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INFLUENCE OF ADDITIONAL MASS RINGS ON FREQUENCIES OF AXI-SYMMETRICAL VIBRATIONS OF CLAMPED CIRCULAR PLATES OF LINEARLY VARIABLE THICKNESS

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The aim of this paper is to analyze the influence of values and radius of an additional mass ring on the continuous distribution of mass of a clamped circular plate of linearly variable thickness. The linear theory of thin plates is used for description of small buckling vibrations. The authors applied the partial discretization method which is based on the discretization of the continuous mass and continuous buckling rigidity function. It is also based on the method of Cauchy's influence function, which gives particularly exact effects for distributed-continuous systems such as that presented in this paper. It is shown that an approximate result leads to the exact value with the discretization degree of less than 5, and it is not dependent on the value and radius of the concentrated mass. Exact results of calculations lead to accurate values discovered by Conway for plates of linearly variable thickness without an additional mass and to accurate values discovered by Roberson for plates of constant thickness with the mass concentrated in the center.

Key words: circular plates, variable thickness, boundary-value problem, partial discretization method

1. Introduction

The model presented in this paper can be applied to numerous structures such as diaprahrgms, bottom parts of boilers and cylindrical containers, etc. Boundary-value problems of axi-symmetrical vibrations of circular plates of variable thickness carrying an additional mass distributed in concentric rings are investigated in this paper. Well known papers by Roberson and Conway are devoted to investigations of a concentrated mass in the centre of a plate of constant thickness or of a plate of variable thickness without any discrete inclusions. The investigation concerns an analogy between the lateral vibration of a conical bar and axi-symmetric vibration of a circular disc of linearly varying thickness (cf. Conway, 1957). Conway analyzed the basic natural frequency of axi-symmetric vibrations of circular plates with clamped edge, whose flexural rigidity D varies with the radius r according to the law $D = D_0 r^m$, where D_0 and m are constants (cf. Conway, 1958). Free, flexural, axi-symmetric vibrations of a clamped circular disc were investigated, solutions in terms of Bessel functions were found for certain values of m and Poisson's ratio ν. Vibrations of a circular plate clamped at its edge and carrying a concentrated mass at its center were considered in Roberson (1951). The plate was excited by motion of the rigid framing, to which it had been clamped. The first four natural frequencies were displayed graphically as functions of the mass ratio and were calculated more precisely for $\mu = 0$, $\mu = 0.05$ and $\mu = 0.10$ (cf. Roberson, 1951). An auxiliary mass μ_0 was concentrated on a concentric radius of the plate, which did not have a thickness dimension and exerted no effect upon the flexibility of the plate.

A method of partial discretization was applied to the investigation of the influence of mass on vibrations of a circular plate of constant thickness (Zoryj and Jaroszewicz, 2000). The effectiveness of that method was evaluated. The paper was focused on analysis of the influence of the mass concentrated at the centre of the plate on the main frequency. In the paper, a method of spectral functions, proposed by Bernštein and Kieropian (1960) was used solely for the analysis of systems characterised by constant parameters with no consideration given to friction.

In this paper, we propose double sided Bernštein-Kieropian estimators for the frequency which should be calculated with different degrees of accuracy. Therefore, the method of characteristic series, which makes use of the Cauchy influence function to solve these problems, appears to be attractive. The direction of the theorem of vibrations of continually-discrete, linearly elastic systems (based on utilizing the Cauchy influence functions and the characteristic series method) is useful for constructing and studying universal frequency equations (cf. Zoryj, 1982; Jaroszewicz and Zoryj, 1994, 1996). Jaroszewicz and Zoryj (2005) proved that FEM is very effective in vibration analysis but in comparison with the Cauchy function method, the functional dependence of mass rigidity is characteristic. It gives directional optimality of such plates.

As it is well known, equations of motion can be presented by forces or displacements which are applied in the partial discretization method. Using the method of partial discretization for a plate with a continuous or discrete-continuous distribution of mass is replaced by discrete systems with one, two or n degrees of freedom dependending on the discretization degree. The rigidity distribution of created discrete systems is the same as in the input system – the masses are concentrated on chosen radii, their radii have no thickness and they do not influence the flexural rigidity of the plate.

This investigation method introduces an additional mass where the distribution depends on the radial coordinate. Jaroszewicz (2000) considered a constant thickness plate only.

