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ON THE PROBLEM OF SOME INTERFACE DEFECT FILLED WITH A COMPRESSIBLE FLUID IN A PERIODIC STRATIFIED MEDIUM[1]

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A periodic two-layered elastic space containing an interface defect filled with a barotropic compressible fluid is considered. At infinity, the composite is subjected to a uniformly distributed load applied perpendicularly to the layering. Faces of the defect are under action of constant internal fluid pressure. An approximate solution to this problem is given within a certain homogenized model. The resulting singular integro-differential equation is obtained and solved for two types of defects by using an analogue of Dyson's theorem. The influence of the filler on the mechanical behaviour of the considered body is analysed and illustrated graphically.

Key words: periodic two-layered space, interface defect, barotropic compressible fluid, singular integro-differential equation

1. Introduction

Natural geological structures are not usually homogeneous. Many types of soils exhibit features of periodic layering (see, for example, Amadei, 1983). In real conditions a soil contains a lot of cracks or cavities, some of which may be filled with a gas or fluid. That is why the investigation of mechanical behaviour of layered structures containing such defects is a very important

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problem involving many geotechnical applications in mining engineering, gas- and oil-producing industry.

Many researchers have studied problems of solids possessing cavities filled with certain substances. It seems that the first article on a general model of a crack filled with a heat-conducting medium was written by Pidstryhach and Kit (1967). Based on this model, the results of the investigation of temperature and stress fields were summarised in two monographs by Kit and Kryvtsun (1983), and Kit and Khay (1989). However, in all these studies, the mechanical influence of the defect filler was neglected. One of the first attempts to take into account this effect was made by Yevtushenko and Sulym (1980). In this work it is suggested that the mechanical action of the filler can be simulaterd by a constant pressure dependent on the crack opening and determined from the equation of state of the fluid. Other researchers used this idea. Kuznetsov (1988) considered static contact of two isotropic bodies with surface gaps filled with a compressible fluid. Recently, the contact interaction of bodies having defects filled with a gas was studied by Martynyak (1998), Machyshyn and Martynyak (2000) and Machyshyn and Nagórko (2003). A combined thermal and mechanical effect of the ideal gas filling a crack was analysed by Matczyński *et al.* (1999) in the case of plane strain.

This contribution is devoted to a three-dimensional problem for a bimaterial periodically layered space containing an interface cavity filled with a barotropic compressible fluid. In Section 2, this problem is formulated and the use of the homogenized model of the composite is demonstrated. Section 3 presents the resulting boundary-value problem and its reduction to a singular integro-differential equation. A detailed analysis is performed and illustrated by graphs for two special types of defects in Section 4. Conclusions are given in the last section.

2. Description of the problem

2.1. Formulation

Let us consider a stratified space, the middle cross of which is given in Fig.1. A repeated fundamental lamina of a small thickness δ consists of two homogeneous isotropic layers (denoted by 1 and 2) of thicknesses δ_1 and δ_2 ($\delta = \delta_1 + \delta_2$), and characterised by the Lamé constants λ_1, μ_1 and λ_2, μ_2, respectively. Let refer the body to the Cartesian coordinate system (x_1, x_2, x_3) with its center on the interface plane and the x_3-axis normal to the layering.

We suppose that there is an interface defect (a cavity) occupying the regular region V_d defined as

$$V_d = \Big\{(x_1, x_2, x_3): \ \ (x_1, x_2) \in S_a \ \ \wedge \ \ -\frac{1}{2}f(x_1, x_2) \leqslant x_3 \leqslant \frac{1}{2}f(x_1, x_2)\Big\}$$
$$(x_1, x_2) \in S_a \Leftrightarrow r^2 \equiv x_1^2 + x_2^2 \leqslant a^2 \tag{2.1}$$

In the above, S_a is the circular median (equatorial) section of the defect with the radius a, and f is a smooth function describing the small initial height of the defect (before applying the load) such that

$$\begin{aligned} & f\Big|_{r=a} = 0 \\ & 0 \leqslant f(x_1, x_2) < \max(\delta_1, \delta_2) && \forall(x_1, x_2) \in S_a \\ & f(x_1, x_2) \ll a && \forall(x_1, x_2) \in S_a \end{aligned} \tag{2.2}$$

This defect is assumed to be completely filled with a barotropic compressible fluid, whose state is governed by the equation (Galanov *et al.*, 1985)

$$V = \frac{m_*}{\rho_0} \exp\Big(-\frac{P_{fl}}{\beta}\Big) \tag{2.3}$$

where ρ_0 stands for the density of the compressed fluid, V is the volume of the fluid of a fixed mass m_* at the internal pressure P_{fl}, and β stands for the coefficient of volume compressibility.

The composite is subjected to a uniform load p applied at infinity and directed perpendicularly to the interface (see Fig. 1). Moreover, perfect bonding between the subsequent layers (excluding the defect region) is assumed.

