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IDENTIFICATION OF BEAM BOUNDARY CONDITIONS IN ILL-POSED PROBLEM

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The purpose of the study is demonstration of the possibility of identification of boundary conditions in an ill-posed problem. The problem is understood as determination of four constants necessary for a description of functions of forced vibration amplitudes from three equations. To this end, the Singular Value Decomposition (SVD) algorithm is used.
After determining the amplitudes of forced vibration, elasticity coefficients of supports can be calculated from the equations describing the boundary conditions.
Verification of the obtained mathematical model (elastically supported Bernoulli-Euler's beam) was done by comparing natural frequencies obtained from analytical and numerical models, and analysing the correlation of forced vibration amplitude vectors for different excitation frequencies.

Key words: Singular Value Decomposition, identification, inverse model

1. Introduction

Analysis of dynamic processes of real objects can be expensive, time-consuming and, in certain cases, impossible, whereas experiments can be easily carried out on models which can be used to simulate dynamic responses. For this purpose, physical and mathematical models of an object should be built and followed by estimation of model parameters and verification. This process is called the identification of mechanical systems (Giergiel and Uhl, 1990).

The bibliography includes many definitions of the identification, such as one given by Bellman (1965), which corresponds best to the analysed problem of identification of boundary conditions:

> "The identification is a process, in which based on certain given information on the system structure and certain information on inputs, outputs and operation of the object the missing information on the structure, inputs and outputs can be obtained."

Here, the analysed system is a beam, described by Bernoulli-Euler's model with unknown boundary conditions, modelled by elastic supports. The mathematical model of the boundary conditions is described by equations coupling respectively the bending moment and angle of rotation of the cross-section as well as the lateral force and amplitude of vibrations in cross sections, in which the beam is supported.

In order to determine support elasticities, an inverse model of the beam has been created. The definition of the inverse model was given by Engel (Engel and Engel, 2005):

> "Inverse modelling is done using the current results of several measurements of visible parameters in order to infer about actual values of the model parameters."

In the paper, the inverse model is understood as determination of functions of amplitudes of forced vibration based on measurements of amplitudes in several points, and then as calculation of support elasticities from equations describing boundary conditions.

The function of vibration amplitudes caused by a given force is described by an equation containing four sought constants. The simplest method of finding the constants is to measure the amplitudes of vibrations caused by a force of a known amplitude and frequency in four points. In order to minimize measuring errors, measurements should be taken in a larger number of points, after which one of standard statistical methods, e.g. regression analysis, can be used. The biggest problem is encountered when the available number of measurement values is lower than the number of constants to be determined the so-called under-determined problem, see Lanczos (1961). This paper concerns such a problem of identification in an ill-posed problem. In such cases, four constants from three equations can be determined by using decomposition of the main matrix by the Singular Value Decomposition (SVD) algorithm (Golub and Van Loan, 1989).

Decomposition by the SVD algorithm is also used in diagnostics of over-determined systems (with excess information about the system) (Cempel and Tobaszewski, 2005; Cholewa and Kiciński, 1997, 2001; Żółtowski and Cempel, 2004), or for determination of inverse models, e.g. in order to find power of acoustic wave sources (Engel, 2004; Engel *et al.*, 2002; Moorhouse, 2003; Stryczniewicz, 2004).

2. The inverse model of a beam

In order to identify boundary conditions of a beam, an inverse model was used. In the paper, it is understood as determination of functions of forced vibration amplitudes based on measurements of vibration amplitudes in several points, and then on calculation of support elasticities from relations describing the boundary conditions.

The model of the beam on elastic supports is shown in Fig. 1.

