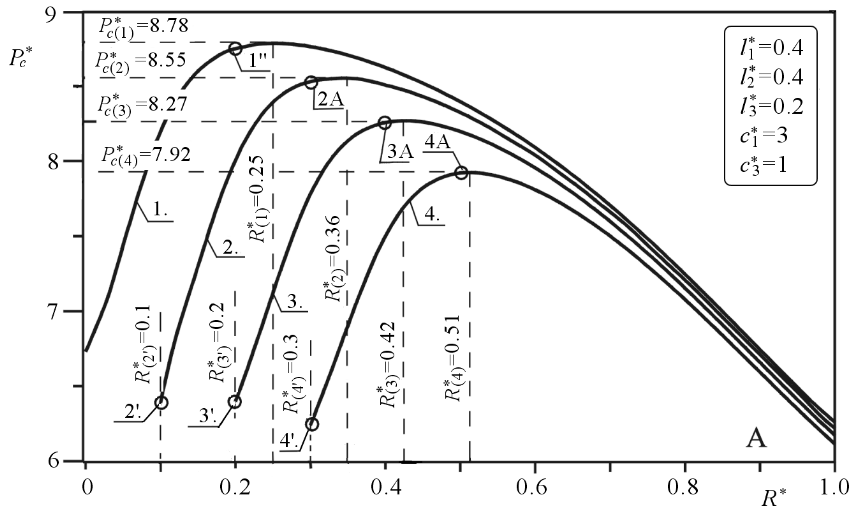

Fig. 3. Changes of the critical parameter of the load P_c^* in relation to R^* within $R^* \in (R_{(j')}^*, 1)$ for the following values of the parameter Δr^*: 1 – $\Delta r^* \to 0$, 2 – $\Delta r^* = 0.1$, 3 – $\Delta r^* = 0.2$, 4 – $\Delta r^* = 0.3$

$R^* = r^*$ (curve 1) the course of changes in the critical parameter of the load corresponds to this parameter for the column loaded by the follower force directed towards the positive pole (see Tomski *et al.*, 1998, 2004; Tomski and Podgórska-Brzdękiewicz, 2006b; Tomski and Szmidla, 2004c). Additionally, points 2A, 3A, 4A, fulfilling $r^* = l_3^*$, determine the critical force for the column loaded by the force directed towards the positive pole (see Timoshenko and Gere, 1961; Gajewski and Życzkowski, 1970), while point $(1'')$ determine the critical load for the system characterised by a hinged mounting at the free end $(R^* = r^* = l_3^*)$ – the case of Euler's load (Timoshenko and Gere, 1961). The load of the system cannot be carried out within the range $R^* \in (0, R_{(j')}^*, 1)$. It is due to the design of the head realising the load $(R^* > r^*)$.

2.3.2. *Free vibrations*

The Lagrange equations (Cannon, 1967; Goldstein, 1950) are taken into account to determine equations of motion of the considered system in the case of the vibration method applied. Considering potential energy (2.13) and kinetic energy (2.8), the following relationships are obtained:

$$\begin{aligned}
&\frac{m_2 l_1^{*2} l^2}{c_2}(1+\gamma)\ddot{\varphi}_1 + \frac{m_2 l_1^* l_2^* l^2 \gamma}{c_2}\ddot{\varphi}_2 + \Big(1 + c_1^* - P^* l_1^* + \frac{P^* l_1^{*2}}{R^* - r^*}\Big)\varphi_1 + \\
&\quad + \Big(\frac{P^* l_1^* l_2^*}{R^* - r^*} - 1\Big)\varphi_2 + \Big[\frac{P^* l_1^*(l_3^* - r^*)}{R^* - r^*}\Big]\varphi_3 = 0 \\
&\frac{m_2 l_1^* l_2^* l^2 \gamma}{c_2}\ddot{\varphi}_1 + \frac{m_2 l_2^{*2} l^2 \gamma}{c_2}\ddot{\varphi}_2 + \Big(\frac{P^* l_1^* l_2^*}{R^* - r^*} - 1\Big)\varphi_1 + \\
&\quad + \Big(1 + c_3^* - P^* l_2^* + \frac{P^* l_2^{*2}}{R^* - r^*}\Big)\varphi_2 + \Big[\frac{P^* l_2^*(l_3^* - r^*)}{R^* - r^*}\Big]\varphi_3 = 0 \\
&\Big(\frac{P^* l_1^*(l_3^* - r^*)}{R^* - r^*}\Big)\varphi_1 + \Big(\frac{P^* l_2^*(l_3^* - r^*)}{R^* - r^*} - c_3^*\Big)\varphi_2 + \\
&\quad + \Big[c_3^* - P^*(l_3^* - r^*) + \frac{P^*(l_3^* - r^*)^2}{R^* - r^*}\Big]\varphi_3 = 0
\end{aligned} \tag{2.17}$$