Figure 1 shows a model of a clamped circular plate with variable thickness and with a mass inclusion. R denotes radius of the plate, h_0 – thickness of the plate on clamping, r_i – radius of the mass ring, m_i, c_i – values of the concentrated masses and elastic supports.

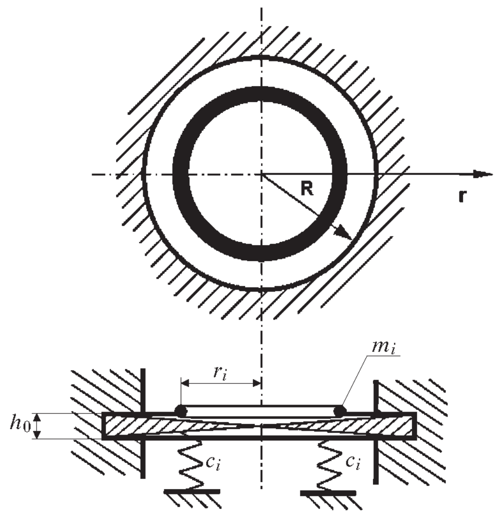

Fig. 1. Model of the plate

2. Formulation of the problem

We consider circular plate of the radius R with discrete inclusions, the location of which depends on the radial coordinate r_i and, its flexural rigidity D is a power function in the following form

$$D = D_0\Big(\frac{r}{R}\Big)^m \qquad 0 < r \leqslant R \tag{2.1}$$

There are two possible variants of changes of the plate thickness:

- $m > 0$ – a diaphragm with thickness decreasing toward the axial center
- $m < 0$ – a disc with thickness increasing toward the axial center.

If $m = 0$. the plate thickness is constant.

Investigation of free axi-symmetrical vibrations of such a plate leads to the following boundary problem (cf. Zoryj and Jaroszewicz, 2002)

$$
\begin{aligned}
&L_0[u] - \frac{\rho h_0 \left(\frac{r}{R}\right)^{\frac{m}{3}}}{D_0 \left(\frac{r}{R}\right)^m} \omega^2 u - \sum_{i=1}^{K} \alpha_i u(r_i) \delta(r - r_i) = 0 \\
&0 < r_1 < r_2 < \ldots < r_K < R
\end{aligned}
\tag{2.2}
$$

for which the following boundary conditions exist

$$
u(R) = 0 \qquad\qquad u'(R) = 0 \tag{2.3}
$$

These conditions correspond to the case of a circumferentially fixed circular plate.

The differential operator has the form (cf. Hondkiewic, 1964)

$$
L_0[u] \equiv u^{\mathrm{IV}} + \frac{2}{r}(m+1)u^{\mathrm{III}} + \frac{1}{r^2}(m^2 + m + \nu m + 1)u^{\mathrm{II}} + \frac{1}{r^2}(m-1)(\nu m - 1)u^{\mathrm{I}}
$$

The following nomenclature is used in (2.1)-(2.4): m – coefficient of the power flexural rigidity function, whose value may assume arbitrary real numbers; $D_0 = Eh_0^3/[12(1-\nu^2)]$ – flexural rigidity at $r = R$; $u = u(r)$ – amplitude of deflection; $h = h_0(r/R)^{m/3}$ – variable thickness; h_0 – thickness at $r = R$; ω – frequency parameter; $\delta(r)$ – Dirac's delta function; ρ – mass density of the plate; ν – Poisson's ratio. The inclusion parameters in discrete masses m_i and rigidity of elastic supports c_i, which are located on the radius circle r_i, are described by the following formula

$$
\alpha_i = \frac{1}{D_0 \left(\frac{r_i}{R}\right)^m}(m_i \omega^2 - c_i) \qquad\qquad (i = \overline{1, K}) \tag{2.4}
$$

In this case, solutions to equation $L_0[u] = 0$ in expressions (2.2) and (2.3) as well as their first derivates at $r = 0$ should be limited. For circular plates of the diaphragm type, we have $h(r) \geqslant 0$ because $0 < r \leqslant R$, $m > 0$ and $h_0 > 0$.

The substitute model of the analyzed plate (Fig. 1) is defined by following assumptions:

- The additional mass ring is distributed on the circle with the radius r_0 from the range 0 to R.
- The linear theory of thin plates and small deflections is used (cf. Timoshenko, 1940).
- The rigidity of elastic supports c_i is neglected.