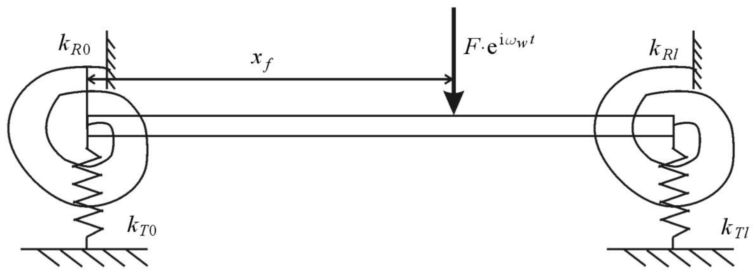

Fig. 1. A beam with general boundary conditions

The differential equation of motion has form

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} + \rho A\frac{\partial^2 y(x,t)}{\partial t^2} = q(x,t) \tag{2.1}$$

The function of load distribution is

$$q(x,t) = F\delta(x,x_f)e^{i\omega_w t}$$

Equation (2.1) can be solved by separating the variables, i.e.: $y(x,t) = X(x)T(t)$.

In the steady-state, the "time" equation may be expressed as

$$T(t) = e^{i\omega_w t}$$

In this case, the differential equation of "space" variable takes form

$$X^{(4)}(x) - \lambda^4 X(x) = \frac{F}{EI}\delta(x,x_f) \tag{2.2}$$

where

$$\lambda^4 = \omega_w^2\frac{\rho A}{EI}$$

and its solution is function (2.3)

$$X(x) = P\cosh\lambda x + Q\sinh\lambda x + R\cos\lambda x + S\sin\lambda x + \\ + \frac{F}{2EI\,\lambda^3}\Big[\sinh\lambda(x-x_f) - \sin\lambda(x-x_f)\Big]H(x,x_f) \tag{2.3}$$

where

EI	–	bending stiffness
A	–	cross-section area
ρ	–	material density
$\delta(x,x_f)$	–	Dirac delta function
$H(x,x_f)$	–	Heaviside step function at $x = x_f$.

Relation (2.3) describes the function (vector) of amplitudes of steady-state vibrations caused by a force with the amplitude F and frequency ω_w applied to the beam at the point with the coordinate $x = x_f$. The constants P, Q, R, S can be determined based on the measurement of vibration amplitudes in several points of the beam. The procedure of determination of the constants is shortly described in the next section of the paper.

After establishing the integration constants P, Q, R, S, the sought values of support elasticity coefficients can be calculated from equations describing the boundary conditions. For the position $x = 0$

$$EI\,X'''(0) = -k_{T0}X(0) \qquad\qquad -EI\,X''(0) = -k_{R0}X'(0) \tag{2.4}$$

hence, the lateral elasticity coefficient is

$$k_{T0} = EI\,\lambda^3\frac{S-Q}{P+R} \tag{2.5}$$

and the rotational elasticity coefficient

$$k_{R0} = EI\,\lambda\frac{P-R}{Q+S} \tag{2.6}$$

The boundary conditions at $x = l$ are described by

$$EI\,X'''(l) = k_{Tl}X(l) \qquad\qquad EI\,X''(l) = -k_{Rl}X'(l) \tag{2.7}$$

hence, the lateral elasticity coefficient is

$$k_{Tl} = EI\,\lambda^3\frac{P\sinh\lambda l + Q\cosh\lambda l - R\sin\lambda l - S\cos\lambda l - f_1}{P\cosh\lambda l + Q\sinh\lambda l + R\cos\lambda l + S\sin\lambda l - f_2} \tag{2.8}$$

where

$$
\begin{aligned}
f_1 &= \frac{F}{2EI\,\lambda^3}[\cosh\lambda(l-x_f)+\cos\lambda(l-x_f)] \\
f_2 &= \frac{F}{2EI\,\lambda^3}[\sinh\lambda(l-x_f)-\sin\lambda(l-x_f)]
\end{aligned}
$$

and the rotational one

$$
k_{Rl} = -EI\,\lambda\frac{P\cosh\lambda l + Q\sinh\lambda l - R\cos\lambda l - S\sin\lambda l - f_3}{P\sinh\lambda l + Q\cosh\lambda l - R\sin\lambda l + S\cos\lambda l - f_4} \tag{2.9}
$$

where

$$
\begin{aligned}
f_3 &= \frac{F}{2EI\,\lambda^2}[\sinh\lambda(l-x_f)+\sin\lambda(l-x_f)] \\
f_4 &= \frac{F}{2EI\,\lambda^2}[\cosh\lambda(l-x_f)-\cos\lambda(l-x_f)]
\end{aligned}
$$

The model has been developed on the assumption that the support elasticities are constant, i.e. they do not depend on the amplitude or vibration frequency.