while

$$\begin{aligned}
&\frac{d}{dt}\Big(\frac{\partial T}{\partial \dot{\varphi}_1}\Big) = (1+\gamma) m_2 l_1^2 \ddot{\varphi}_1 + m_2 l_1 l_2 \gamma \ddot{\varphi}_2 \\
&\frac{d}{dt}\Big(\frac{\partial T}{\partial \dot{\varphi}_2}\Big) = m_2 l_1 l_2 \gamma \ddot{\varphi}_1 + m_2 l_2^2 \gamma \ddot{\varphi}_2 \qquad \frac{d}{dt}\Big(\frac{\partial T}{\partial \dot{\varphi}_3}\Big) = 0
\end{aligned} \tag{2.18}$$

and

$$\ddot{\varphi}_i = \frac{\partial^2 \varphi_i}{\partial t^2} \qquad \gamma = \frac{m_3}{m_2} \tag{2.19}$$

The system of equations (2.17) after separation of variables

$$\varphi_i(t) = \Phi_i \sin(\omega t) \tag{2.20}$$

where ω is the frequency of free vibrations, makes it possible to obtain a transcendental equation describing of frequency of free vibrations Ω in relation to the load P^*, where

$$\Omega = \frac{m_2 l^2 \omega^2}{c_2} \tag{2.21}$$

The course of curves in the plane P^*-Ω for given geometrical and physical parameters of the column is presented in Fig. 4. Curves 1 and 3 present the change of the frequency of free vibrations of the column in relation to the function of external load, adequately: for the system loaded by the force directed towards the positive pole (curves 3) and follower force directed towards the positive pole (curves 1). The critical load for each eigenvalue is determined for $\Omega = 0$. The maximum values of the load parameter are obtained for the column loaded by the follower force directed towards the positive pole. The course of presented curves is characteristic for systems of the divergence pseudo-flutter type.

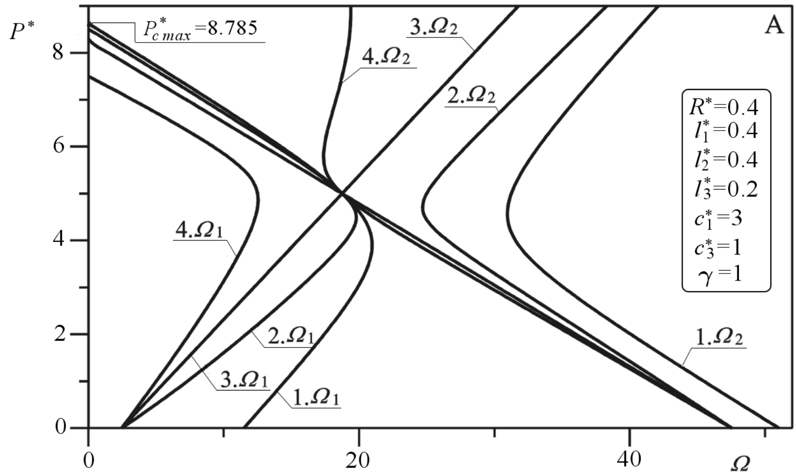

Fig. 4. Curves in the plane: load P^* – frequency of free vibrations Ω for the following values of the parameter Δr^*: 1 – $\Delta r^* \to 0$, 2 – $\Delta r^* = 0.1$, 3 – $\Delta r^* = 0.2$, 4 – $\Delta r^* = 0.3$

