JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
43, 3, pp. 675-693, Warsaw 2005

APPLICATION OF POLI-CRITERIAL LINEARIZATION FOR CONTROL PROBLEM OF STOCHASTIC DYNAMIC SYSTEMS

Lesław Socha

Department of Mathematics and Sciences, Cardinal Stefan Wyszynski University in Warsaw
e-mail: leslawsocha@poczta.onet.pl

The problem of the determination of response characteristics and quasi-optimal control for nonlinear stochastic dynamic systems by using a multi-criteria linearization technique is presented in this paper. This idea was first introduced in previous author's paper (Socha, 1999a) for a simple dynamic system. In this paper, it is extended, and detailed analysis is given for a nonlinear oscillator with Gaussian external excitations and for a few criteria of statistical linearization. The obtained results are illustrated by a numerical example for Duffing's oscillator.

Key words: Stochastic control of nonlinear systems, LQG control problem, stochastic linearization, Pareto optimal solution

1. Introduction

Linearization methods are the most versatile methods for analysis of nonlinear systems and structures under stochastic excitations. Different criteria of linearization connected with well known linearization methods such as statistical linearization, equivalent linearization or exact linearization in three basic spaces were separately considered in the literature. They were discussed in:

- space of moments of stochastic processes – Elishakoff (2000)
- space of probability density functions of stochastic processes – Socha (1999b)
- space of spectral density functions of stochastic processes – Bernard and Taazount (1994).

The number of different linearization criteria is large (more than 100). For details see, for instance, a survey papers Elishakoff (2000) or Socha and Soong (1991). It is clear that the choice of a group of criteria of linearization depends on a considered problem but in one group, for instance, in the space of moments of stochastic processes the problem is open and the choice requires analysis. To study multicriteria problems, special approaches called multicriteria optimization methods were developed in the literature. In the field of mechanics, it was reviewed in Stadler (1984). The objective of this paper is to show a relationship between these criteria using two approaches of poly-criterial optimization techniques, namely the scalarization method and Pareto-optimal solution (Sawaragi *et al.*, 1985; Skulimowski, 1996) in application to the determination of the response characteristics and quasi-optimal control for nonlinear stochastic dynamic systems. This idea was first introduced in previous author's paper (Socha, 1999a) for a simple dynamic system. In this paper, it is extended and detailed analysis is given for a nonlinear oscillator with Gaussian external excitations and for a few criteria of statistical linearization. The obtained results are illustrated by a numerical example for Duffing's oscillator.

Consider a nonlinear stochastic model of a dynamic system described by the Ito vector differential equation

$$d\boldsymbol{x}(t) = \boldsymbol{\Phi}(\boldsymbol{x})\ dt + \sum_{k=1}^{M} \boldsymbol{G}_k\ d\xi_k(t) \tag{1.1}$$

where $\boldsymbol{x} = [x_1, \dots, x_n]^\top$ is the state vector, $\boldsymbol{\Phi} = [\Phi_1, \dots, \Phi_n]^\top$ is a nonlinear vector function such that $\boldsymbol{\Phi}(0) = \mathbf{0}$, $\boldsymbol{G}_k = [G_{k1}, \dots, G_{kn}]^\top$ are deterministic vectors, ξ_k are independent standard Wiener processes. We assume that a unique solution to equation (1.1) exists.

2. Linearization techniques for stochastic systems

2.1. Statistical and equivalent linearization

There are two basic groups of linearization methods for stochastic dynamic systems, namely the statistical (or local) linearization and equivalent linearization. In the case of statistical linearization, the objective is to find for a nonlinear vector $\boldsymbol{\Phi} = [\Phi_1, \dots, \Phi_n]^\top$ an equivalent one "in the sense of a linearization criterion", i.e., replacing

$$\boldsymbol{Y} = \boldsymbol{\Phi}(x, t) \tag{2.1}$$

in equation (1.1) by a linearized form

$$\boldsymbol{Y} = \boldsymbol{\Phi}_0(\boldsymbol{m_x}, \boldsymbol{\Theta_x}, t) + \mathbf{K}(\boldsymbol{m_x}, \boldsymbol{\Theta_x}, t)\boldsymbol{x}^0 \tag{2.2}$$

where

$$\boldsymbol{m_x} = E[\boldsymbol{x}] \qquad\qquad \boldsymbol{\Theta_x} = [\theta_{ij}] = E[x_i^0 x_j^0] \tag{2.3}$$

with $x_i^0 = x_i - m_{x_i}$ being the centralized stochastic process

$$\boldsymbol{\Phi}_0 = [\Phi_{01}, \dots, \Phi_{0n}]^\top$$

is a nonlinear vector function of the moments of $\boldsymbol{x}$ and $\mathbf{K} = [k_{ij}]$ is a $n \times n$ matrix of statistical linearization coefficients.

In the case of equivalent linearization, the objective is to find for nonlinear dynamic system (1.1) an equivalent one in the sense of a linearization criterion based on response properties for nonlinear system (1.1) and for the following linearized system

$$d\boldsymbol{x}(t) = [\mathbf{A}(t)\boldsymbol{x} + \boldsymbol{C}(t)]dt + \sum_{k=1}^{M}[\mathbf{D}_k\boldsymbol{x} + \boldsymbol{G}_k]d\xi_k(t) \tag{2.4}$$

where $\mathbf{A} = [a_{ij}]$, $\mathbf{D}_k = [d_{kij}]$, $i,j = 1,\dots,n$, $k = 1,\dots,M$ are matrices and $\boldsymbol{C} = [C_1,\dots,C_n]^\top$, $\boldsymbol{G}_k = [G_{k1},\dots,G_{kn}]^\top$ are vectors of linearization coefficients.