The assumption of the partial discretization method is the substitution of continuous distribution of the plate mass by a sequence of concentrated masses m_i located on circles with radii r_i, which can be determined by the following formula (cf. Jaroszewicz and Zoryj, 2006)

$$r_i = \frac{R}{2K}(2i-1) \qquad (i=\overline{1,K}) \tag{2.5}$$

where K denotes the degree of discretization, which is equal to the amount of concentrated masses that substitute the continuous mass of the plate. The number of distributed masses depends on the thickness variability coefficient (m).

For a circular plate of linearly variable thickness $(m=3)$, the concentrated masses equivalent to the degree of discretization K have following form

$$M_i = \frac{m_i}{2\pi} = \frac{K^3-(K-1)^3}{3K^3}\rho h_0 R^2 \qquad (i=\overline{1,K}) \tag{2.6}$$

The sum of masses received from the discretization is equal to the total mass of the plate

$$M = \frac{2}{3}\pi R^2 \rho h_0$$

For a plate with one additional mass ring m_0 of the radius r_0, the relative value of the additional mass with respect to the total plate mass μ_0 and its relative radius χ_0 are defined by the following formulas

$$\mu_0 = \frac{m_0}{\pi R^2 \rho h_0} \qquad \chi_0 = \frac{r_0}{R} \tag{2.7}$$

3. Determination of the influence matrix for plates with linearly variable thickness

The elements of the flexibility influence matrix are found by considering an appropriate static problem (cf. Jaroszewicz and Zoryj, 2005)

$$\begin{aligned} &L_0[u] \equiv u^{\mathrm{IV}} + \frac{8}{r}u^{\mathrm{III}} + \frac{12}{r^2}u^{\mathrm{II}} = \frac{F_j R^3}{r_j^4 D_0}\delta(r-r_j) \\ &u(R)=0 \qquad u'(R)=0 \end{aligned} \tag{3.1}$$

The differential operator $L_0[u]$ in (3.1) has been obtained for the coefficient of the power flexural rigidity function $m=3$, which corresponds to linearly variable thickness and Poisson's ratio $\nu=1/3$, which corresponds to steel.

The value of F_j characterises a flexural force concentrated on a circle with the radius $r_j(r_j \geqslant r_i)$.

A limited solution to equation (3.1) is determined by the formula

$$u = C_0 + C_1 r + \frac{F_j R^3}{r_j^4 D_0} K_0(r, r_j)\Theta(r - r_j) \tag{3.2}$$

where: C_0, C_1 are arbitrary constants; $K_0(r, r_j) = K_0(r, \alpha)\big|_{\alpha=r_j}$; $K_0(r, \alpha)$ – fundamental function of the operator $L_0[u]$; $\Theta(r)$ – Heaviside's function.

From (3.2) we obtain

$$u' = C_1 + \frac{F_j R^3}{r_j^4 D_0} K_0'(r, r_j)\Theta(r - r_j) \tag{3.3}$$

Substituting the right-hand sides of (3.2) and (3.3) to the boundary conditions of (3.1), we determine the constants C_0, C_1 after which, from formula (3.2), we obtain a solution to the mentioned problem in the form

$$u_j = \frac{F_j R^3}{r_j^4 D_0}[K_{Rj}'(R - r) - K_{Rj} + K_{rj}\Theta_{rj}] \tag{3.4}$$

where the following denotes

$$\begin{aligned} K_{Rj} &\equiv K_0(R, r_j) \qquad & K_{Rj}' &\equiv K_0'(R, r_j) \\ K_{rj} &\equiv K_0(r, r_j) \qquad & \Theta_{Rj} &\equiv \Theta(R - r_j) \end{aligned} \tag{3.5}$$

Hence, considerin that $\beta_{ij} = u_j(r_i)$ for $F_j \equiv 1$ (Zoryj and Jaroszewicz, 1982), we arrive from (3.4) at the following formula

$$\beta_{ij} = \frac{R^3}{r_j^4 D_0}[K_{Rj}'(R - r_i) - K_{Rj}] \tag{3.6}$$

since $r_i \leqslant r_j$, then $\theta_{rj} = 0$. It is know (cf. Zoryj and Jaroszewicz, 2002) that the fundamental function of the operator $L_0[u]$ in equation (3.1) has the form

$$K_0(r, \alpha) = \frac{1}{6}(r\alpha^2 - \alpha^5 r^{-2}) + \frac{1}{2}(\alpha^4 r^{-1} - \alpha^3) \tag{3.7}$$