3. The system loaded by the follower force directed towards the positive pole (system B)

3.1. Realization of the load by the follower force directed towards the positive pole

The realisation of the load of the discrete system by a force directed towards the positive pole (see Timoshenko and Gere, 1961; Gajewski and Życzkowski, 1970) is presented in Fig. 5a. It is the load applied to the free end of the column, where the direction of action passes through the constant point O located on the non-deformed axis of the column, lying below its free end. Beck's load (follower one) (see Beck, 1952; Bogacz and Janiszewski, 1986) – it is the load by a concentrated force (Fig. 5b), whose direction is tangential to the deflection angle of the free end of the column. The direction line of the force action crosses the non-deformed axis of the column at different points O_1, O_2. The systems presented in Figs. 5a and 5b have three degrees of freedom. Connecting the features of the above systems (Tomski *et al.*, 1998, 2004;

Tomski and Podgórska-Brzdękiewicz, 2006a,b; Tomski and Szmidla, 2004a,c) made structures (compare Fig. 1) for loading the columns, which realise the load by the follower force directed foward the positive pole. This is a load by a concentrated force. Its direction of action coincides with the tangent to the free end of the column and passes through a constant point (pole – point O) placed on the non-deformed axis of the column below its free end. The system presented in Fig. 5c has two degrees of freedom because the follower force is directed towards the pole, what takes away one degree of freedom. This will be proved in the further part of the paper.

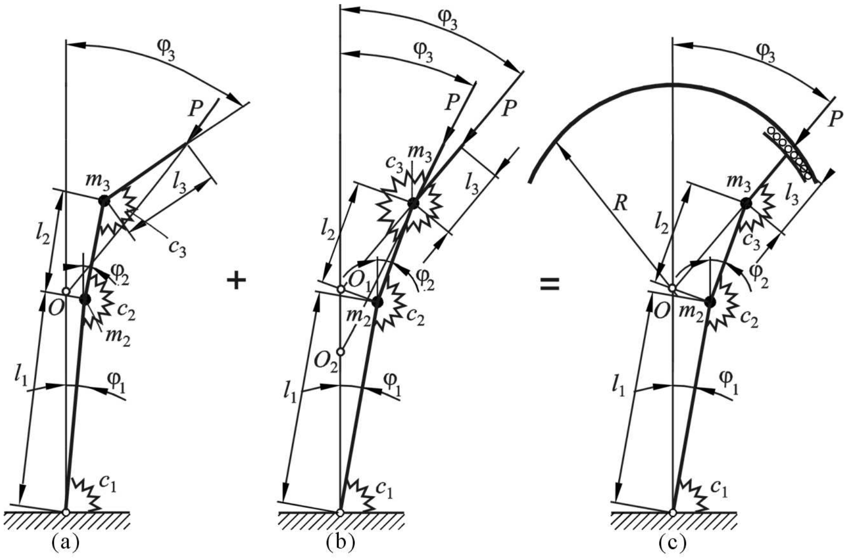

Fig. 5. Realisation of the load by the follower force directed towards the positive pole; (a) load by the force directed towards the positive pole, (b) follower load (Beck's one), (c) load by the follower force directed towards the positive pole

3.2. Physical model of the system

The above presented structure (Fig. 6) loaded by the follower force directed towards the positive pole is identically built as the system subjected to the generalised load by the force directed towards the positive pole (Fig. 2), concerning rods of the system, concentrated masses m_2 and m_3, springs modelling the finite rigidity of structural nodes of the column and its mounting rigidity. The difference relies on the design of receiving head ② (see Tomski *et al.*, 2004; Tomski and Szmidla, 2004a,c), whose constant radius of curvature r has the same value as the radius of curvature of loading head ①. As a result, the direction of the external force P passes through the constant point O located at the distance $r = R$ from the free end of the column. As in the case of the previous system, it is assumed that the elements of loading and receiving heads and rods of the system are infinitely rigid.