2.2. Basic linearization criteria for stochastic dynamic systems

Criterion 1a – Criterion of Equality of the First and Second Moments of Nonlinear and Linearized Variables (Kazakov, 1956)

$$\begin{aligned} &E[Y_i] = \Phi_{i0} \\ &E[(Y_i - E[Y_i])(Y_j - E[Y_j])] = \sum_{i=1}^{n}\sum_{j=1}^{n} k_i k_j \theta_{ij} \end{aligned} \tag{2.5}$$

Criterion 1b – Criterion of Moment Error of Approximation

$$E\Big[\Big(\Phi_i^{2p}(x,t) - \Big(\Phi_{0i} + \sum_{j=1}^{n} k_j x_j^0\Big)^{2p}\Big)^2\Big] \qquad p = 0,1,2,\dots \tag{2.6}$$

where for $p = 0$ and $p = 1$ it is known in the literature as Criterion of Mean-square Error of Approximation (Kazakov, 1956) and Energy Criterion (Elishakoff and Zhang, 1984)

Equivalency of response probability densities (Socha, 1999b):

Criterion 2a – Pseudomoment metric

$$I_{2a} = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \boldsymbol{x}^{[2r]} |g_N(\boldsymbol{x}) - g_L(\boldsymbol{x})| \, dx_1 \dots dx_n \tag{2.7}$$

where $\boldsymbol{x} = [x_1, \dots, x_n]^\top$, $\boldsymbol{x}^{[2r]} = x_1^{2r_1} x_2^{2r_2} \dots x_n^{2r_n}$, $r_1, \dots r_n \in \boldsymbol{N}$, $\sum_{i=1}^n 2r_i = 2r$.

Criterion 2a – Square probability metric

$$I_{2b} = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} [g_N(\boldsymbol{x}) - g_L(\boldsymbol{x})]^2 \, dx_1 \dots dx_n \tag{2.8}$$

where $g_N(\boldsymbol{x})$ and $g_L(\boldsymbol{x})$ are probability density functions of solutions to nonlinear (1.1) and linearized systems (2.4), respectively.

2.3. Poly-criteria optimization methods

In this section, we quote some basic definitions and facts from multicriteria optimization theory (Socha, 1999a,b).

A subset Θ of a linear space B is called a *convex cone* if and only if

$$\forall \alpha_1 \geqslant 0 \quad \forall \alpha_2 \geqslant 0 \quad \forall \boldsymbol{x}^1 \boldsymbol{x}^2 \in \Theta \quad (\alpha_1 \boldsymbol{x}^1 + \alpha_2 \boldsymbol{x}^2) \in \Theta \tag{2.9}$$

To every convex cone Θ, there corresponds an ordering relation R in B defined by

$$\boldsymbol{x}^1 \leqslant \boldsymbol{x}^2 \iff \boldsymbol{x}^2 - \boldsymbol{x}^1 \in \Theta \tag{2.10}$$

The relations of partial order induced by convex cones are generalization of the natural order in R^n defined as follows

$$\boldsymbol{x} \leqslant \boldsymbol{y} \iff \forall i = 1, \dots, n \quad x_i \leqslant y_i \tag{2.11}$$

where $\boldsymbol{x} = [x_1, \dots, x_n]^\top$ and $\boldsymbol{y} = [y_1, \dots, y_n]^\top$. This is equivalent to the relation $\boldsymbol{y} - \boldsymbol{x} \in R_+^n$. The positive orthant R_+^n satisfies all properties of the convex cones.

The general problem of multicriteria optimization is

$$(F : \, U_d \to B) \;\to\; \min(\Theta) \tag{2.12}$$

where the set of admissible controls U_d is a subset of a linear space U, the goal space B is partially ordered Banach space with a closed convex cone Θ. Moreover, it is assumed that the admissible set $F(U_d)$ is non-empty and closed.

2.4. Pareto-optimal approach

A control u_{opt} is said to be *nondominated or Pareto-optimal, or Θ-optimal* if and only if

$$(F(u_{opt}) - \Theta) \cap F(U_d) = \{F(u_{opt})\} \tag{2.13}$$

Condition (2.13) means that no element of the admissible set is better than u_{opt} in the sense of the partial order relation.

Relation (2.10) plays the fundamental role in classical problems of multi-criteria optimization which can be reduced to the simultaneous minimization of scalar functions

$$(F_1, F_2, \dots, F_m) \to \min \tag{2.14}$$

2.5. Scalarization methods

The most frequently used scalarization method for the problem

$$(F:\ U_d \to R^n) \ \to\ \min(R^n_+) \tag{2.15}$$

is a positive convex combination of the criteria

$$F_w(u) = \sum_{i=1}^{N} w_i F_i(u) \tag{2.16}$$

where $u \in U_d$, $w_i > 0$ for $1 \leqslant i \leqslant N$ and $\sum_{i=1}^{N} w_i = 1$. The parameters $w_i > 0$, $1 \leqslant i \leqslant N$ are weight coefficients.

The scalarization by distance

$$F_d(u) = d(q, F(u)) \tag{2.17}$$

where d is a metric in the goal space, q is a fixed unattainable element of the goal space which dominates at least one point from $F(U_d)$, for instance

$$F_p(u) = \|q - F(u)\|_p^p \tag{2.18}$$

where $\|\cdot\|_p^p$ is a p-th power of the norm in the L_p space. For instance, as the scalarizing family for the finite-dimensional multicriteria optimization problem with respect to the natural partial order in R^n, one can consider the following family of functionals

$$N_p(u, w) = \sum_{i=1}^{N} w_i (F_i(u) - q_i)^p \qquad w \in R^n_+ \setminus \{0\} \qquad 1 \leqslant p \leqslant \infty \tag{2.19}$$

In a particular case, when $n = 2$, $U_d = R^1 = \{k : -\infty < k < +\infty\}$ and $F_n = I_n$, $n = 1, 2$ an illustration of a dominated point q and relation (2.17) is given in Fig. 1. In this case, I_1 and I_2 are two criteria (for instance linearization criteria) and the convex curve is parametrized by k (for instance, a linearization coefficient). The points of the curve are defined by $(I_1(k), I_2(k))$.