3. The method of determination of the integration constants

The simplest method of finding the constants P, Q, R, S is to measure vibration amplitudes caused by a force of a known amplitude and frequency at four points. In that case, the four constants can be determined from four equations (2.3) describing the vibration amplitudes at the four measuring points.

In order to minimize measuring errors, the measurements should be taken at a larger number of points, and then one of statistical methods, e.g. regression analysis, can be used (equation (2.3) is a linear equation due to P, Q, R, S constants).

In many cases of diagnostics or identification, it is not possible to obtain full information on the analysed object. In the case analysed here, the "incomplete information" is to be understood that the measurements of vibration amplitudes were taken only in three measuring points.

By assuming that the measuring points are points with the coordinates $x = a$, $x = b$ and $x = c$ (values of the forced vibration amplitudes in these points are marked respectively as $X(a)$, $X(b)$, $X(c)$), we obtain three algebraic equations in form (2.3), which can be written in matrix form

$$\mathbf{M}\boldsymbol{C} = \boldsymbol{B}$$

hence

$$\begin{bmatrix} \cosh\lambda a & \sinh\lambda a & \cos\lambda a & \sin\lambda a \\ \cosh\lambda b & \sinh\lambda b & \cos\lambda b & \sin\lambda b \\ \cosh\lambda c & \sinh\lambda c & \cos\lambda c & \sin\lambda c \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \\ S \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{3.1}$$

where

$$\begin{aligned} b_1 &= X(a) - \frac{F}{2EI\,\lambda^3}[\sinh\lambda(a-x_f) - \sin\lambda(a-x_f)]H(a,x_f) \\ b_2 &= X(b) - \frac{F}{2EI\,\lambda^3}[\sinh\lambda(b-x_f) - \sin\lambda(b-x_f)]H(b,x_f) \\ b_3 &= X(c) - \frac{F}{2EI\,\lambda^3}[\sinh\lambda(c-x_f) - \sin\lambda(c-x_f)]H(c,x_f) \end{aligned}$$

Equation (2.10) is a matrix equation a solution to which can be obtained by inversion of the main matrix, i.e. determination of the matrix $\mathbf{M}^{-1}$. In the case of a rectangular matrix, the inverse matrix can be calculated by decomposing the main matrix according to the Singular Value Decomposition algorithm (Golub and Van Loan, 1989)

$$\mathbf{M} = \mathbf{U}\mathbf{W}\mathbf{V}^\top$$

where

- $\mathbf{U}$ – square matrix of the 3×3 rank, having 3 orthogonal columns corresponding to 3 singular values given in the matrix $\mathbf{W}$, such that $\mathbf{U}^\top\mathbf{U} = \mathbf{1}$
- $\mathbf{W}$ – pseudo-diagonal matrix of the rank 3×4, having non-negative singular values of the matrix $\mathbf{M}$ on its diagonal

$$\mathbf{W} = \begin{bmatrix} w_{1,1} & 0 & 0 & 0 \\ 0 & w_{2,2} & 0 & 0 \\ 0 & 0 & w_{3,3} & 0 \end{bmatrix}$$

- $\mathbf{V}$ – square matrix of thee rank 4×4, having 4 orthogonal columns, such that $\mathbf{V}^\top\mathbf{V} = \mathbf{1}$

Thus, the matrix $\mathbf{M}^{-1}$ can be calculated

$$\mathbf{M}^{-1} = (\mathbf{U}\mathbf{W}\mathbf{V}^\top)^{-1} = (\mathbf{V}^\top)^{-1}\mathbf{W}^{-1}\mathbf{U}^{-1} = \mathbf{V}\mathbf{W}^{-1}\mathbf{U}^\top$$

and sought constant vector can be determined based on matrix obtained as result of decomposition

$$\boldsymbol{C} = \mathbf{V}\mathbf{W}^{-1}\mathbf{U}^{\top}\boldsymbol{B}$$

where

$$\mathbf{W}^{-1} = \begin{bmatrix} \frac{1}{w_{1,1}} & 0 & 0 & 0 \\ 0 & \frac{1}{w_{2,2}} & 0 & 0 \\ 0 & 0 & \frac{1}{w_{3,3}} & 0 \end{bmatrix}$$

After computing the integration constants P, Q, R, S, the sought values of the support elasticities can be calculated from relations (2.5), (2.6), (2.8) and (2.9).