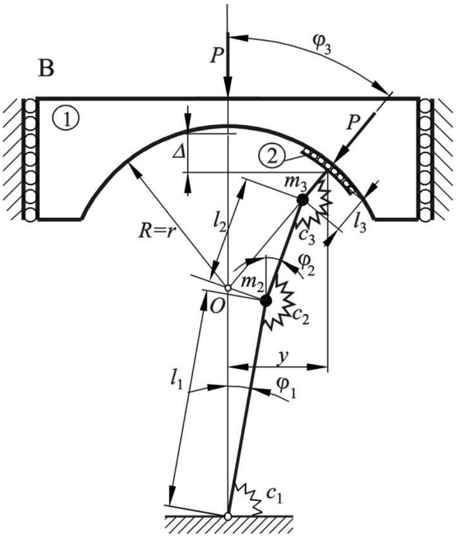

Fig. 6. Physical model of the column loaded by the follower force directed towards the positive pole

3.3. Potential energy of the system

For the presented physical model of the column, considering three variables $\varphi_i(t)$, appropriate relationships for the loading head and rods of the system take form

$$y = R\sin\varphi_3 \qquad\qquad y = l_1\sin\varphi_1 + l_2\sin\varphi_2 + l_3\sin\varphi_3 \tag{3.1}$$

The longitudinal displacement $\varDelta$ of the free end of the column stays in good agreement with relationship $(2.4)_1$. The components of potential energy for the thus defined dependences are:

— potential energy of the component of the vertical force P

$$V_2 = -P\varDelta = -P[l_1(1-\cos\varphi_1) + l_2(1-\cos\varphi_2) + l_3(1-\cos\varphi_3)] \tag{3.2}$$

— potential energy of the component of the horizontal force P

$$V_3 = \frac{P}{2}y\sin\varphi_3 = \frac{P}{2}\sin\varphi_3[l_1\sin\varphi_1 + l_2\sin\varphi_2 + l_3\sin\varphi_3] \tag{3.3}$$

where $\varphi_i = \varphi_i(t)$.

The elastic strain energy V_1 is defined by equation (2.5).

3.4. Stability and free vibrations of the column

3.4.1. Stability of the system

Taking into account dimensionless quantities (2.10) in the total potential energy, assumption of small displacements of structure (2.12) and a condition resulting from equality of transverse displacement of the receiving head and rods (3.1) in the form

$$\varphi_3 = \frac{l_1^*\varphi_1 + l_2^*\varphi_2}{R^* - l_3^*} \tag{3.4}$$

one can obtain the final form of potential energy of the considered column described by two generalized coordinates φ_j $(j = 1, 2)$

$$\begin{aligned} V(\varphi_j) = V_1 + V_2 + V_3 = \frac{1}{2}c_2\Big\{&c_1^*\varphi_1^2 + (\varphi_2 - \varphi_1)^2 + c_3^*[a_1\varphi_1 + (a_2 - 1)\varphi_2]^2 + \\ &+P^*(a_1\varphi_1 + a_2\varphi_2)[(l_1^* + a_1l_3^*)\varphi_1 + (l_2^* + a_2l_3^*)\varphi_2] + \\ &-P^*[l_1^*\varphi_1^2 + l_2^*\varphi_2^2 + l_3^*(a_1\varphi_1 + a_2\varphi_2)^2]\Big\} \end{aligned} \tag{3.5}$$

while

$$a_j = \frac{l_j^*}{R^* - l_3^*} \tag{3.6}$$

In the case of the system loaded by the follower force directed towards the positive pole, derivatives of potential energy (3.6) over generalized coordinates φ_j are represented by the following expressions