Substituting dependencies (3.4), (3.5) to (3.6), we determine the elements of the flexibility influence matrix $\boldsymbol{\beta} = [\beta_{ij}]$ (cf. Zoryj and Jaroszewicz, 2005)

$$\beta_{ij} = \frac{R^2}{3D_0\chi_i}\Big[\frac{3}{2}\chi_i(\chi_i + \chi_j) + \frac{3}{2}\frac{\chi_i}{\chi_j} - \frac{1}{2}\Big(\frac{\chi_i}{\chi_j}\Big)^2 - 3\chi_i - \chi_i^2\chi_j\Big] \qquad \chi_i = \frac{r_i}{R} \tag{3.8}$$

In particular, the diagonal monomial elements

$$\beta_{ij} = \frac{R^2}{3D_0\chi_i}(1-\chi_i)^3 \qquad (i=\overline{1,K}) \tag{3.9}$$

are consistent with the formula obtained by Zoryj and Jaroszewicz (2002). It should be noticed from (3.6) that the multinominal elements β_{ij} could be arrived at equally similarly, and it might be proved that $\beta_{ji} = \beta_{ij}$, which confirms symmetry of the matrix $\boldsymbol{\beta}$.

It is not difficult to find formulas for β_{ji} for plates with constant thickness

$$\beta_{ij} = \Big(1 - \frac{r_j^2 - r_i^2}{R^2} - \frac{r_j^2 r_i^2}{R^2} + 2\frac{r_j^2 + r_i^2}{R^2}\ln\frac{r_j}{R}\Big)\frac{R^2}{8} \qquad (i \leqslant j) \tag{3.10}$$

and $\beta_{ij} = \beta_{ji}$, $0 < r_1 < r_2 < \ldots < r_k < R$.

4. The characteristic equation and the double estimator for the base frequency

Formulas (3.8) and (3.9) allow us to write down an equation for small vibrations of a given model for arbitrarily limited number K, in an reverse form that is expressed through dislocations (Zoryj, 1982)

$$\sum_{j=1}^{K} M_j\beta_{ij}\frac{d^2q_j}{dt^2} + q_i = 0 \qquad (i=\overline{1,K}) \tag{4.1}$$

where q_j and M_j represent generalised coordinates and concentrated masses defined in (2.7), respectively.

The natural frequencies and forms of the system described by equation (4.1) can be calculated by well known methods (cf. Jaroszewicz and Zoryj, 2005). For the formulated influence matrix and equation of motion of the discretizated system for $K = 2$, a characteristic equation for plates without additional masses has been obtained in a power series form

$$\varDelta = 1 - a_1\varLambda + a_2\varLambda^2 - \ldots = 0 \tag{4.2}$$

where

$$\varLambda = \omega^2\frac{R^4\rho h}{D} \qquad\qquad a_0 = 1$$

$$a_1 = \frac{m_1}{2\pi}\beta_{11} + \frac{m_2}{2\pi}\beta_{22} \qquad\qquad a_2 = \frac{m_1m_2}{4\pi^2}(\beta_{11}\beta_{22} - \beta_{12}^2)$$

The estimators of the base frequency ω_1 can be calculated by Bernstein's double estimators for frequency coeffcients in equation (4.2) (cf. Jaroszewicz *et al.*, 2004)

$$\omega_1 = \gamma_{\pm}\frac{1}{R^2}\sqrt{\frac{D_0}{\rho h_0}} \tag{4.3}$$

where

$$\gamma_- = \sqrt{\frac{a_0}{\sqrt{a_1^2 - 2a_0a_2}}} \qquad \gamma_+ = \sqrt{\frac{2a_0}{a_1 + \sqrt{a_1^2 - 4a_0a_2}}}$$
$$\gamma_{\pm} = \frac{1}{2}(\gamma_- + \gamma_+)$$

By taking into consideration (4.2) and replacing in it a with $\widetilde{a}$, it is easy to get an equation for the plate with an additional mass ring

$$\widetilde{a}_0 - \widetilde{a}_1\Lambda + \widetilde{a}_2\Lambda^2 = 0 \tag{4.4}$$

where

$$\widetilde{a}_0 = 1 \qquad \widetilde{a}_1 = \frac{m_0}{2\pi}\beta_{00} + a_1$$
$$\widetilde{a}_2 = \frac{m_0m_1}{4\pi^2}(\beta_{00}\beta_{11} - \beta_{01}^2) + \frac{m_0m_2}{4\pi^2}(\beta_{00}\beta_{22} - \beta_{02}^2) + a_2$$