4. Comparative criteria for the models

Verification of the model based on data obtained from an experiment is one of the main problems of the identification. The model obtained from the identification of the system with "incomplete information" (under-determined problem) is an approximation of the real system. The verification stage of the identification involves checking of the obtained approximation for sufficiency of the objective for which the model was created.

The identification objective adopted in the paper is the determination of amplitude vectors of steady state vibrations caused by a force of any frequency below the second eigenfrequency.

The first basic criterion for comparison of the mechanical models is the comparison of their natural frequency (Uhl, 1997). Due to limitations of excitation frequency, the first two free vibration frequencies will be compared here.

The second criterion used in the analysis of the correlation between the models is the visual comparison of amplitude vectors of vibrations caused by forces of different excitation frequencies. Very commonly, a diagram in the Cartesian coordinate system is created, where amplitudes obtained from the experimental model are placed on the x axis, and the same amplitudes from the analytical model are placed on the y axis. If the models are coincident, the corresponding points should be located on the straight line inclined by an angle of $45°$. The mathematical notation of this type of comparison (described and used in modal analysis for finding the correlation between the eigenvectors) is done using the MAC (Modal Assurance Criterion) (Uhl, 1997) coefficient

$$MAC(\boldsymbol{x},\boldsymbol{y}) = \frac{|\boldsymbol{x}^{*\top}\mathbf{W}_g\boldsymbol{y}|^2}{(\boldsymbol{y}^{*\top}\mathbf{W}_g\boldsymbol{y})(\boldsymbol{x}^{*\top}\mathbf{W}_g\boldsymbol{x})}$$

where $\boldsymbol{x}^*$, $\boldsymbol{x}$ and $\boldsymbol{y}^*$, $\boldsymbol{y}$ are two vectors of forced vibration amplitudes obtained from the analytical and experimental model; $\mathbf{W}_g$ is a weight matrix indicating which coordinates of the vector are the most important during the comparison. The analysis is performed on the assumption that the matrix $\mathbf{W}_g$ is a unit matrix. i.e. amplitudes in all points of the beam are identically important.

5. Numerical examples

The subject of the analysis is an elastically supported beam shown in Fig. 1 with the following material data: Young's modulus $E = 2.1 \cdot 10^{11}$ Pa; material density $\rho = 7860\,\mathrm{kg/m^3}$ and geometric data: cross-section $b \times h = 0.03 \times 0.03\,\mathrm{m}$; beam length $l = 1.3\,\mathrm{m}$.

The "experimental" data required for the identification and verification come from vibration analysis using the finite element method. For this purpose, amplitudes were computed for vibrations excited by a force of the amplitude $F = 100\,\mathrm{N}$ applied to the beam at the point with a coordinate $x_f = 0.9\,\mathrm{m}$ and frequency $\omega = 2\pi f$ (for different frequencies f).

The elasticity constants of supports were determined based on the measurement (obtained from FEM analysis) of the vibration amplitudes at three points, where two of them are located at the beam ends ($a = 0$, $c = l$). After determining the elasticity coefficients and decomposing the main matrix according to the Singular Value Decomposition algorithm, the analytical model was verified against the criteria specified above.

The free vibration frequencies were compared by finding deviation defined by the following formula

$$\delta_i = \frac{|\omega_{ie} - \omega_{ia}|}{\omega_{ie}} \cdot 100\% \qquad i = 1, 2 \tag{5.1}$$

where ω_{ie} denotes the ith natural frequency obtained from the experimental model (from FEM analysis here), ω_{ia} – ith natural frequency obtained from the analytical model, where $i = 1, 2$ due to limitations of the excitation frequency (below the second eigenfrequency), i.e. only the first two natural frequencies will be compared.