$$\begin{aligned} \frac{\partial V}{\partial \varphi_1} &= c_2[1 + c_1^* + a_1^2c_3^* - P^*l_1^*(1 - a_1)]\varphi_1 + \\ &\quad +c_2\Big\{\frac{P^*l_1^*a_2}{2} + a_1\Big[c_3^*(a_2 - 1) + \frac{P^*l_2^*}{2}\Big] - 1\Big\}\varphi_2 \\ \frac{\partial V}{\partial \varphi_2} &= c_2\Big\{\frac{P^*l_1^*a_2}{2} + a_1\Big[c_3^*(a_2 - 1) + \frac{P^*l_2^*}{2}\Big] - 1\Big\}\varphi_1 + \\ &\quad +c_2[1 + c_3^*(a_2 - 1)^2 + P^*l_2^*(a_2 - 1)]\varphi_2 \end{aligned} \tag{3.7}$$

which leads into a transcendental equation for the critical load in the form

$$\begin{aligned} &4[1 + c_1^* + a_1^2c_3^* - P^*l_1^*(1 - a_1)][1 + c_3^*(a_2 - 1)^2 + P^*l_2^*(a_2 - 1)] + \\ &-\{P^*l_1^*a_2 + a_1[2c_3^*(a_2 - 1) + P^*l_2^*] - 2\}^2 = 0 \end{aligned} \tag{3.8}$$

The solution to the above equation is presented in Fig. 7, which illustrates changes of the critical load parameter P_c^* in relation to the radius of the

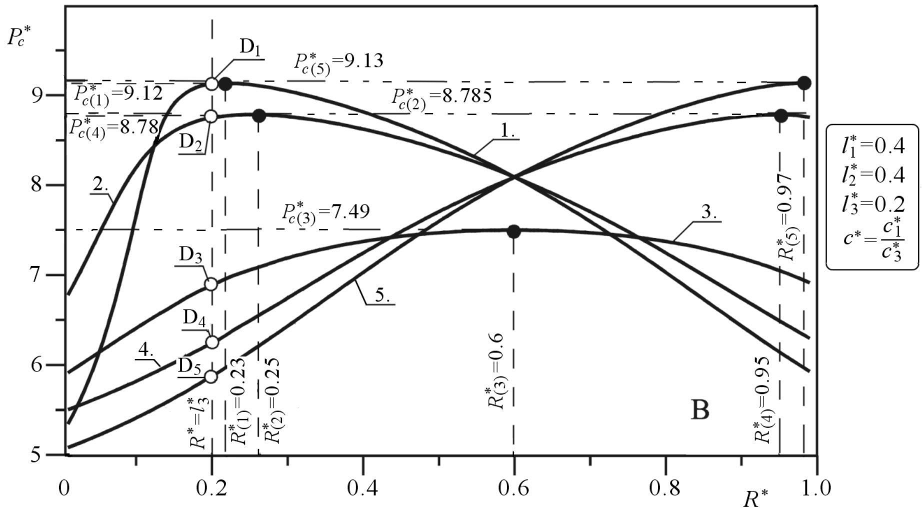

Fig. 7. Changes of the critical load P_c^* with respect to R^* in the range $R^* \in (0,1)$ for the following values of the parameter c^*: $1 - c^* = 4.714$, $2 - c^* = 3$, $3 - c^* = 1$, $4 - c^* = 0.333$, $5 - c^* = 0.212$

receiving head R^* within $R^* \in (0,1)$. Every curve of critical load changes, determined for different relations between rigidities of structural nodes of the system, is characterised by the maximum value of the critical load $P_{c(k)}^*$ (black circles) at $R^* = R_{(k)}^*$, $(k = 1, \dots, 5)$. For $R^* = l_3^*$ (white circles), the considered system realises Euler's load. Selection of identical parameters for lengths of individual rods of the column as in the case of the system subjected to the generalised load by the force directed towards the pole (system A) was aimed at comparing the critical load for both systems presented in the paper. The comparison of the results presented in Fig. 3 and Fig. 7 leads to the conclusion that at identical rigidities of springs (parameters c_i^*) the higher value of the critical load characterises the system loaded by the follower force directed towards the positive pole.