5. Results of calculations

As an example of calculation, a linearly variable thickness plate with one additional mass ring is given. Bellow, calculations for a simple case of two-degree discretization with one additional mass, which corresponds to $K+1$, are presented. This single additional mass increases the degree of discretization by 1. Using formulas (2.5)-(2.7), (3.8), (4.2)-(4.4), the following values have been obtained

$$\chi_0 = 0.2 \qquad \chi_1 = 0.25 \qquad \chi_2 = 0.75$$
$$m_0 = 0.1\pi\rho h_0R^2 \qquad m_1 = \frac{1}{12}\pi\rho h_0R^2 \qquad m_2 = \frac{7}{12}\pi\rho h_0R^2$$

$$\widetilde{a}_0 = 1 \qquad \widetilde{a}_1 \Lambda = \frac{1}{2\pi}(m_0\beta_{00} + m_1\beta_{11} + m_2\beta_{22}) = 0.068129\frac{R^4\rho h_0}{D_0}$$

$$\widetilde{a}_2 \Lambda = \frac{1}{4\pi^2}\Big[m_0 m_1 \begin{vmatrix} \beta_{00} & \beta_{01} \\ \beta_{01} & \beta_{11} \end{vmatrix} + m_0 m_2 \begin{vmatrix} \beta_{00} & \beta_{02} \\ \beta_{02} & \beta_{22} \end{vmatrix} + m_1 m_2 \begin{vmatrix} \beta_{11} & \beta_{12} \\ \beta_{12} & \beta_{22} \end{vmatrix}\Big] =$$

$$= 2.35492 \cdot 10^{-5}\Big(\frac{R^4\rho h_0}{D_0}\Big)^2$$

$$\beta_{00} = \frac{R^2(1-\chi_0)^3}{3D_0\chi_0} = 0.8533\frac{R^2}{D_0}$$

$$\beta_{11} = \frac{R^2(1-\chi_1)^3}{3D_0\chi_1} = 0.5625\frac{R^2}{D_0}$$

$$\beta_{22} = \frac{R^2(1-\chi_2)^3}{3D_0\chi_2} = 0.00694\frac{R^2}{D_0}$$

$$\beta_{01} = \frac{R^2}{3D_0\chi_0}\Big[\frac{3}{2}\chi_0(\chi_0+\chi_1) + \frac{3}{2}\frac{\chi_0}{\chi_1} - \frac{1}{2}\Big(\frac{\chi_0}{\chi_1}\Big)^2 - 3\chi_0 - \chi_0^2\chi_1\Big] = 0.675\frac{R^2}{D_0}$$

$$\beta_{02} = \frac{R^2}{3D_0\chi_0}\Big[\frac{3}{2}\chi_0(\chi_0+\chi_2) + \frac{3}{2}\frac{\chi_0}{\chi_2} - \frac{1}{2}\Big(\frac{\chi_0}{\chi_2}\Big)^2 - 3\chi_0 - \chi_0^2\chi_2\Big] = 0.0324\frac{R^2}{D_0}$$

$$\beta_{12} = \frac{R^2}{3D_0\chi_1}\Big[\frac{3}{2}\chi_1(\chi_1+\chi_2) + \frac{3}{2}\frac{\chi_1}{\chi_2} - \frac{1}{2}\Big(\frac{\chi_1}{\chi_2}\Big)^2 - 3\chi_1 - \chi_1^2\chi_2\Big] = 0.03009\frac{R^2}{D_0}$$

$$\omega_1 = 3.841\frac{1}{R^2}\sqrt{\frac{D_0}{\rho h_0}}$$

$$(\gamma_1)_- = 3.8409557$$

$$(\gamma_1)_+ = 3.8409809$$

Calculations were carried out for the above-mentioned example for many values of coefficients μ_0 and χ_0. The results are presented in tables and figures below. The natural frequency is displayed graphically as a function of the mass ring coefficients μ_0 and χ_0 (Fig. 2) and as a function of the discretization degree K (Fig. 3-Fig. 5). The results of calculations are presented in Table 1 to Table 5. Table 1 and Table 2, and in Fig. 2 give results for plates with linearly variable thickness ($m = 3$, $\nu = 1/3$).