Figure 2 shows the above defined deviation in the determination of the first and second natural frequency as a function of location of the third measurement point (other two points at the beam ends) for the excitation frequency $\omega = 2\pi \cdot 25 = 157.1\,\mathrm{rad/s}$.

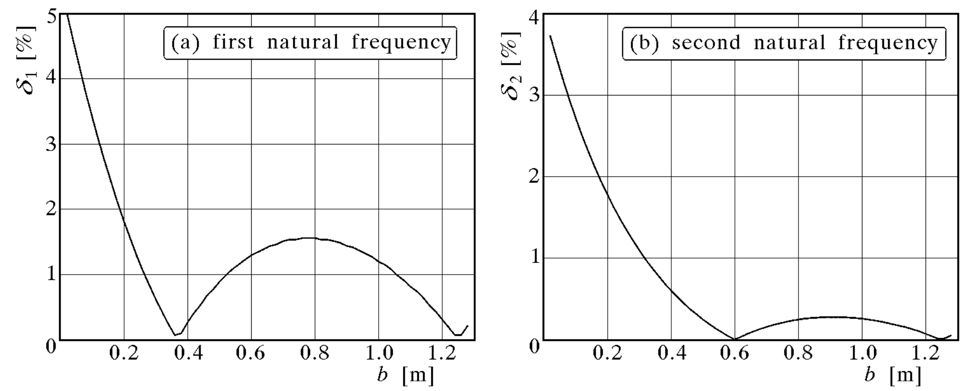

Fig. 2. Deviation in the determination of the first two natural frequencies as a function of location of the third sensor; $\omega = 157.1\,\mathrm{rad/s}$

Natural frequencies from FEM analysis are: $\omega_1 = 262.1\,\mathrm{rad/s}$ and $\omega_2 = 1039.4\mathrm{rad/s}$.

Figure 3 shows the above defined deviation in the determination of the first and second natural frequency as a function of location of the third measurement point for the excitation frequency $\omega = 2\pi \cdot 50 = 314.2\,\mathrm{rad/s}$ (higher than the first eigenfrequency).

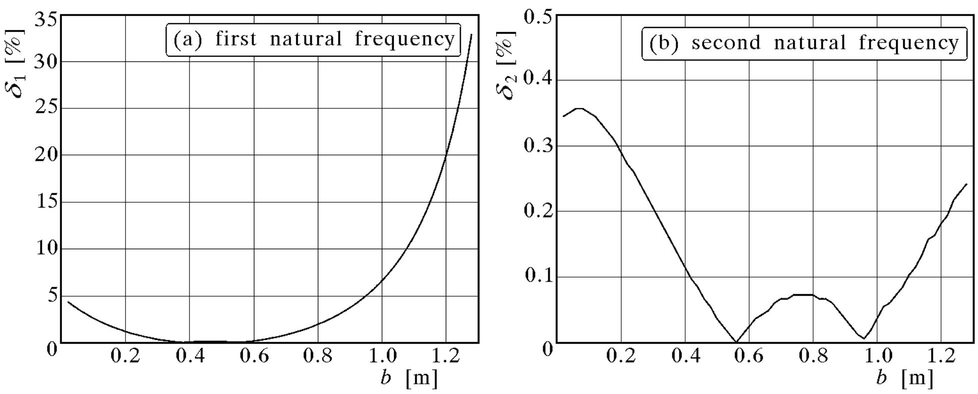

Fig. 3. Deviation in the determination of the first two natural frequencies as a function of location of the third sensor; $\omega = 314.2\,\mathrm{rad/s}$

According to the analysis of the results shown in Figs 2 and 3, smaller identification errors are made when the system is loaded by a frequency lower than the first frequency of free vibration. In such a case, location of the central sensor (other two are located on the beam ends) has no significant effect on uncertainty of the determination of the first two eigenfrequencies (deviations δ_1 and δ_2 below 5%).