3.4.2. Free vibrations of the system

Substitution of derivatives (2.18) of kinematic energy (2.8) and derivatives (3.7) of potential energy (3.6) into Lagrange's equations (see Cannon, 1967; Goldstein, 1950) leads to the following equations of motion

$$\begin{aligned}
&\frac{m_2 l_1^{*2} l^2}{c_2}(1+\gamma)\ddot{\varphi}_1 + \frac{m_2 l_1^* l_2^* l^2 \gamma}{c_2}\ddot{\varphi}_2 + [1 + c_1^* + a_1^2 c_3^* - P^* l_1^*(1-a_1)]\varphi_1 + \\
&\quad + \Big\{\frac{P^* l_1 a_2}{2} + a_1\Big[c_3^*(a_2-1) + \frac{P^* l_2^*}{2}\Big] - 1\Big\}\varphi_2 = 0 \\
&\frac{m_2 l_1^* l_2^* l^2 \gamma}{c_2}\ddot{\varphi}_1 + \frac{m_2 l_2^{*2} l^2 \gamma}{c_2}\ddot{\varphi}_2 + \Big\{\frac{P^* l_1 a_2}{2} + a_1\Big[c_3^*(a_2-1) + \frac{P^* l_2^*}{2}\Big] - 1\Big\}\varphi_1 + \\
&\quad + [1 + c_3^*(a_2-1)^2 + P^* l_2^*(a_2-1)]\varphi_2 = 0
\end{aligned} \tag{3.9}$$

for which, after separation of variables in the form

$$\varphi_j = \Theta_j \sin(\omega t) \tag{3.10}$$

a transcendental equation is obtained. The transcendental equation describes the natural frequency Ω (compare Eq. (2.21)) in relation to the external load P^*. The effect of changes in the radius of the receiving head on the parameter Ω is presented in Fig. 8 for the assumed geometry and physical constants of the system of rods. The course of changes in eigenvalues corresponds to the system of the divergent (curves 5, 6) or divergence-pseudo-flutter (curves 1-4) type.

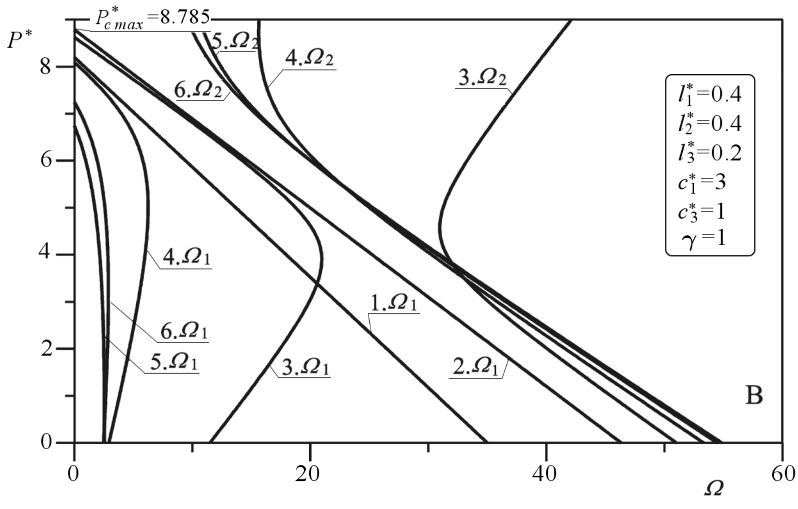

Fig. 8. Curves in the plane: load P^* – natural frequency Ω for the following values of the parameter R^*: 1 $-R^* = 0.1$, 2 – $R^* = 0.25$, 3 – $R^* = 0.4$, 4 – $R^* = 0.6$, 5 – $R^* = 0.8$, 6 – $R^* = 0.9$