In order to check the method with Roberson's results, particular cases with constant thickness of different values and radii of the additional mass ring have been investigated. The results are presented in Table 3 to Table 5 and in Fig. 3 to Fig. 5.

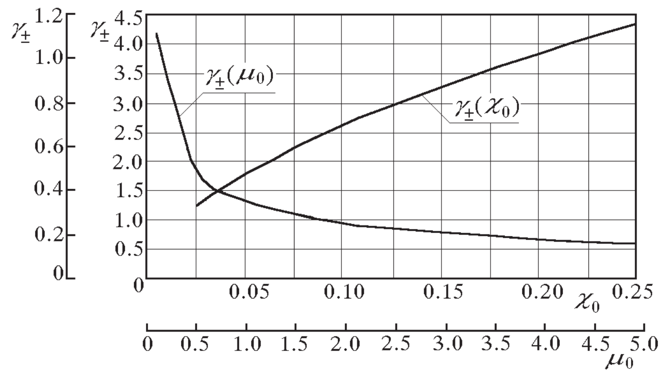

Fig. 2. Double estimators for the natural frequency parameter $(\gamma_\pm)$ of the plate with linearly variable thickness with one additional mass versus coefficients μ_0 and χ_0

Table 1. Calculations of $\gamma_\pm(\mu_0)$ for $\chi_0 = 0.02$, $K = 2$ (when $\mu_0 = 0.001$, the discretization degree was $K = 10$)

μ_0	0.001	0.5	0.99	2	2.5
$\gamma_\pm$	7.7918	0.5035	0.3583	0.2523	0.2257
μ_0	3	3.5	4	4.5	5
$\gamma_\pm$	0.2061	0.1908	0.1785	0.1683	0.1597

Table 2. Calculations of $\gamma_\pm(\chi_0)$ for $\mu_0 = 0.1$, $K = 2$

μ_0	0.025	0.05	0.075	0.1	0.125
$\gamma_\pm$	1.2471	1.7936	2.2308	2.6119	2.9564
μ_0	0.15	0.175	0.2	0.225	0.25
$\gamma_\pm$	3.2732	3.5672	3.841	4.0961	4.3336

Table 3. Calculations for $\chi_0 = 0.001$; $\mu = 0$

K	2	3	4	5
a_1	0.0118	0.011	0.0107	0.0106
a_2	$9.16097 \cdot 10^{-6}$	$9.2951 \cdot 10^{-6}$	$8.8804 \cdot 10^{-6}$	$8.6301 \cdot 10^{-6}$
γ_-	9.5127	9.9196	10.0501	10.1074
γ_+	9.5262	9.94	10.071	10.1282
$\gamma_\pm/\Delta$	9.5194/6.8005%	9.9298/2.7824%	10.0605/1.5028%	10.1178/0.9418%
γ_s/Δ	9.1860/10.0646%	9.5175/6.8191%	9.6403/5.6168%	9.6975/5.0568%

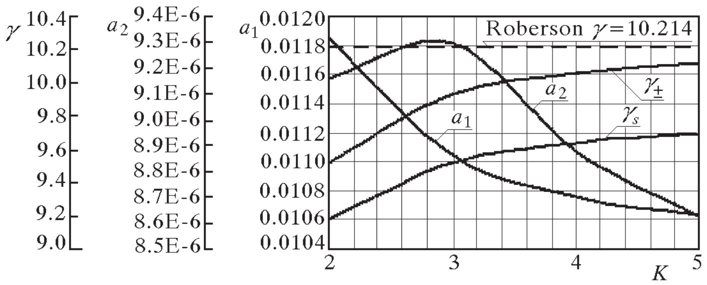

Fig. 3. Natural frequency parameter (γ) and terms of the characteristic series (a_1, a_2) of the constant thickness plate without the additional mass versus discretization degree (K) for $\chi_0 = 0.001$; $\mu = 0$