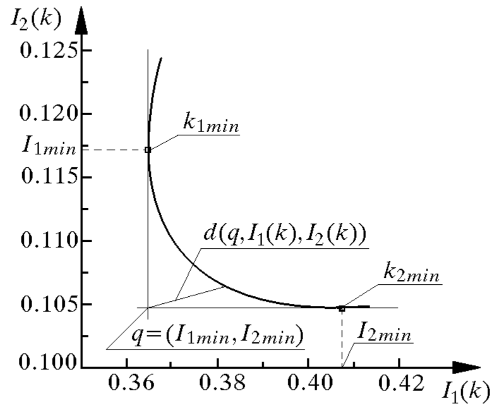

Fig. 1. A geometric illustration of condition (2.17)

3. Applications for single degree of freedom systems

Consider a single degree-of-freedom system described by

$$\begin{aligned} dx_1 &= x_2\,dt \\ dx_2 &= [-f(x_1) - 2hx_2]\,dt + \sigma\,d\xi \end{aligned} \tag{3.1}$$

where h and σ are constant parameters, f is a nonlinear function such that $f(0) = 0$. Then, the mean value of the stationary solution is equal to zero, i.e $E[x_1] = 0$.

An equivalent linearized system has the form

$$\begin{aligned} dx_1 &= x_2\,dt \\ dx_2 &= [-kx_1 - 2hx_2]\,dt + \sigma\,d\xi \end{aligned} \tag{3.2}$$

where k is a linearization coefficient.

The most frequently used scalarization method is a positive convex combination of considered criteria, i.e.

$$I_{opt}(k) = \sum_{i=1}^{N} \alpha_i I_i(k) \tag{3.3}$$

where I_{opt} and I_i, $i = 1, \dots, N$ are multiobjective criteria of the linearization, and partial criteria of the linearization, respectively, $\alpha_i > 0$, $i = 1, \dots, N$ are weight coefficients such that $\sum_{i=1}^{N} \alpha_i = 1$.

The idea of finding the Pareto-optimal solution is to determine a nondominated point $\boldsymbol{q}$ whose coordinates are defined by minimal values of considered criteria, i.e.

$$\boldsymbol{q}(\boldsymbol{k}) = \boldsymbol{q}(I_{i_{min}}(k)) \tag{3.4}$$

where

$$I_{i_{min}}(\boldsymbol{k}) = \min_{\boldsymbol{k}} I_i(\boldsymbol{k}) \tag{3.5}$$

The scalarization distance d_w is defined, for instance,by

$$d_w = \sqrt{\sum_{i=1}^{N} \alpha_i (I_i(\boldsymbol{k}) - I_{i_{min}}(\boldsymbol{k}))^2} \tag{3.6}$$

where $\alpha_i > 0$, $i = 1, \dots, N$ are weight coefficients such that $\sum_{i=1}^{N} \alpha_i = 1$.

To illustrate an application of the Pareto-optimal approach and scalarization method in the determination of response characteristics, we use two criteria of the statistical linearization $(N = 2)$. In further consideration, we analyse two cases of the moment criteria and criteria in the probability density space.

The corresponding criteria and linearization coefficients have the following forms.

3.1. Statistical linearization criteria in state space

- Mean-square Criterion

$$\begin{aligned} I_{MS} &= \frac{E[f^2(x_1)]E[x_1^2] - (E[f(x_1)x_1])^2}{E[x_1^2]} \\ k_{MS} &= \frac{E[f(x_1)x_1]}{E[x_1^2]} \end{aligned} \tag{3.7}$$

- Energy Criterion

$$
\begin{aligned}
I_E &= \frac{E\Big[\Big(\int\limits_0^{x_1} f(x)\,dx\Big)^2\Big]\frac{1}{4}E[x_1^4]}{\frac{1}{4}E[x_1^4]} - \frac{\Big(E\Big[\frac{x_1^2}{2}\int\limits_0^{x_1} f(x)\,dx\Big]\Big)^2}{\frac{1}{4}E[x_1^4]} \\
k_E &= \frac{E\Big[\frac{x_1^2}{2}\int\limits_0^{x_1} f(x)\,dx\Big]}{\frac{1}{4}E[x_1^4]}
\end{aligned}
\tag{3.8}
$$

3.2. Probability density linearization criteria

- Pseudomoment metric

$$
I_{PMM} = \int\limits_{-\infty}^{+\infty} \dots \int\limits_{-\infty}^{+\infty} x_1^{2p} x_2^{2q} |g_N(x_1,x_2) - g_L(x_1,x_2)|\ dx_1 dx_2 \tag{3.9}
$$

where $p+q=r$, $p,q=0,1,\dots$.