We shall consider two cases separately:

Case (i): The defect degenerates into a crack. It means that its initial height is equal to zero, i.e. $f(x_1, x_2) \equiv 0$, and the region of occupation S_a does not alter under the external load. The surfaces of the crack are under the action of the fluid pressure P_{fl}.

Case (ii): We deal with a cavity (described by Eqs (2.1) and (2.2)), filled with a barotropic compressible fluid, and a situation when owing to the external load p its faces are in frictionless contact in a certain annulus $b < r \leqslant a$. Hence, $S_b = \{(x_1, x_2): \ x_1^2 + x_2^2 \leqslant b^2\}$ is the circular noncontacting domain with the radius b, subjected to the fluid pressure P_{fl}.

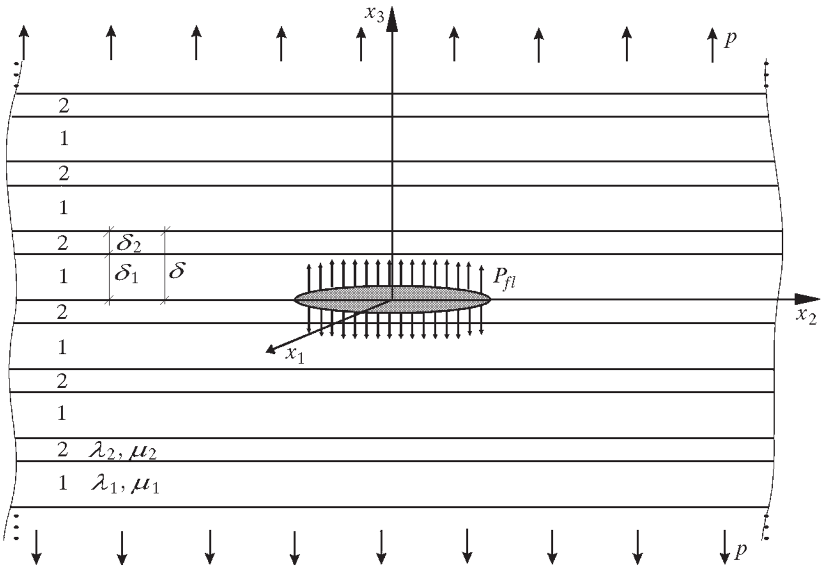

Fig. 1. Periodic two-layer space weakened by an interface defect filled with a compressible barotropic fluid

Notice that P_{fl} is an unknown parameter and is constant (in view of the Pascal law). Besides, we assume that the fluid does not transmit tensile forces, i. e. the pressure of the fluid P_{fl} may be compressive only.

The problem lies in the determination of the fields of displacements and stresses in the stratified space. Of prime interest is to analyse the effect of the filler on the stresses. The unknown fluid pressure P_{fl} is found during solving the problem by using Eq. (2.3). Moreover, in Case (ii) the parameter b is determined by the condition of finiteness of the normal stress on $r = b$.

2.2. Governing equations

To solve the problem under study (with an infinite number of thin layers), a direct analytical approach becomes more intricated and the numerical formulation more unstable. A classic idea is the use of the homogenization process to replace the heterogeneous medium by a continuous, equivalent one, which gives the average behaviour of the medium at the macroscopic scale. We base on a specific non-asymptotic procedure called the microlocal modelling (cf Woźniak, 1987; Matysiak and Woźniak, 1988). The final governing equations (in a static case with neglecting body forces) and constitutive relations for the stresses $\sigma_{ij}^{(l)}$ of the homogenized model, given in terms of the unknown macrodisplacements w_i, take the form[1] (for details, see Kaczyński, 1993)

$$\begin{aligned}
&\frac{1}{2}(c_{11}+c_{12})w_{\gamma,\gamma\alpha}+\frac{1}{2}(c_{11}-c_{12})w_{\alpha,\gamma\gamma}+c_{44}w_{\alpha,33}+(c_{13}+c_{44})w_{3,3\alpha}=0\\
&(c_{13}+c_{44})w_{\gamma,\gamma3}+c_{44}w_{3,\gamma\gamma}+c_{33}w_{3,33}=0
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
\sigma_{\alpha3}^{(l)} &= c_{44}(w_{\alpha,3}+w_{3,\alpha}) & \sigma_{33}^{(l)} &= c_{13}w_{\gamma,\gamma}+c_{33}w_{3,3}\\
\sigma_{12}^{(l)} &= \mu_l(w_{1,2}+w_{2,1}) & \sigma_{11}^{(l)} &= d_{11}^{(l)}w_{1,1}+d_{12}^{(l)}w_{2,2}+d_{13}^{(l)}w_{3,3}\\
\sigma_{22}^{(l)} &= d_{12}^{(l)}w_{1,1}+d_{11}^{(l)}w_{2,2}+d_{13}^{(l)}w_{3,3}
\end{aligned} \tag{2.5}$$

All coefficients appearing above are given in Appendix A. They depend on the material and geometrical characteristics of subsequent layers. It is noteworthy here that the condition of perfect bonding between the layers is satisfied (the components σ_{3j} do not depend on l implying the continuity of the stress vector at the interfaces). Finally, by assuming $\lambda_1=\lambda_2\equiv\lambda$, $\mu_1=\mu_2\equiv\mu$ we get $c_{11}=c_{33}=\lambda+2\mu$, $c_{12}=c_{13}=\lambda$, $c_{44}=\mu$, passing directly to the well-known equations of elasticity for a homogeneous isotropic body with Lamé's constants λ, μ.