In the case of identification with excitation by a force of a frequency higher than the first natural frequency, it is essential (due to the identification error) to find the "appropriate" position of the measuring element (deviation of the frequency determination varies from 0 to 33%).

Afterwords, we will determine the correlation coefficients for the vectors of amplitudes of forced vibration obtained from the experiment $\boldsymbol{X}_e$ and the analytical model $\boldsymbol{X}_a$

$$MAC(\boldsymbol{X}_a, \boldsymbol{X}_e) = \frac{|\boldsymbol{X}_a^\top \boldsymbol{X}_e|^2}{(\boldsymbol{X}_a^\top \boldsymbol{X}_a)(\boldsymbol{X}_e^\top \boldsymbol{X}_e)}$$

Table 1 summarizes the MAC coefficients computed by comparing the forced vibration vectors obtained for 7 different frequencies of the excitation force. The boundary conditions necessary to calculate the vectors of vibrations from the analytical model are obtained from the identification measurements with the excitation frequency $\omega = 2\pi \cdot 25 = 157.1\text{rad/s}$ and $\omega = 2\pi \cdot 50 = = 314.2\text{rad/s}$.

Table 1. Correlation coefficients MAC for vibration amplitude vectors obtained from the analytical and experimental models

Identification	Verification for $\omega = 2\pi f$						
with $\omega = 2\pi f$	$f = 10$	$f = 25$	$f = 50$	$f = 75$	$f = 100$	$f = 125$	$f = 150$
$f = 25$ Hz	0.9983	0.9982	0.9975	0.9989	0.9930	0.9965	0.9953
$f = 50$ Hz	0.9998	0.9997	0.9976	0.9992	0.9992	0.9907	0.9811

The identification and verification measurements were performed for the case in which the central measuring point was in the beam center $(b = l/2)$.

The verification results for the analytical model indicate the correct identification of the boundary conditions of the beam, at least in the assumed band of thr excitation frequency (below the second eigenfrequency).

Another example of identification in the ill-posed problem can be a system shown in Fig. 4 characterised by the following parameters: Young's modulus $E = 2.1 \cdot 10^{11}$ Pa; material density $\rho = 7860\,\text{kg/m}^3$; cross-section dimensions $b \times h = 0.03 \times 0.03$ m; beam length $l = 1.3$ m.

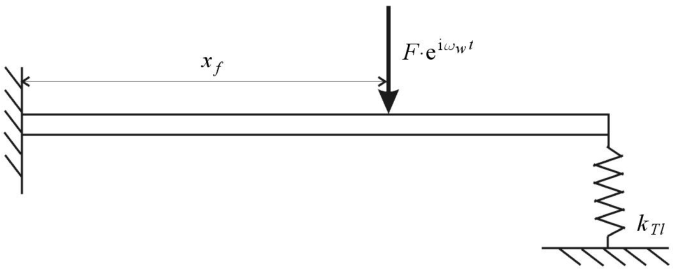

Fig. 4. A cantilever beam with an elastic support

The purpose of the identification measurements is the determination of the lateral elasticity coefficient k_{Tl} by measuring forced vibration amplitudes. As thr identification quantities, two boundary conditions for the beam left end

$$X(0) = 0 \qquad\qquad X'(0) = 0$$

and forced vibration amplitudes in one point of the beam have been adopted.

Using the above mentioned procedure (SVD), we can determine the integration constants P, Q, R, S, which describe the amplitude of forced vibrations (2.3). Having determined the integration constants, the sought value of the elastic coefficient is calculated from the relation

$$k_{Tl} = EI\,\lambda^3 \frac{P\sinh\lambda l + Q\cosh\lambda l - R\sin\lambda l - S\cos\lambda l - f_1}{P\cosh\lambda l + Q\sinh\lambda l + R\cos\lambda l + S\sin\lambda l - f_2} \tag{5.2}$$

where

$$f_1 = \frac{F}{2EI\,\lambda^3}[\cosh\lambda(l - x_f) + \cos\lambda(l - x_f)]$$
$$f_2 = \frac{F}{2EI\,\lambda^3}[\sinh\lambda(l - x_f) - \sin\lambda(l - x_f)]$$

Figure 5 shows the above defined deviation, (5.1), in the determination of the first and second natural vibration frequency as a function of location of the measurement point $(x = a)$ for the excitation frequency $\omega = 2\pi \cdot 20 = 125.7$ rad/s (below the first eigenfrequency).