4. Conclusions

Physical models of discrete systems subjected to specific loads, including the generalised load by the force directed towards the positive pole as well as by the follower force directed towards the positive pole, are presented in the paper. For the considered column, the total mechanical energy of the system is determined on the basis of the external load of the structure whose direction of action depends on geometry of the loading and receiving heads. Applying the energetic and vibration method, relationships for changes of the critical load and natural frequency in relation to the external load are determined for chosen values regarding geometry and physical constants of the considered discrete systems. The presented character of changes of the critical load allows one to determine such values of parameters for which the considered load has its maximum value. For chosen dependences between geometrical parameters of heads realising the given specific load and lengths of column rods, other known cases of the conservative load of columns are obtained: Euler's one and that loaded by a force directed towards the positive pole. The obtained changes in natural frequencies in relation to the external load allows one to qualify the considered systems to one of the two types of systems, that is divergent or divergence-pseudo-flutter for given parameters describing the geometry and physical constants of the structure.

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References

1. BECK M., 1952, Kniclast des einsetig eingespanuten tangential gedrücken Stabes, *Zeit. Ang. Math. Phys.*, **4**, 225-228

2. BOGACZ R., JANISZEWSKI R., 1986, *Problems of Analysis and Synthesis of Column Loaded by Follower Forces Considering Stability*, Institute of Fundamental Technological Research, Reports 6 [in Polish]

3. CANNON R.H., 1967, *Dynamics of Physical Systems*, MCGraw-Hill Inc.

4. Elishakoff I., 1980, Remarks of the static and dynamic imperfection – sensitivity of nonsymmetric structures, *Journal of Applied Mechanics*, **47**, 111-115
5. Elishakoff I., Birman V., Singer J., 1984, Effect of imperfections on the vibrations of loaded structures, *Journal of Applied Mechanics*, **51**, 191-193
6. Elishakoff I., Marcus S., Starnes J.H., 1996, On vibrational sensitivity of Augustis Model Structure in the vicinity of a non-linear static state, *International Journal Non-Linear Mechanics*, **31**, 2, 229-236
7. Gajewski A., 1972, On the destabilizing effect in a non-conservative system with slight internal and external damping, *Proc. Vibr. Problems*, **13**, 2, 187-198
8. Gajewski A., Życzkowski M., 1970, Optimal design of elastic columns subject to the general conservative behaviour of loading, *ZAMP*, **21**, 806-818
9. Gajewski A., Życzkowski M., 1972, Influence of simultaneous nonhomogeneous external and internal friction on the stability of nonconservative systems, *Mechanika Teoretyczna i Stosowana*, **10**, 1, 127-142 [in Polish]
10. Gajewski A., Życzkowski M., 1988, *Optimal Structural Design under Stability Constraints*, Dordrecht, Kluwer Academic Publishers, 103-107
11. Goldstein H., 1950, *Classical Mechanics*, Addison-Wesley Publishing Company, Inc., Cambridge, Mass, 38-40
12. Herrmann G., Jong I.C., 1965, On the destabilizing effect of damping in nonconservative elastic systems, *J. Appl. Mech.*, **32**, 3, 592-597
13. Herrmann G., Jong I.C., 1966, On nonconservative stability problems of elastic systems with slight damping, *J. Appl. Mech.*, **33**, 1, 125-133
14. Kordas Z., 1963, Stability of the elastically clamped compressed bar in the general case of behaviour of the loading, *Bulletin de L'Academie Polonaise des Sciences*, **XI**, 419-427
15. Leipholz H.H.E., 1974, On conservative elastic systems of the first and second kind, *Ingenieur-Archiv*, **43**, 255-271
16. Roorda J., 1981, Modern methods of analysis of construction stability, Chapter 2 in: *Problems of Stability of Elastic Constructions*, Edited by Z. Waszczyszyn, Zakład Narodowy Im. Ossolińskich, Wydawnictwo Polskiej Akademii Nauk, 81-192 [in Polish]
17. Thompson J.M.T, Hunt G.W., 1973, *A General Theory of Elastic Stability*, John Wiley & Sons, London, New York, Sydney, Toronto
18. Timoshenko S.P., Gere J.M., 1961, *Theory of Elastic Stability*, New York: McGraw-Hill
19. Tomski L., Gołębiowska-Rozanow M., Przybylski J., Szmidla J., 1995, Stability and vibration of a two-member frame under generalised load, *Stability of Steel Structures*, Edited by M. Ivanyi, Vol. 1, Akademiai Kiado, Publ. House of Hungarian Academy of Science, Budapest, 493-500