3. The boundary-value problem and method of its solution

Bearing Eqs (2.1) and (2.2) in mind, the problem under consideration is now posed within the homogenized model as follows: find the fields w_i, σ_{ij} symmetric about the interface plane $x_3=0$ and suitably smooth on R^3-S_a such that Eqs (2.4) holds, subject to the following boundary conditions

$$\left.\begin{aligned}
&\sigma_{33}(x_1,x_2,+\infty)=p\\
&\sigma_{31}(x_1,x_2,+\infty)=\sigma_{32}(x_1,x_2,+\infty)=0
\end{aligned}\right\} \qquad \forall(x_1,x_2)\in R^2 \tag{3.1}$$

[1]Throughout this paper, the Latin indices i,j run over 1,2,3 while the Greek indices α,β run over 1,2, and the summation over repeated subscripts is taken for granted. Subscripts preceded by a comma indicate partial differentiation with respect to the corresponding coordinates. The index l or (l), assuming values 1 or 2, is associated with layer 1 and 2, respectively.

— Case (i)

$$\begin{aligned}
&\sigma_{33}(x_1,x_2,0^+) = -P_{fl} && \forall(x_1,x_2) \in S_a \\
&\sigma_{31}(x_1,x_2,0^+) = \sigma_{32}(x_1,x_2,0^+) = 0 && \forall(x_1,x_2) \in R^2 \\
&w_3(x_1,x_2,0^+) = 0 && \forall(x_1,x_2) \notin S_a
\end{aligned} \tag{3.2}$$

— Case (ii)

$$\begin{aligned}
&\sigma_{33}(x_1,x_2,0^+) = -P_{fl} && \forall(x_1,x_2) \in S_b \\
&\sigma_{31}(x_1,x_2,0^+) = \sigma_{32}(x_1,x_2,0^+) = 0 && \forall(x_1,x_2) \in R^2 \\
&w_3(x_1,x_2,0^+) = \frac{1}{2}f(x_1,x_2) && \forall(x_1,x_2) \in S_a - S_b \\
&w_3(x_1,x_2,0^+) = 0 && \forall(x_1,x_2) \notin S_a
\end{aligned} \tag{3.3}$$

The procedure for obtaining the solution follows along the same line of reasoning as that used in the classical crack theory. Applying the principle of superposition, we construct the solution as the sum of a trivial solution for the homogenized space without any defects, loaded by the given external load $\sigma_{33} = p$ at infinity, and a corrective solution to the problem involving perturbations caused by the defect V_d, which tends to zero at infinity. Attention will be drawn then on finding the corrective solution. In view of the symmetry, we can reduce the perturbed problem (having tilde) to a boundary-value problem for the half-space $x_3 \geqslant 0$ defined by the boundary conditions

$$\left.\begin{aligned}
&\widetilde{\sigma}_{33}(x_1,x_2,+\infty) = 0 \\
&\widetilde{\sigma}_{31}(x_1,x_2,+\infty) = \widetilde{\sigma}_{32}(x_1,x_2,+\infty) = 0
\end{aligned}\right. \qquad \forall(x_1,x_2) \in R^2 \tag{3.4}$$

— Case (i)

$$\begin{aligned}
&\widetilde{\sigma}_{33}(x_1,x_2,0^+) = -p - P_{fl} && \forall(x_1,x_2) \in S_a \\
&\widetilde{\sigma}_{31}(x_1,x_2,0^+) = \widetilde{\sigma}_{32}(x_1,x_2,0^+) = 0 && \forall(x_1,x_2) \in R^2 \\
&\widetilde{w}_3(x_1,x_2,0^+) = 0 && \forall(x_1,x_2) \notin S_a
\end{aligned} \tag{3.5}$$

— Case (ii)

$$\begin{aligned}
&\widetilde{\sigma}_{33}(x_1,x_2,0^+) = -p - P_{fl} && \forall(x_1,x_2) \in S_b \\
&\widetilde{\sigma}_{31}(x_1,x_2,0^+) = \widetilde{\sigma}_{32}(x_1,x_2,0^+) = 0 && \forall(x_1,x_2) \in R^2 \\
&\widetilde{w}_3(x_1,x_2,0^+) = \frac{1}{2}f(x_1,x_2) && \forall(x_1,x_2) \in S_a - S_b \\
&\widetilde{w}_3(x_1,x_2,0^+) = 0 && \forall(x_1,x_2) \notin S_a
\end{aligned} \tag{3.6}$$