- Square probability metric

$$
I_{PSM} = \int\limits_{-\infty}^{+\infty} \dots \int\limits_{-\infty}^{+\infty} [g_N(x_1,x_2) - g_L(x_1,x_2)]^2\ dx_1 dx_2 \tag{3.10}
$$

where $g_N(\boldsymbol{x})$ and $g_L(\boldsymbol{x})$ are probability density functions of solutions to nonlinear (3.1) and linearized systems (3.2), respectively. $g_N(x_1,x_2)$ is defined by the Gramm-Charlier expansion (Pugacev and Sinicyn, 1985)

$$
g_N(x_1,x_2) = g_G(x_1,x_2)\Big[1 + \sum_{k=3}^{N} \sum_{\sigma(\nu)=k} \frac{c_{\nu_1\nu_2} H_{\nu_1\nu_2}(x_1,x_2)}{\nu_1!\nu_2!}\Big] \tag{3.11}
$$

where

$$
g_G(x_1,x_2) = \frac{1}{2\pi\sqrt{k_{11}k_{22} - k_{12}^2}} \exp\Big[-\frac{k_{11}x_2^2 - 2k_{12}x_1x_2 + k_{22}x_1^2}{2(k_{11}k_{22} - k_{12}^2)}\Big] \tag{3.12}
$$

and $c_{\nu_1\nu_2} = E[G_{\nu_1\nu_2}(x_1,x_2)]$, are quasimoments $\nu_1,\nu_2 = 0,1,\dots,N$, $\nu_1+\nu_2 = 3,4,\dots,N$, $H_{\nu_1\nu_2}(x_1,x_2)$ and $G_{\nu_1\nu_2}(x_1,x_2)$ are Hermite's polynomials defined by

$$\begin{aligned} H_{pq}(x_1,x_2) &= (-1)^{p+q}\exp\Big(\frac{1}{2}A\Big)\frac{\partial^{p+q}}{\partial x_1^p \partial x_2^q}\exp\Big(-\frac{1}{2}A\Big) \\ G_{pq}(x_1,x_2) &= (-1)^{p+q}\exp\Big(\frac{1}{2}A\Big)\cdot \\ &\cdot\Big[\frac{\partial^{p+q}}{\partial y_1^p \partial y_2^q}\exp\Big(-\frac{1}{2}(k_{11}y_1^2+2k_{12}y_1y_2+k_{22}y_2^2)\Big)\Big]_{\boldsymbol{y}=\boldsymbol{V}\boldsymbol{x}} \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} & A = v_{11}x_1^2 + 2v_{12}x_1x_2 + v_{22}x_2^2 \\ & \mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \qquad \mathbf{K}^{-1} = \boldsymbol{V} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \\ & k_{ij} = E[x_ix_j] \qquad i,j=1,2 \end{aligned} \tag{3.14}$$

For the stationary probability density function the corresponding moments are as follows

$$\begin{aligned} k_{12} &= k_{21} = 0 \qquad & k_{22} &= \frac{1}{v_{22}} \\ v_{12} &= v_{21} = 0 \qquad & k_{11} &= \frac{1}{v_{11}} \end{aligned} \tag{3.15}$$

The moment k_{11} has to be found from moment equations.

4. Example

Consider the Duffing oscillator described by

$$\begin{aligned} dx_1 &= x_2\,dt \\ dx_2 &= [-\omega_0^2x_1 - \varepsilon x_1^3 - 2hx_2]\,dt + \sigma\,d\xi \end{aligned} \tag{4.1}$$

where ω_0^2, ε, h and σ are constant parameters. The parameters selected for calculations are: $\omega_0^2 = 0.5$, $\varepsilon = 0.1$, $h = 0.05$ and $\sigma^2 = 0.2$.

The corresponding linearized system has form (3.2).

4.1. Moment criteria in statistical linearization

- Mean-square Criterion

$$I_{MS} = (\omega_0^2 - k)^2 E[x_1^2] + 15\varepsilon^2 (E[x_1^2])^3 + 6\varepsilon(\omega_0^2 - k)(E[x_1^2])^2 \tag{4.2}$$

- Energy Criterion

$$I_E = \frac{3}{4}(\omega_0^2 - k)^2 (E[x_1^2])^2 + 105\Big(\frac{\varepsilon}{4}\Big)^2 (E[x_1^2])^4 + \frac{15}{4}\varepsilon(\omega_0^2 - k)(E[x_1^2])^3 \tag{4.3}$$

where

$$E[x_1^2] = \frac{\sigma^2}{4hk} \tag{4.4}$$

Then, the set of dominating points is presented in Fig. 2

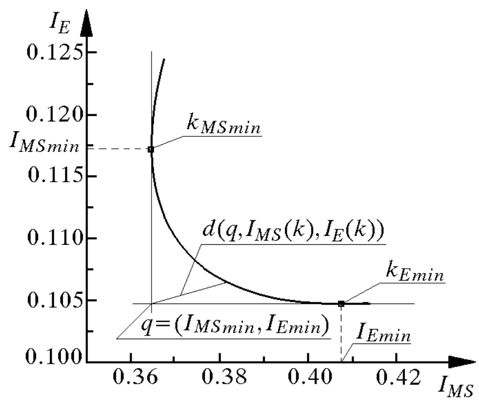

Fig. 2. A graphical illustration of the set of dominating points determined by (4.2)-(4.4)

- Convex combination

$$I_{opt} = \alpha I_{MS}(k) + (1 - \alpha) I_E \qquad 0 \leqslant \alpha \leqslant 1 \tag{4.5}$$

The characteristics $k_{min} = k_{min}(\alpha)$ and $I_{opt} = I_{opt}(\alpha)$ determined by relation (4.5) are shown in Fig. 3a,b, respectively.