20. Tomski L., Gołębiowska-Rozanow M., Szmidla J., 1994, Free vibration of a column loaded by a force and bending moment, *Vibrations and WaVES'94*, Poznań-Błażejewko, 317-319

21. Tomski L., Gołębiowska-Rozanow M., Szmidla J., 2004, Vibrations and stability of a two-rod column loaded by the sector of a rolling bearing, *Journal of Theoretical and Applied Mechanics*, **42**, 4, 905-926

22. Tomski L., Podgórska-Brzdękiewicz I., 2006a, Vibrations and stability of columns loaded by three side surfaces of circular cylinders considering type of friction in loading heads, *International Journal of Structural Stability and Dynamics*, **6**, 3, 297-315

23. Tomski L., Podgórska-Brzdękiewicz I., 2006b, Vibrations and stability of columns loaded by four side surfaces of circular cylinders, *Journal of Theoretical and Applied Mechanics*, **44**, 4, 907-927

24. Tomski L., Przybylski J., Gołębiowska-Rozanow M., Szmidla J., 1996, Vibration and stability of an elastic column subject to a generalized load, *Archive of Applied Mechanics*, **67**, 105-116

25. Tomski L., Przybylski J., Gołębiowska-Rozanow M., Szmidla J., 1998, Vibration and stability of a cantilever column subject to a follower force passing through a fixed point, *Journal of Sound and Vibration*, **214**, 1, 67-81

26. Tomski L., Szmidla J., 2004a, Free vibrations and stability of columns loaded by specific load – rigid constructional nods of the forced and taking the load systems, Chapter 3 in: *Vibrations and Stability of Slender Systems*, Edited by L. Tomski, WNT Warsaw, 68-133 [9n Polish]

27. Tomski L., Szmidla J., 2004b, Introduction into the problem of stability and free vibrations of columns, Chapter 1 in: *Vibrations and Stability of Slender Systems*, Edited by L. Tomski, WNT Warsaw, 15-39 [in Polish]

28. Tomski L., Szmidla J., 2004c, Theoretical and experimental investigations of the natural vibrations of the divergence and divergence pseudoflutter type systems, *PAMM*, **4**, 1, 418-419

29. Tomski L., Szmidla J., 2004d, Vibration and stability of column subjected to generalised load by a force directed towards a pole, *Journal of Theoretical and Applied Mechanics*, **42**, 1, 163-193

30. Ziegler H., 1956, On the concept of elastic stability, *Adv. Appl. Mechanics*, **4**, 351-403

31. Ziegler H., 1968, *Principles of Structural Stability*, Waltham Blaisdell Publishing Company

Drgania swobodne i stateczność układów dyskretnych poddanych działaniu obciążenia swoistego

Streszczenie

W pracy prezentuje się układy dyskretne poddane działaniu obciążenia swoistego uogólnionego z siłą skierowaną do bieguna dodatniego oraz siłą śledzącą skierowaną do bieguna dodatniego. Dla zaprezentowanych struktur analizuje się wpływ parametrów geometrycznych głowic realizujących obciążenie oraz sztywności sprężyn rotacyjnych modelujących skończoną sztywność węzłów konstrukcyjnych układu na wartość obciążenia krytycznego oraz na przebieg zmian częstości drgań własnych w funkcji obciążenia zewnętrznego. Odpowiednie związki opisujące stateczność rozważanych kolumn uzyskuje się biorąc pod uwagę energię potencjalną układów (metoda energetyczna) lub całkowitą energię mechaniczną (metoda drgań).

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