Table 4. Calculations for $\chi_0 = 0.001$; $\mu = 0.05$

K	2	3	4	5
a_1	0.015	0.0142	0.0139	0.0137
a_2	$1.5778 \cdot 10^{-5}$	$1.6373 \cdot 10^{-5}$	$1.6061 \cdot 10^{-5}$	$1.5841 \cdot 10^{-5}$
γ_-	8.4873	8.7851	8.882	8.9248
γ_+	8.5017	8.8063	8.9046	8.9477
$\gamma_\pm/\Delta$	8.4945/5.7423%	8.7957/2.4001%	8.8933/1.3171%	8.9363/0.84%
γ_s/Δ	8.1716/9.3253%	8.4023/6.7654%	8.4864/5.8322%	8.5254/5.3995%

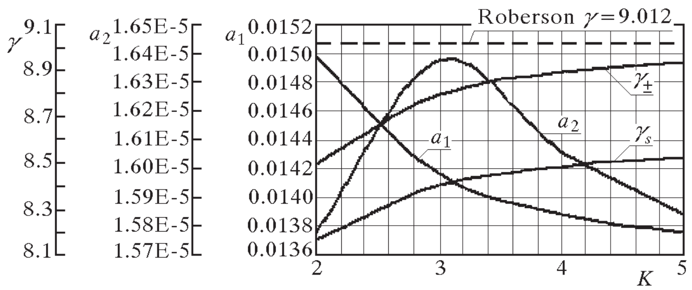

Fig. 4. Natural frequency parameter (γ) and terms of the characteristic series (a_1, a_2) of the constant thickness plate with one additional mass versus discretization degree (K) for $\chi_0 = 0.001$; $\mu = 0.05$

Table 5. Calculations for $\chi_0 = 0.001$; $\mu = 0.1$

K	2	3	4	5
a_1	0.0181	0.0173	0.017	0.0169
a_2	$2.239 \cdot 10^{-5}$	$2.345 \cdot 10^{-5}$	$2.3242 \cdot 10^{-5}$	$2.3053 \cdot 10^{-5}$
γ_-	7.71107	7.9367	8.0105	8.0431
γ_+	7.7233	7.9541	8.0292	8.0621
$\gamma_\pm/\Delta$	7.7172/4.8563%	7.9454/2.0429%	8.0198/1.1256%	8.0526/0.7212%
γ_s/Δ	7.4328/8.3626%	7.6052/6.2371%	7.6674/5.4703%	7.6961/5.1164%

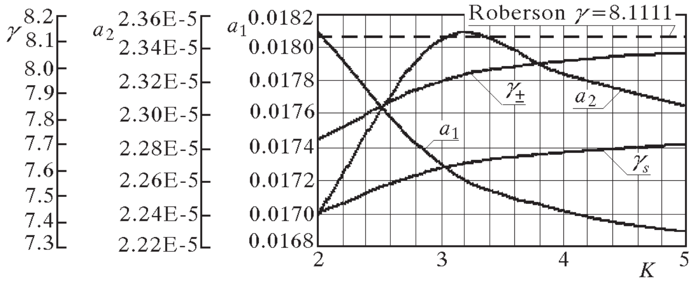

Fig. 5. Natural frequency parameter (γ) and terms of the characteristic series (a_1, a_2) of the constant thickness plate with one additional mass versus discretization degree (K) for $\chi_0 = 0.001$; $\mu = 0.1$

6. Conclusions

The suggested method takes into consideration not only the mass but also the radius of an additional mass ring. It is shown that if the radii of additional masses increase, the main frequency increases as well. The radius of the mass ring is not taken into consideration in the method applied by Roberson. However, the presented method gives a deviation less than 1% for a small discretization degree $(K = 5)$ in comparison to the precise solution obtained by Roberson. As the mass value grows from 0 to 0.1, the deviation decreases from 0.94% to 0.72% (Tables 3-5). We discovered that solutions for the mass values illustrated in Fig. 4. and Fig. 5 coincide with Roberson's solutions. Roberson did not take into account the change of thickness of the plate, therefore the obtained results cannot be compared. However, the results for cases without additional masses correspond to Conway's results. As shown in Table 1, the result for the case when $\mu = 0.001$ and $\chi_0 = 0.02$ (7.7918) is different from the exact value obtained by Conway (8.75) by 1.1% for the degree of discretization $K = 10$. Simultaneous analysis of the variable thickness and additional mass by analytical methods has not been encountered in the literature on the subject.