The characteristics $d_E = d_E(k)$ and $d_w = d_w(\alpha)$ obtained by *Scalarization by distance* have the form

$$\begin{aligned} d_E(k) &= \sqrt{[I_{MS_{min}} - I_{MS}(k)]^2 + [I_{E_{min}} - I_E(k)]^2} \\ d_w(\alpha) &= \sqrt{\alpha[I_{MS_{min}} - I_{MS}(k)]^2 + (1-\alpha)[I_{E_{min}} - I_E(k)]^2} \end{aligned} \tag{4.6}$$

for $0 \leqslant \alpha \leqslant 1$. A graphical illustration of these characteristics is given in Fig. 4a,b, respectively.

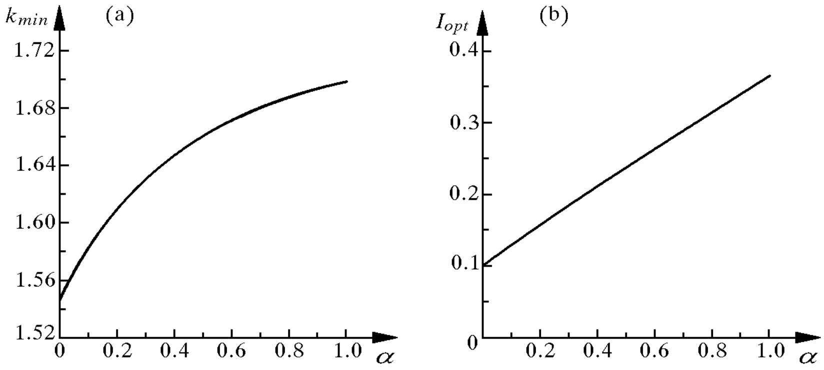

Fig. 3. (a) Characteristics $k_{min} = k_{min}(\alpha)$ determined by relation (4.5); (b) characteristics $I_{opt} = I_{opt}(\alpha)$ determined by relation (4.5)

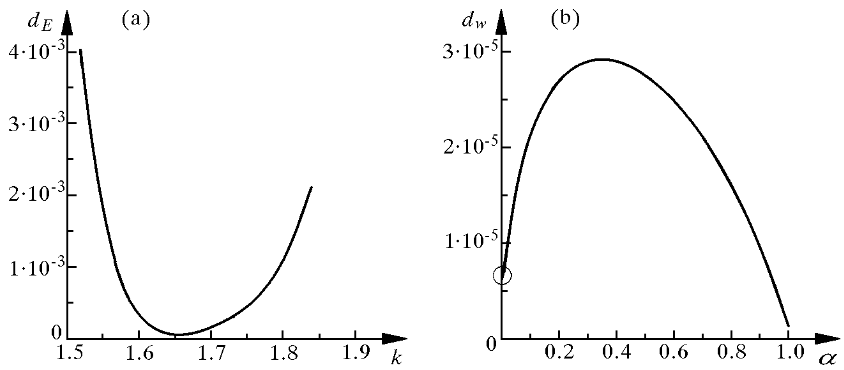

Fig. 4. (a) Characteristics $d_E = d_E(k)$ determined by relation $(4.6)_1$; (b) chracteristics $d_w = d_w(\alpha)$ determined by relation $(4.6)_2$

4.2. Probability density linearization criteria in equivalent linearization

- Pseudomoment metric

$$I_{PMM} = \int\limits_{-\infty}^{+\infty} \dots \int\limits_{-\infty}^{+\infty} x_1^{2p} x_2^{2q} |g_N(x_1, x_2) - g_L(x_1, x_2)| \, dx_1 dx_2 \tag{4.7}$$

where $p + q = r$, $p, q = 0, 1, \dots$.

- Square probability metric

$$I_{PSM} = \int\limits_{-\infty}^{+\infty} \dots \int\limits_{-\infty}^{+\infty} [g_N(x_1, x_2) - g_L(x_1, x_2)]^2 \, dx_1 dx_2 \tag{4.8}$$

where $g_N(\boldsymbol{x})$ and $g_L(\boldsymbol{x})$ are probability density functions defined by

$$
\begin{aligned}
g_N(x_1,x_2) &= \frac{1}{c_N}\exp\Big[-\frac{2h}{q_1^2}(\omega_0^2x_1^2+\frac{\varepsilon}{2}x_1^4+x_2^2)\Big]\\
g_L(x_1,x_2,k) &= \frac{1}{c_L}\frac{4h\sqrt{k}}{q_1^2}\exp\Big[-\frac{2h}{q_1^2}(kx_1^2+x_2^2)\Big]
\end{aligned}
\tag{4.9}
$$

where c_N and c_L are normalized constants.

Then, the set of dominating points for parameters $\omega_0^2 = 0.5$, $h = 0.05$, $\varepsilon = 0.1$, $\sigma^2 = 0.2$, $p = 1$, $q_1 = 0$ is presented in Fig. 5.

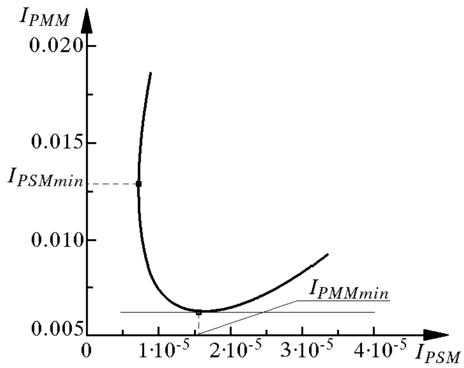

Fig. 5. A graphical illustration of the set of dominating points determined by (4.7) and (4.8)

5. Applications in control problems

The multicriteria analysis was also used in the determination of optimal control for linear stochastic systems by Piunovski (1996) and by Radievski (1993). In this Section, we extend these results for a class of nonlinear systems combining the multicriteria approach, statistical or equivalent linearization with the LQG technique.