The method of solving the above problem is based on the results given by Kaczyński (1993). A convenient representation of the solution is expressed by a single harmonic function denoted by

$$\varphi = \begin{cases} \widehat{\varphi} & \text{for} \quad \mu_1 \neq \mu_2 \\ \overline{\varphi} & \text{for} \quad \mu_1 = \mu_2, \quad \lambda_1 \neq \lambda_2 \end{cases}$$

as follows[2]:

— Case $\mu_1 \neq \mu_2$

$$\begin{aligned} \widetilde{w}_\alpha(x_1,x_2,x_3) &= \sum_{\gamma=1}^{2} (-1)^\gamma [t_\gamma(1+m_\gamma)]^{-1} \widehat{\varphi}_{,\alpha}(x_1,x_2,z_\gamma) \\ \widetilde{w}_3(x_1,x_2,x_3) &= \sum_{\gamma=1}^{2} (-1)^\gamma m_\gamma (1+m_\gamma)^{-1} \frac{\partial}{\partial z_\gamma} \widehat{\varphi}(x_1,x_2,z_\gamma) \\ \widetilde{\sigma}_{3\alpha}(x_1,x_2,x_3) &= \sum_{\gamma=1}^{2} (-1)^\gamma c_{44} \Big[\frac{\partial}{\partial z_\gamma} \widehat{\varphi}(x_1,x_2,z_\gamma)\Big]_{,\alpha} \\ \widetilde{\sigma}_{33}(x_1,x_2,x_3) &= \sum_{\gamma=1}^{2} (-1)^\gamma c_{44} t_\gamma^{-1} \frac{\partial^2}{\partial z_\gamma^2} \widehat{\varphi}(x_1,x_2,z_\gamma) \end{aligned} \tag{3.7}$$

— Case $\mu_1 = \mu_2 \equiv \mu$, $\lambda_1 \neq \lambda_2$

$$\begin{aligned} \widetilde{w}_\alpha &= \frac{\mu}{B+\mu} \overline{\varphi}_{,\alpha} & \qquad \widetilde{w}_3 &= -\frac{B+2\mu}{B+\mu} \overline{\varphi}_{,3} + x_3 \overline{\varphi}_{,33} \\ \widetilde{\sigma}_{3\alpha} &= 2\mu x_3 \overline{\varphi}_{,\alpha 33} & \qquad \widetilde{\sigma}_{33} &= 2\mu(-\overline{\varphi}_{,33} + x_3 \overline{\varphi}_{,333}) \end{aligned} \tag{3.8}$$

where

$$B = \frac{\lambda_1\lambda_2 + 2\mu[(1-\eta)\lambda_2 + \eta\lambda_1]}{(1-\eta)\lambda_1 + \eta\lambda_2 + 2\mu}$$

Expressions for the remaining stresses have been omitted.

Note that the above suitable representation automatically satisfies the conditions appearing in Eqs $(3.5)_2$ and $(3.6)_2$. The quantities of immediate interest are

$$\widetilde{w}_3(x_1,x_2,0^+) = L\varphi_{,3}\Big|_{x_3=0^+} \qquad \widetilde{\sigma}_{33}(x_1,x_2,0^+) = M\varphi_{,33}\Big|_{x_3=0^+} \tag{3.9}$$

[2] All constants appearing in the following equations are given in Appendix B; $z_\alpha = t_\alpha x_3$, $\alpha = 1,2$

where

$$L = \begin{cases} \dfrac{m_2}{1+m_2} - \dfrac{m_1}{1+m_1} & \text{for} \quad \mu_1 \neq \mu_2 \\ -\dfrac{B+2\mu}{B+\mu} & \text{for} \quad \mu_1 = \mu_2 \equiv \mu \end{cases}$$

$$M = \begin{cases} c_{44}\Big(\dfrac{1}{t_2} - \dfrac{1}{t_1}\Big) & \text{for} \quad \mu_1 \neq \mu_2 \\ -2\mu & \text{for} \quad \mu_1 = \mu_2 \equiv \mu \end{cases} \tag{3.10}$$

The problem described by Eqs (3.4)-(3.6) is then reduced to a mixed problem of finding a harmonic function φ in the half-space $x_3 \geqslant 0$, decaying at infinity (in view of (3.4)) and satisfying the boundary the conditions:
— Case (i)

$$\begin{aligned} M\varphi_{,33}\Big|_{x_3=0^+} &= -p - P_{fl} && \forall (x_1,x_2) \in S_a \\ L\varphi_{,3}\Big|_{x_3=0^+} &= 0 && \forall (x_1,x_2) \notin S_a \end{aligned} \tag{3.11}$$