The natural frequencies found by FEM analysis are: $\omega_1 = 214.0$ rad/s and $\omega_2 = 622.9$rad/s.

Figure 6 shows the deviation in the determination of the first and second natural frequency as a function of location of the measurement point $(x = a)$ for excitation frequency $\omega = 2\pi \cdot 60 = 377.0$ rad/s (above the first eigenfrequency).

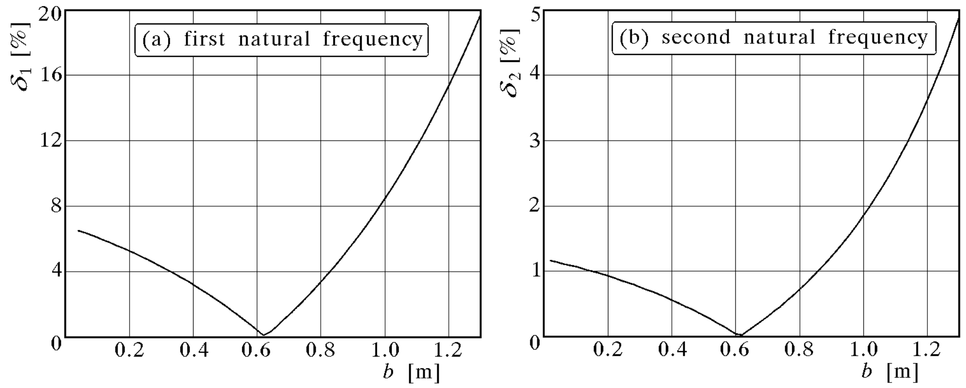

Fig. 5. Deviation in the determination of the first two natural frequencies as a function of the sensor location; $\omega = 125.7\,\mathrm{rad/s}$

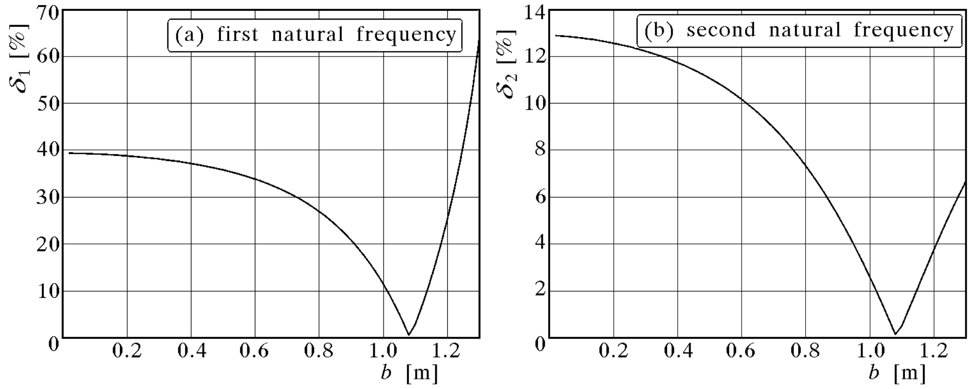

Fig. 6. Deviation in the determination of the first two natural frequencies as a function of the sensor location; $\omega = 377.0\,\mathrm{rad/s}$

Similarly as in the previous example, smaller identification errors are made when the system is loaded by a frequency lower than the first eigenfrequency of the system. However, it is essential to find such a position of the sensor, which will ensure the minimal identification error (deviation varies from 0 to 20%).

Table 2 summarizes the MAC coefficients computed by comparison of the forced vibration vectors obtained for 7 different frequencies of the excitation force. The boundary conditions necessary to calculate the vibration vectors from the analytical model are obtained from the identification measurements with the excitation frequency $\omega = 2\pi \cdot 20 = 125.7\,\mathrm{rad/s}$ and $\omega = 2\pi \cdot 60 = 377.0\,\mathrm{rad/s}$.