Consider the following optimal control problem. The nonlinear stochastic model of a dynamic system is described by

$$
d\boldsymbol{x}(t) = [\mathbf{A}\boldsymbol{x}(t) + \boldsymbol{\Phi}(\boldsymbol{x}) + \mathbf{B}\boldsymbol{u}(t)]\,dt + \sum_{k=1}^{M}\boldsymbol{G}_k\,d\xi_k(t) \tag{5.1}
$$

where $\boldsymbol{x} \in R^n$ and $\boldsymbol{u} \in R^m$ are the state and control vectors, respectively. $\mathbf{A}$ and $\mathbf{B}$ are time invariant matrices of appropriate dimensions, $\boldsymbol{\Phi} = [\Phi_1, \dots, \Phi_n]^\top$ is a nonlinear vector function such that $\boldsymbol{\Phi}(\mathbf{0}) = \mathbf{0}$ are time invariant deterministic vectors, ξ_k are independent standard Wiener processes for $k = 1, 2, \dots, M$. We assume that the unique solution to equation (5.1) exists and the system is controllable.

The control strategy is designed to minimize the criterion

$$I = E[\boldsymbol{x}^\top \mathbf{Q}\boldsymbol{x} + \boldsymbol{u}^\top \mathbf{R}\boldsymbol{u}] \tag{5.2}$$

where $\mathbf{Q}$ and $\mathbf{R}$ are time-invariant positive definite symmetric matrices.

5.1. Quasi-optimal control

We assume that the nonlinear vector $\boldsymbol{\Phi}(\boldsymbol{x})$ can be substituted by a linearized form

$$\boldsymbol{\Phi}(\boldsymbol{x}) = \mathbf{A}_e \boldsymbol{x} \tag{5.3}$$

where $\mathbf{A}_e$ is a $n \times n$ matrix of linearization coefficients such that $(\mathbf{A}+\mathbf{A}_e, \mathbf{B})$ is stabilizable and detectable. Then, the optimal control for the linearized system

$$d\boldsymbol{x}_L(t) = [(\mathbf{A} + \mathbf{A}_e)\boldsymbol{x}_L(t) + \mathbf{B}\boldsymbol{u}(t)]\;dt + \sum_{k=1}^{M} \boldsymbol{G}_k\;d\xi_k(t) \tag{5.4}$$

can be found by a standard method (Kwakernak and Sivan, 1972) in the linear feedback form

$$\boldsymbol{u} = -\mathbf{K}\boldsymbol{x}_L \qquad\qquad \mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^\top\mathbf{P} \tag{5.5}$$

where $\mathbf{K}$ is the gain matrix and $\mathbf{P}$ is a positive solution to the algebraic Riccati equation

$$\mathbf{P}(\mathbf{A} + \mathbf{A}_e) + (\mathbf{A} + \mathbf{A}_e)^\top\mathbf{P} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top\mathbf{P} + \mathbf{Q} = \mathbf{0} \tag{5.6}$$

Substituting (5.5) into equation (5.4) yields

$$d\boldsymbol{x}_L(t) = [(\mathbf{A} + \mathbf{A}_e - \mathbf{B}\mathbf{K})\boldsymbol{x}_L(t)]\;dt + \sum_{k=1}^{M} \boldsymbol{G}_k\;d\xi_k(t) \tag{5.7}$$

The corresponding covariance equation and criterion have the form

$$(\mathbf{A} + \mathbf{A}_e - \mathbf{B}\mathbf{K})\boldsymbol{V}_L + \mathbf{V}_L(\mathbf{A} + \mathbf{A}_e - \mathbf{B}\mathbf{K})^\top + \sum_{k=1}^{M} \boldsymbol{G}_k\boldsymbol{G}_k^\top = \mathbf{0} \tag{5.8}$$

and

$$I_L = E[\boldsymbol{x}_L^\top(\mathbf{Q} + \mathbf{K}^\top\mathbf{R}\mathbf{K})\boldsymbol{x}_L] = \operatorname{tr}[(\mathbf{Q} + \mathbf{K}^\top\mathbf{R}\mathbf{K})\boldsymbol{V}_L] \tag{5.9}$$

where the subindex L corresponds to the linearized problem, tr denotes the trace of the matrix

$$\mathbf{V}_L = E[\boldsymbol{x}_L \boldsymbol{x}_L^\top] \tag{5.10}$$

In the case of statistical linearization, the elements of the nonlinear vector $\boldsymbol{\Phi}(\boldsymbol{x})$ have to be replaced by corresponding equivalent elements "in the sense of a given criterion" in a linear form.

The following two moment criteria for scalar functions are considered:

Criterion 1. Mean-square error of displacements (Kazakov, 1956)

$$E[(c_1 x - \phi(x))^2] \to \min \tag{5.11}$$

Criterion 2. Mean-square error of potential energies (Elishakoff and Zhang, 1984)

$$E\Big[\Big(\int_0^x [c_2 y - \phi(y)]\, dy\Big)^2\Big] \to \min \tag{5.12}$$

In the case of application of the linear feedback gain obtained for the linearized system to the nonlinear system, the state equation and the corresponding criterion have the form

$$d\boldsymbol{x}_N(t) = [(\mathbf{A} - \mathbf{BK})\boldsymbol{x}_N(t) + \boldsymbol{\Phi}(\boldsymbol{x}_N(t))]\, dt + \sum_{k=1}^{M} \boldsymbol{G}_k\, d\xi_k(t) \tag{5.13}$$

and

$$I_N = \operatorname{tr}[(\mathbf{Q} + \mathbf{K}^\top \mathbf{RK})\boldsymbol{V}_N] \qquad \mathbf{V}_N = E[\boldsymbol{x}_N \boldsymbol{x}_N^\top] \tag{5.14}$$

where the subindex N denotes the original nonlinear problem with

$$\boldsymbol{V}_N = E[\boldsymbol{x}_N \boldsymbol{x}_N^\top] \tag{5.15}$$

In general, the covariance matrix $\mathbf{V}_N$ can be found approximately. To obtain the linearization matrix $\mathbf{A}_e$ and quasi-optimal control, one of the four proposed criteria should be selected and used with an iterative procedure.