— Case (ii)

$$\begin{aligned} M\varphi_{,33}\Big|_{x_3=0^+} &= -p - P_{fl} && \forall (x_1,x_2) \in S_b \\ L\varphi_{,3}\Big|_{x_3=0^+} &= \frac{1}{2} f(x_1,x_2) && \forall (x_1,x_2) \in S_a - S_b \\ L\varphi_{,3}\Big|_{x_3=0^+} &= 0 && \forall (x_1,x_2) \notin S_a \end{aligned} \tag{3.12}$$

The harmonic function φ can be represented as a potential of a single layer:
— Case (i)

$$\varphi(x_1,x_2,x_3) = -\frac{1}{2\pi L} \iint\limits_{S_a} \frac{h(\xi_1,\xi_2)\; d\xi_1 d\xi_2}{\sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + x_3^2}} \tag{3.13}$$

— Case (ii)

$$\begin{aligned} \varphi(x_1,x_2,x_3) \;&=\; -\frac{1}{2\pi L} \iint\limits_{S_a} \frac{h(\xi_1,\xi_2)\; d\xi_1 d\xi_2}{\sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + x_3^2}} + \\ &+\; \frac{1}{2\pi L} \iint\limits_{S_b} \frac{f(\xi_1,\xi_2)\; d\xi_1 d\xi_2}{\sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + x_3^2}} \end{aligned} \tag{3.14}$$

where h is the unknown layer density that can be identified as the defect face displacement: $\widetilde{w}_3(x_1, x_2, 0^+)$ in Case (i) and $\widetilde{w}_3(x_1, x_2, 0^+) + 0.5f(x_1, x_2)$ in Case (ii). Expressions (3.13) and (3.14) satisfy boundary conditions $(3.11)_2$ and $(3.12)_{2,3}$ identically due to the property of the normal derivative of the single-layer potential. Substitution (3.13) and (3.14) into remaining conditions $(3.11)_1$ and $(3.12)_1$ leads to the following integro-differential singular equation for the function h

— Case (i)

$$\nabla^2 \iint\limits_{S_a} \frac{h(\xi_1, \xi_2)\, d\xi_1 d\xi_2}{\sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}} = \frac{2\pi L(-p - P_{fl})}{M} \tag{3.15}$$

— Case (ii)

$$\begin{aligned} \nabla^2 \iint\limits_{S_b} \frac{h(\xi_1, \xi_2)\, d\xi_1 d\xi_2}{\sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}} &= \\ = \frac{2\pi L(-p - P_{fl})}{M} &+ \nabla^2 \iint\limits_{S_a} \frac{f(\xi_1, \xi_2)\, d\xi_1 d\xi_2}{\sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}} \end{aligned} \tag{3.16}$$

where $\nabla^2 \equiv \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ stands for the two-dimensional Laplace operator.

To find the two unknown parameters of our problem, namely the pressure P_{fl} and the radius of the non-contacting zone b, we have to use state equation (2.3) and the condition of boundedness of the normal stress $\sigma_{33}|_{r=b}$, respectively. Once the function h is known from the solution to Eqs (3.15) and (3.16), the stresses and displacements in any point of the composite can be found from the harmonic potential φ, determined from Eqs (3.13) or (3.14), and formulas (3.7) or (3.8).

4. Examples

In this Section, we present and discuss solutions to the boundary-value problem defined by (3.4)-(3.6) for Case (i) and Case (ii).

4.1. Case (i)

In this extreme case, when the defect is a circular crack S_a, the problem can be solved in elementary functions.

The solution to Eq. (3.15) is known (see, for example, Khay, 1993), and given as

$$h(x_1,x_2)=\frac{2L(p+P_{fl})}{\pi M}\sqrt{a^2-r^2} \tag{4.1}$$

Substituting Eq. (4.1) into Eq. (3.13) and evaluating the integral, as outlined by Fabrikant (1989), one obtains the governing potential φ in the form

$$\varphi(x_1,x_2,x_3)=-\frac{p+P_{fl}}{2\pi M}\Big[(2a^2+2x_3^2-r^2)\sin^{-1}\frac{a}{l_+}-\frac{2a^2-3l_-^2}{a}\sqrt{l_+^2-a^2}\Big] \tag{4.2}$$

where

$$l_\mp\equiv\frac{1}{2}\Big[\sqrt{(r+a)^2+x_3^2}\mp\sqrt{(r-a)^2+x_3^2}\Big] \tag{4.3}$$

To find P_{fl}, we take into account Eq. (2.3), in which

$$V=\iint\limits_{S_a}[w_3(x_1,x_2,0^+)-w_3(x_1,x_2,0^-)]\;dx_1dx_2=2\iint\limits_{S_a}h(x_1,x_2)\;dx_1dx_2 \tag{4.4}$$

Substituting Eq. (4.1) into Eq. (4.4), and performing integration, we get

$$V=\frac{8L(p+P_{fl})a^3}{3M} \tag{4.5}$$

and, according to Eq. (2.3), one arrives at the following transcendental equation for the unknown pressure P_{fl}

$$\frac{8L(p+P_{fl})a^3}{3M}=\frac{m_*}{\rho_0}\exp\Big(-\frac{P_{fl}}{\beta}\Big) \tag{4.6}$$