Table 2. Correlation coefficients MAC for vibration amplitude vectors obtained from analytical and experimental models

Identification	Verification for $\omega = 2\pi f$						
with $\omega = 2\pi f$	$f = 10$	$f = 20$	$f = 40$	$f = 50$	$f = 60$	$f = 80$	$f = 90$
$f = 20\,\mathrm{Hz}$	1.0	1.0	1.0	1.0	1.0	1.0	0.9999
$f = 60\,\mathrm{Hz}$	0.956	0.9538	0.9441	0.935	0.9213	0.8731	0.8713

Verification results for the analytical model indicate correct identification of the boundary conditions of the beam (it is assumed that good coincidence of vectors is for $MAC > 0.8$ (Uhl, 1997). In all calculations, it has been adopted that the measuring point is located in the beam center.

6. Conclusions

The purpose of the study was to demonstrate the possibility of identification of beam boundary conditions in an ill-posed problem. In the analysed cases, the problem was to be understood as determination of four constants necessary for description of functions of forced vibration amplitudes from three equations. In the first case, these equations described amplitudes of vibrations in points where measuring elements (sensors) were located, or in the second case, two equations were available for the description of the boundary conditions and vibration amplitude at the measuring point.

Verification of the obtained mathematical model (elastically supported Bernoulli-Euler's beam) was done by comparing natural frequencies obtained from the analytical model and the numerical experiment, and analysing the correlation of forced vibration amplitude vectors for different excitation frequencies.

In both analysed cases, the deviation in the determination of the first and second free vibration frequency was computed as a function of location of one measuring element. Identification of boundary conditions was done for two frequencies of the excitation force: below and above the first eigenfrequency.

According to the results of analysis presented in Figs 2, 3, 5 and 6, smaller identification errors are made when the system is loaded by a frequency lower than the first eigenfrequency. However, it is essential to find such a position of the measuring element, which will ensure the minimal identification error.

Another criterion used in the analysis for studying the correlation between the models (described and used in modal analysis for finding the correlation between eigenvectors) is the MAC (Modal Assurance Criterion) coefficient. MAC coefficients for vectors of forced vibration obtained from the experiment and the analytical model have been summarized in Tables 1 and 2. All MAC coefficients are greater than 0.8, above which, good coincidence of vectors is assumed.

The MAC coefficient defines only the similarity of forms between vectors – its value is not affected by vibration amplitudes, i.e. multiple vectors (e.g. amplitudes of vibrations obtained from the experiment can be n-times greater than those obtained from the analytical model in each cross-section of the beam) have the same shape (i.e. $MAC = 1$).

Therefore, it is only necessary to analyse the vibration amplitude of any single cross-section of the beam. The amplification factor (for a given excitation frequency) depends on the location of resonant frequencies. So selection of the position of measuring sensors which ensures the minimal error of determination of the natural frequency, also ensures the minimal error of determination of the amplitudes of forced vibrations.

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Identyfikacja warunków brzegowych belki w układach o niepełnej informacji

Streszczenie

Praca dotyczy identyfikacji warunków brzegowych belki w przypadkach, w których nie ma możliwości uzyskania pełnej informacji o układzie. Niepełna informacja wynika, w rozważanym w pracy przypadku, z problemu wyznaczenia czterech stałych całkowania, niezbędnych do opisania funkcji amplitud drgań wymuszonych belki, z trzech równań. Do tego celu wykorzystano algorytm rozkładu macierzy względem wartości szczegónych (*Singular Value Decomposition*).

Po wyznaczeniu stałych całkowania i funkcji amplitud drgań wymuszonych, uogólnione współczynniki sprężystości podparcia wyznaczono z równań opisujących warunki brzegowe.

Weryfikacji tak uzyskanego modelu matematycznego dokonano poprzez porównanie częstości drgań własnych i wyznaczenie wspólczynników korelacji wektorów drgań wymuszonych.

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