5.2. Iterative procedure for multicriteria nonlinear control problem

The following procedure is an extended version of a standard one given in (Yoshida, 1984):

Step 1. Choose a parameter of the nonlinear system (an element of Eq. (5.1)) and criteria of linearization, then select $\mathbf{A}_e = \mathbf{0}$ in (5.3). Next, for every criterion repeat Steps 2 - 11.

Step 2. Solve (5.6). The solution to (5.7) is $\boldsymbol{P}$.

Step 3. Substitute $\mathbf{P}$ obtained in Step 2 into (5.6) and find the matrix $\mathbf{K}$. Next, substitute $\mathbf{K}$ and $\mathbf{A}_e = \mathbf{0}$ into equation (5.8) and solve the equation. The solution of equation (5.8) is $\mathbf{V}_L$.

Step 4. Substitute $\mathbf{P}$ obtained in Step 2 into (5.9) and find I_L.

Step 5. For each nonlinear element find the linearization coefficient which minimizes the selected criterion, for instance, (5.11) or (5.12).

Step 6. Substitute the matrix of linearization coefficients $\mathbf{A}_e(\mathbf{V}_L)$ obtained in Step 5 into equation (5.8) and then solve the equation.

Step 7. If the error is greater than a given parameter ε_1, then repeat Steps 3 - 6 until $\mathbf{V}_L$ converges.

Step 8. Substitute the matrix of linearization coefficients $\mathbf{A}_e(\mathbf{V}_L)$ into Riccati equation (5.6) and then solve the equation.

Step 9. Substitute the matrix $\mathbf{P}$ obtained in Step 8 into covariance equation (5.8) and then solve the equation.

Step 10. If the error is greater than a given ε_2, then repeat Steps 3 - 9 until $\mathbf{V}_L$ and $\mathbf{P}$ converge.

Step 11. Calculate criteria I_L and I_N given by (5.9) and (5.14), respectively.

Step 12. Calculate a measure of the multicriteria optimization problem based on criteria calculated in Step 11 for different linearization criteria chosen in Step 1.

5.3. Example (Duffing oscillator)

Consider the Duffing oscillator described by

$$\begin{aligned} dx_1 &= x_2\, dt \\ dx_2 &= [-\omega_0^2 x_1 - \varepsilon x_1^3 - 2h x_2 + bu]\, dt + \sigma\, d\xi \end{aligned} \tag{5.16}$$

where $\omega_0^2, \varepsilon, h, b$ and σ are constant parameters, u is a scalar control, ξ is the standard Wiener process, and the mean-square criterion is

$$I = E[\boldsymbol{x}^\top \mathbf{Q} \boldsymbol{x} + r u^2] \tag{5.17}$$

where $\boldsymbol{x} = [x_1, x_2]^\top$, $\mathbf{Q} = \operatorname{diag}[Q_i]$, $i = 1,2$; Q_i, r are positive constants. The linearized system has the following form

$$
\begin{aligned}
dx_1 &= x_2 \, dt \\
dx_2 &= [-\omega_0^2 x_1 - \varepsilon c x_1 - 2h x_2 + bu] \, dt + \sigma \, d\xi
\end{aligned}
\tag{5.18}
$$

where c is a linearization coefficient. The coordinates of solutions to algebraic Riccati and covariance equations denoted by $\mathbf{P} = [p_{ij}]$ and $\boldsymbol{V}_L = [v_{L_{ij}}]$, respectively, for $i,j = 1,2$ are the following

$$
\begin{aligned}
p_{11} &= 2hp_{12} + cp_{22} + \beta p_{12} p_{22} \\
p_{12} &= \frac{1}{\beta}(-c + \sqrt{c^2 + Q_1 \beta}) \\
p_{22} &= \frac{1}{\beta}\sqrt{4h^2 + \beta(Q_2 + 2p_{12})}
\end{aligned}
\tag{5.19}
$$

and

$$
v_{L_{22}} = \frac{g^2}{2(2h + \beta p_{22})} \qquad v_{L_{12}} = 0 \qquad v_{L_{11}} = \frac{v_{22}}{\gamma + \beta p_{12}}
\tag{5.20}
$$

where $\beta = b^2/r$. The optimal value of the criterion for linearized system is

$$
I_L = (Q_1 + \beta p_{12}^2) v_{L_{11}} + (Q_2 + \beta p_{22}^2) v_{L_{22}}
\tag{5.21}
$$

Applying the obtained linear feedback control to nonlinear system we obtain the state equation and the corresponding criterion

$$
\begin{aligned}
dx_1 &= x_2 \, dt \\
dx_2 &= [-2h x_2 - \omega_0^2 - \varepsilon x_1^3 - \beta(x_1 p_{12} + x_2 p_{22})] \, dt + \sigma \, d\xi
\end{aligned}
\tag{5.22}
$$

and

$$
I_{N_{opt}} = (Q_1 + \beta p_{12}^2) v_{N_{11}} + (Q_2 + \beta p_{22}^2) v_{N_{22}}
\tag{5.23}
$$

where the second order moments $v_{N_{11}}$ and $v_{N_{22}}$ can be found in an analytical form from

$$
v_{N_{ii}} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_i^2 g_N(x_1, x_2) \, dx_1 dx_2 \qquad i = 1,2
\tag{5.24}
$$

where

$$g_N(x_1,x_2) = \frac{1}{c_N}\exp\Big[-\frac{2h+\beta p_{22}}{\sigma^2}(\omega_0^2+\beta p_{12})x_1^2 + \varepsilon\frac{x_1^4}{2} + x_2^2\Big] \tag{5.25}$$

and c_N is a normalized constant.