Now we can discuss the singular behaviour at the crack edge in the above symmetrical problem. By means of Eqs (3.9) and (4.2), one finds

$$\widetilde{\sigma}_{33}=\frac{2(p+P_{fl})}{\pi}\Big(\frac{a}{\sqrt{r^2-a^2}}-\sin^{-1}\frac{a}{r}\Big) \qquad r>a, \quad x_3=0 \tag{4.7}$$

If the stress intensity factor (SIF) is defined as

$$K_I=\lim_{r\to a^+}\sqrt{2\pi(r-a)}\,\widetilde{\sigma}_{33}(r,0) \tag{4.8}$$

then we have

$$K_I=2\sqrt{\frac{a}{\pi}}(p+P_{fl}) \tag{4.9}$$

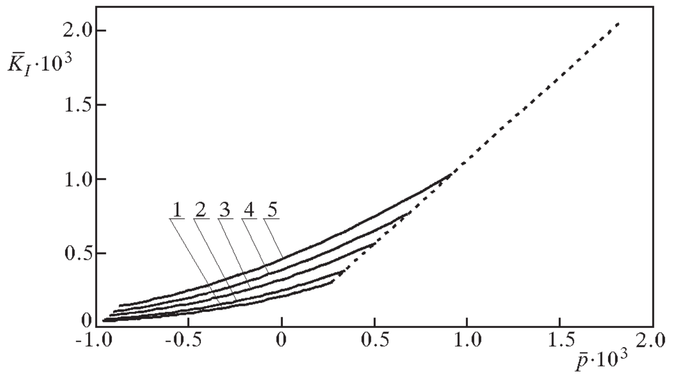

Fig. 2. Dependence of the stress intensity factor of mode I on the external load ($1 - \gamma = 0.5,\ \eta = 0.1,\ 2 - \gamma = 0.5,\ \eta = 0.5,\ 3 - \gamma = 1,\ \eta = 1$ corresponds to the homogeneous isotropic material, $4 - \gamma = 2,\ \eta = 0.5,\ 5 - \gamma = 2,\ \eta = 0.1$)

It is interesting to note that this factor turns out to be independent of the elastic properties of the laminated body.

Figure 2 demonstrates the change of the SIF K_I under the loading. The calculations were performed on the simplifying assumptions $\lambda_1 = \mu_1$ and $\lambda_2 = \mu_2$ for the following dimensionless quantities

$$\begin{array}{lll} \eta = \dfrac{\delta_1}{\delta} & \gamma = \dfrac{\mu_2}{\mu_1} & \overline{\beta} = \dfrac{\beta}{\mu_1} = 5 \cdot 10^{-4} \\ \overline{p} = \dfrac{p}{\mu_1} & \overline{P}_{fl} = \dfrac{P_{fl}}{\mu_1} & \dfrac{m_*}{a^3 \rho_0} = 10^{-3} \end{array} \tag{4.10}$$

It can be seen that even for negative values of the parameter p, i.e. for a compressive pressure at infinity, the stress intensity factor has a positive magnitude. As the compressive load increases, SIF K_I monotonically decreases and asymptotically tends to zero. For this range of the external load, the dependence of K_I as a function of p is strictly nonlinear. This fact is a consequence of the presence of the filler in the defect. For the tensile external pressure p the internal pressure of the fluid P_{fl} decreases, and hence the effect of the defect filler weakens. There is some critical value of the tensile external load under which the pressure of the filler becomes zero. If the external pressure is greater than this critical value, the filler of the cavity does not affect the mechanical behaviour of the body. The dashed lines in Fig. 2 correspond to this range of the external load. Then the dependence of K_I on p is linear and all curves coincide. It means that this stress intensity factor does not depend on mechanical and geometrical parameters of the body. This result is similar to that of the classical theory of homogeneous isotropic solid (see, for example, Khay, 1993).

4.2. Case (ii)

Let us consider the cavity V_d (see (2.1) and (2.2)), the initial shape of which is of a specific form, given by the function

$$f(x_1, x_2) = h_0 \sqrt{\Big(1 - \frac{x_1^2 + x_2^2}{a^2}\Big)^3} \qquad h_0 \ll a \tag{4.11}$$

In this case, the faces of this defect may contact through an unknown ring-shaped zone $b < r = \sqrt{x_1^2 + x_2^2} \leqslant a$ adjacent to its rim.