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References

1. BERNSTEIN S.A., KIEROPIAN K.K., 1960, *Opredelenie chastot kolebanij sterzhnevykh system metodom spektralnoi funkcii*, Gosstroiizdat, Moskwa, p. 281

2. Conway H.D., 1957, An analogy between the flexural vibrations of a cone and a disc of linearly varying thickness, *Z. Angew. Math. Mech.*, **37**, 9/10, 406-407

3. Conway H.D., 1958, Some special solutions for the exural vibrations of discs of varying thickness, *Ing. Arch.*, **26**, 6, 408-410

4. Hondkiewič W.S., 1964, *Sobstvennye kolebaniya plastin i obolochek*, Kiev, Nukowa Dumka, p. 288

5. Jaroszewicz J., 2000, Drgania swobodne utwierdzonej płyty kołowej obłożonej masami, *Prace Naukowe Instytutu Technicznego Wojsk Lotniczych*, **9**, 37-44

6. Jaroszewicz J., Zoryj L.M., 1994, Izgibnyje kolebanija i dinamičeskaja ustojčivost' balok s peremennymi parametrami, *Prikladnaja Mechanika*, Kijev, **30**, 9, 75-81

7. Jaroszewicz J., Zoryj L.M., 1996, Critical Euler load for cantilever tapered beam, *Journal of Theoretical and Applied Mechanics*, **34**, 4, 843-851

8. Jaroszewicz J., Zoryj L.M., 1997, *Metody analizy drgań i stateczności kontynualno-dyskretnych układów mechanicznych*, Monografia, Białystok

9. Jaroszewicz J., Zoryj L.M., 2005, *Metody analizy drgań osiowosymetrycznych płyt kołowych z zastosowaniem funkcji wpływu Cauchy'ego*, Monografia, Białystok

10. Jaroszewicz J., Zoryj L.M., 2006, The method of partial discretization in free vibration problems of circular plates with variable distribution of parameters, *International Applied Mechanics*, **42**, 3, 364-373

11. Jaroszewicz J., Zoryj L.M., Katunin A., 2004, Dwustronne estymatory częstości własnych drgań osiowosymetrycznych płyt kołowych o zmiennej grubości, *Materiały III Konferencji Naukowo-Praktycznej "Energia w Nauce i Technice"*, Suwałki, 45-56

12. Roberson R.E., 1951, Vibration of a clamped circural plate carrying concentrated mass, *J. Appl. Mech.*, **18**, 4, 349-352

13. Timoshenko S.P., 1940, *Theory of Plates and Shells*, New York, p. 283

14. Zoryj L.M., 1982, Ob universalnykh kharakteristicheskikh uravneniyakh w zadachakh kolebanij i ustoichivosti uprugikh sistem, *Mekhanika Tverdogo Tela*, **6**, 155-162

15. Zoryj L.M., Jaroszewicz J., 2000, Infuence of concentrated mass on vibrations of the circular plate, *Mechanical Engineering*, Lviv, **9**, 17-18 (in Ukrainian)

16. Zoryj L., Jaroszewicz J., 2002, Main frequencies of axial symmetric vibrations of the thin plates with variable parameters distribution, *Mechanical Engineering*, Lviv, **5**, 37-38 (in Ukrainian)

Wpływ dodatkowych pierścieni masowych na częstości drgań osiowosymetrycznych płyty o liniowo-zmiennej grubości utwierdzonej na obwodzie

Streszczenie

W pracy zbadano wpływ wartości i promienia dodatkowej masy pierścieniowej występującej w ciągłym rozkładzie masy płyty o liniowo-zmiennej grubości. Pominięto przy tym wpływ pierścienia masowego na sztywność giętną płyty. Zastosowano liniową teorię cienkich płyt do opisu małych drgań giętnych. Zastosowana metoda dyskretyzacji polega na dyskretyzacji masy ciągłej płyty i pozostawieniu ciągłej funkcji sztywności. Bazuje ona na metodzie funkcji wpływu Cauchy'ego, która daje szczególnie dobre efekty dla układów dyskretno-ciągłych. Pokazano, że rozwiązanie przybliżone zbiega do wartości ścisłej już przy stopniu dyskretyzacji mniejszym niż 5, niezależnie od wartości i promienia rozłożenia masy skupionej. Uzyskane wyniki obliczeń zmierzają do ścisłych wartości uzyskanych przez Conway'a dla płyty o liniowo-zmiennej grubości bez dodatkowej masy i do wyników Robersona dla płyty o stałej grubości z dodatkową masą skupioną w środku symetrii.

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