One can show (Elishakoff and Zhang, 1984; Kazakov, 1956) that the linearization coefficients for two considered criteria have the form

$$c_1 = 3E[x_1^2] \qquad\qquad c_2 = 2.5E[x_1^2] \tag{5.26}$$

To obtain the quasi-optimal controls and corresponding mean-square criteria I_1 and I_2 depending on the choice of linearization coefficients c_1 and c_2, one can use the iterative procedure proposed in the previous section. To illustrate the obtained results, a comparison of the considered criteria is discussed. The set of dominating points for parameters $\omega_0^2 = 1$, $h = 0.05$, $b = 1$, $\varepsilon = 1$, $\sigma = 1$, $Q_1 = Q_2 = 1$, $r = 100$ is presented in Fig. 6.

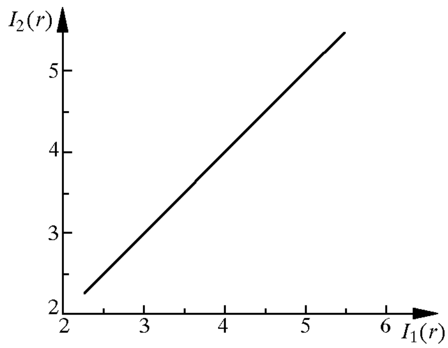

Fig. 6. A graphical illustration of the set of dominating points

Figure 8 shows that the considered mean-square criteria I_1 and I_2 are linearly dependent and there are no dominated points. It means that the quasi-optimal controls obtained for both criteria are the same. This confirms an earlier observation presented in an earlier author's paper (Socha, 2000).

6. Conclusions

Numerical studies show that in the response analysis there are significant differences between the obtained linearization coefficients and corresponding criteria in contrast to control problems where for a given mean-square criterion of minimization (5.2) there are no significant differences between the applied linearization methods. It means that the mean-square criteria corresponding to different linearization criteria are linearly dependent. This linear dependence also appears in application of linearization techniques with criteria in probability density space.

References

1. Bernard P., Taazount M., 1994, Random dynamics of structures with gaps: simulation and spectral linearization, *Nonlinear Dynamics*, **5**, 313-335
2. Elishakoff I., 2000, Multiple combination of the stochastic linearization criteria by the moment approach, *J. Sound Vib.*, **237**, 550-559
3. Elishakoff I., Zhang R., 1984, Comparison of the new energy-based version of stochastic linearization technique, in N.Beilimo and F.Casciati (edit.) *Nonlinear Stochastic Mechanics*, Springer, Berlin, 201-212
4. Kazakov I., 1956, Approximate probabilistic analysis of the accuracy of operation of essentially nonlinear systems, *Avtom. i Telemekhan.*, **17**, 423-450
5. Kwakernak H., Sivan R., 1972, *Linear Optimal Control Systems*, Wiley-Interscience, New York
6. Piunowski A., 1996, A multicriteria model of optimal control of a stochastic linear system, *Avtom. i Telemekhan.*, **57**, 71-91
7. Pugacev W.S., Sinicyn I.N., 1985, *Stochastic Differential Systems*, Willey, Chichester
8. Radievskii A., 1993, Stochastic control problem of one class, *Kibern. Vychisl. Tekh.*, Ukraine
9. Sawaragi Y. Nakayama H., Tanino T., 1985, *Theory Multiobjective Optimization*, Academic Press, New York
10. Skulimowski A., *Decision Support Systems Based on Reference Sets Theory Multiobjective Optimization*, AGH-Press, Krakow
11. Socha L., 1999, Poli-criterial equivalent linearization, *Z. Angew. Math. Mech.*, **79**, S2, 369-382
12. Socha L., 1999, Probability density equivalent linearization and non-linearization Techniques, *Archive Mechanics*, **51**, 487-507
13. Socha L., 2000, Application of statistical linearization techniques in design of quasi-optimal active control of nonlinear systems, *J. Theor. and Appl. Mech.*, **28**, 591-605

14. Socha L., Soong T.T., 1991, Linearization of nonlinear stochastic systems, *Appl. Mech. Rev.*, **44**, 399-422

15. Stadler W., 1984, Multicriteria optimization in mechanics (A Survey), *Appl. Mech. Rev.*, **37**, 277-286

16. Yoshida K., 1984, A method of optimal control of non-linear stochastic systems with non-quadratic criteria, *Int. J. Control*, **39**, 279-291

Zastosowanie wielokryterialnej linearyzacji w problemie sterowania stochastycznych nieliniowych układów dynamicznych

Streszczenie

W pracy przedstawiono problem wyznaczania quasi-optymalnego sterowania w nieliniowych stochastycznych układach dynamicznych za pomocą wielokryterialnej metody linearyzacji stochastycznej. Pomysł wielokryterialnej linearyzacji został zasygnalizowany we wcześniejszej pracy autora (Socha, 1999a). W niniejszym artykule jest on rozwinięty i zastosowany do problemu sterowania, a szczegółowa analiza jest przeprowadzona dla nieliniowego oscylatora z addytywnym wymuszeniem Gaussa.

Manuscript received December 20, 2004; accepted for print, March 18, 2005