Having calculated the integral in the right-hand side with the use of (4.11) (see Khay, 1993), integro-differential equation (3.16) takes the form

$$\nabla^2 \iint\limits_{S_b} \frac{h(\xi_1, \xi_2)\; d\xi_1 d\xi_2}{\sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}} = \frac{2\pi L(-p - P_{fl})}{M} - \frac{3\pi^2 h_0}{2a^3} \Big[a^2 - \frac{3}{2}(x_1^2 + x_2^2)\Big] \tag{4.12}$$

To solve it, an analogue of Dyson's theorem (Khay, 1993) will be applied. Thus, we seek a solution to Eq. (4.12) in the following form

$$h(x_1, x_2) = \sqrt{b^2 - x_1^2 - x_2^2}\; (c_{00} + c_{10}x_1 + c_{01}x_2 + c_{20}x_1^2 + c_{11}x_1x_2 + c_{02}x_2^2) \tag{4.13}$$

where c_{ij} are unknown coefficients.

Substituting (4.13) into (4.12) and calculating the resulting integrals (see Khay, 1993), we arrive at the equality of two polynomials of the second order. Hence, the coefficients c_{ij} can be easily found by comparing the coefficients, and the sought function turns out to be

$$h(x_1, x_2) = \sqrt{b^2 - r^2}\, \Big[\frac{2L(p + P_{fl})}{\pi M} + \frac{3h_0^2(a^2 + b^2)}{2a^3} + \frac{h_0(b^2 - r^2)}{a^3}\Big] \tag{4.14}$$

This expression contains two unknown parameters: b and P_{fl}. Using the requirement that the normal stresses $\widetilde{\sigma}_{33}$ are bounded at $r = b$ or, equivalently, the corresponding SIF K_I has to be zero, we obtain the equation

$$2\sqrt{\frac{b}{\pi}}\, (p + P_{fl}) + \frac{3h_0 M\sqrt{\pi b}}{2a^3 L}(a^2 + b^2) = 0 \tag{4.15}$$

The parameter P_{fl} is found from state equation (2.3), similarly to Case (i) with one exception. Here we assume that the fluid fills the whole initial defect without any internal pressure. It means that the volume of the unpressed fluid of the mass m_* is equal to the volume of the initial defect. In other words, the following relationship takes place

$$\frac{m_*}{\rho_0} \equiv V_0 = \iint\limits_{S_a} f(x_1, x_2)\, dx_1 dx_2 \tag{4.16}$$

The calculation of the volume of the cavity after application of the load, and insertion into Eq. (2.3), yields the following transcendental algebraic equation

$$4\pi\Big[\frac{2L}{\pi M}(p + P_{fl}) + \frac{3h_0}{2a^3}(a^2 + b^2)\Big] + 4\pi\frac{h_0 b^5}{5a^3} = 4\pi\frac{h_0 a^2}{5}\exp\Big(-\frac{P_{fl}}{\beta}\Big) \tag{4.17}$$

Using the system of two equations (4.15) and (4.17), we find b and P_{fl} and then, substituting them into formula (4.14), we get the solution via the harmonic function φ, given by (3.14).

Some results of numerical calculations are shown in Fig. 3 and Fig. 4 by using the following parameters as in Case (i), see Eq. (4.10)

$$\begin{array}{lll} \overline{\beta} = \dfrac{\beta}{\mu_1} & \overline{p} = \dfrac{p}{\mu_1} & \overline{P}_{fl} = \dfrac{P_{fl}}{\mu_1} \\ \overline{\sigma}_{33} = \dfrac{\widetilde{\sigma}_{33}}{\mu_1} & \overline{b} = \dfrac{b}{a} & \eta = \dfrac{\delta_1}{\delta} = 1 \\ \gamma = \dfrac{\mu_2}{\mu_1} = 1 & \overline{h}_0 = \dfrac{h_0}{a} = 10^{-3} & \end{array} \tag{4.18}$$

The simplifying assumptions $\lambda_1 = \mu_1$ and $\lambda_2 = \mu_2$ have been used as well.

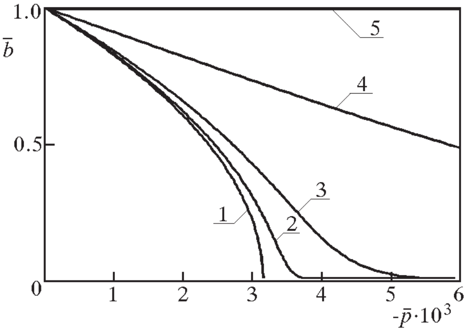

Fig. 3. Dependence of the inner radius of the contacting zone on the external load ($1 - \overline{\beta} = 0,\ 2 - \overline{\beta} = 2.5 \cdot 10^{-5},\ 3 - \overline{\beta} = 10^{-4},\ 4 - \overline{\beta} = 10^{-3},\ 5 - \overline{\beta} = \infty$)

Figure 3 illustrates the dependence of the inner radius of contacting zone b on the external load p. The graphs obtained show that this zone decreases monotonically as the tensile external load increases. The less is the coefficient of compressibility of the fluid β, the faster decreases the curve. For the range $0 < \beta < \infty$, while the parameter p tends to infinity, the curves show an asymptotic decrease to zero, non-intersecting the abscissa axis. This effect is a consequence of the assumption on the fixed mass of the fluid in the defect. If $\beta = \infty$ (the fluid is imcompressible), the shape and size of the cavity do not change under the loading.

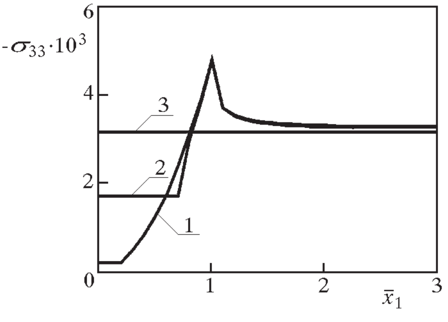

Fig. 4. Normal stress distribution on the interface $x_3 = 0$ versus $\overline{x}_1 = r/a$ $(1 - \overline{\beta} = 2.5 \cdot 10^{-5},\ 2 - \overline{\beta} = 10^{-3},\ 3 - \overline{\beta} = \infty)$

The normal stress distribution on the interface plane is shown in Fig. 4. The curves correspond to the value of the tensile external load $\overline{p} = -3.142 \cdot 10^{-3}$ and different values of the coefficient of compressibility of the fluid. Referring to Fig. 4, it can be noted that the difference between the greatest and the least magnitudes of the normal stress, so-called dispersion, is less for the bigger value of the parameter β. In the limit case $\overline{\beta} = \infty$, which corresponds to an imcompressible fluid, the dispersion of the normal stresses is equal to zero. In this case, the normal stresses are distributed uniformly on the interface.

5. Conclusions

As a result of the analysis made it was revealed that the compressible barotropic fluid filling the interface defect induces severe non-linear behaviour of the composite structure. It is true for such a range of the external load, under which the internal pressure of the filler is greater than zero.

A. Appendix

Denoting by $b_l = \lambda_l + 2\mu_l$ $(l = 1,2)$, $b = (1-\eta)b_1 + \eta b_2$, the positive coefficients in governing equations (2.4) and (2.5) are given by the following formulae

$$
\begin{aligned}
c_{11} &= \frac{b_1 b_2 + 4\eta(1-\eta)(\mu_1-\mu_2)(\lambda_1-\lambda_2+\mu_1-\mu_2)}{b} \\
c_{12} &= \frac{\lambda_1\lambda_2 + 2[\eta\mu_2 + (1-\eta)\mu_1][\eta\lambda_1 + (1-\eta)\lambda_2]}{b} \\
c_{13} &= \frac{(1-\eta)\lambda_2 b_1 + \eta\lambda_1 b_2}{b} & c_{33} &= \frac{b_1 b_2}{b} \\
c_{44} &= \frac{\mu_1\mu_2}{(1-\eta)\mu_1 + \eta\mu_2} & d_{11}^{(l)} &= \frac{4\mu_l(\lambda_l+\mu_l) + \lambda_l c_{13}}{b_l} \\
d_{12}^{(l)} &= \frac{2\mu_l\lambda_l + \lambda_l c_{13}}{b_l} & d_{13}^{(l)} &= \frac{\lambda_l c_{33}}{b_l}
\end{aligned}
$$

B. Appendix

The constants appearing first in Eqs. (3.7) are given as follows

$$
\begin{aligned}
t_1 &= \frac{1}{2}(t_+ - t_-) & t_2 &= \frac{1}{2}(t_+ + t_-) \\
m_\alpha &= \frac{c_{11}t_\alpha^{-2} - c_{44}}{c_{13} + c_{44}} & &\forall\alpha \in \{1,2\}
\end{aligned}
$$

where

$$
t_\pm = \sqrt{\frac{(A_\pm \pm 2c_{44})A_\mp}{c_{33}c_{44}}} \qquad A_\pm = \sqrt{c_{11}c_{33}} \pm c_{13}
$$

Note that

$$
t_1 t_2 = \sqrt{\frac{c_{11}}{c_{33}}} \qquad m_1 m_2 = 1
$$

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O zagadnieniu pewnego defektu międzywarstwowego wypełnionego ściśliwym płynem w periodycznym ośrodku warstwowym

Streszczenie

W pracy rozważono zagadnienie periodycznej dwuwarstwowej przestrzeni zawierającej defekt na granicy warstw, który wypełniony jest ściśliwym płynem. Zakłada się, że kompozyt jest pod działaniem stałego obciążenia w nieskończoności prostopadłego do uwarstwienia, a powierzchnie defektu są pod wpływem stałego i nieznanego wewnętrznego ciśnienia płynu. Analizowany problem sprowadza się w ramach pewnego zhomogenizowanego modelu do poszukiwania rozwiązania osobliwego równania całkowo-różniczkowego. Wyniki otrzymano dla dwóch typów defektu z użyciem analogu twierdzenia Dysona. Zbadano i zilustrowano wpływ wypełniacza defektu na mechaniczne zachowanie rozważanego ciała.

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