JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
41, 3, pp. 509-520, Warsaw 2003

HYBRID ASYMPTOTIC METHOD FOR THE EFFECT OF LOCAL THICKNESS DEFECTS AND INITIAL IMPERFECTIONS ON THE BUCKLING OF CYLINDRICAL SHELLS

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A double asymptotic method for the effect of local thickness defects and initial imperfections on the buckling problems of some mechanical systems is proposed. Asymptotic formulas for the critical buckling load parameter are obtained with the perturbation technique.

Key words: cylindrical shell, asymptotic method, local thickness defects, initial imperfections, asymptotic formulas

1. Introduction

It is well known that initial imperfections of shells may have a remarkable effect on the buckling load. Many authors discussed the effect of imperfections on the buckling of cylindrical shells under axial compression. One can refer to the works by Koiter (1945, 1978), Amazigo and Budiansky (1972), Kan (1966), Volmir (1967), Gristchak (1976), Koiter *et al.* (1994) in details.

In this paper, following the considerations of Gristchak (1976), Koiter *et al.* (1994), a hybrid perturbation method is proposed for the estimation of effect of local thickness defects and initial imperfections on the buckling of an axially compressed cylindrical shell.

In order to obtain an approximate solution to the problem we use the double asymptotic expansion technique that includes two steps. In the first step the stress function is presented as an expansion of the magnitude of the thickness defects in terms of a small parameter ε. In the second step the first term in the obtained expansion is presented as an asymptotic expansion in a small parameter μ^2/m^2 that indicates the thickness defect spread area.

In spite of the fact that the proposed asymptotic method is used only for the buckling problem of a compressed cylindrical shell in this paper, it can be applied for the investigation of behaviour of thin-walled structures under combined loadings as well.

2. Description of the hybrid perturbation method

We consider a circular cylindrical shell of the radius R, length L that is made of an isotropic, elastic material with Young's modulus E and Poisson's ratio ν. It is subjected to an axial compressive load P_0. The coordinate system is assumed as shown in Figure 1.

The governing equations of the pressurized non-uniform cylindrical shell are as follows (Koiter *et al.*, 1994)

$$\begin{aligned}
&\frac{Eh^3(x)}{12(1-\nu^2)}\Big(\frac{\partial^4 W}{\partial x^4}+2\frac{\partial^4 W}{\partial x^2\partial y^2}+\frac{\partial^4 W}{\partial y^4}\Big)+\\
&+\frac{3E}{12(1-\nu^2)}[2h(x)h'^2(x)+h^2(x)h''(x)]\Big(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\Big)+\\
&+\frac{1}{R}\frac{\partial^2 F}{\partial x^2}+\frac{6E}{12(1-\nu^2)}h^2(x)h'(x)\Big(\frac{\partial^3 W}{\partial x^3}+\frac{\partial^3 W}{\partial x\partial y^2}\Big)+\\
&+P_0\frac{\partial^2 W}{\partial x^2}-\frac{3E}{12(1-\nu^2)}(1-\nu)[2h(x)h'^2(x)+h^2(x)h''(x)]\frac{\partial^2 W}{\partial y^2}=0
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
&\frac{1}{Eh(x)}\Big(\frac{\partial^4 F}{\partial x^4}+2\frac{\partial^4 F}{\partial x^2\partial y^2}+\frac{\partial^4 F}{\partial y^4}\Big)-\frac{1}{E}\Big(\frac{h''(x)h(x)-2h'^2(x)}{h^3(x)}\Big)\Big(\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}\Big)-\\
&-\frac{1}{R}\frac{\partial^2 W}{\partial x^2}-\frac{2}{E}\frac{h'(x)}{h^2(x)}\Big(\frac{\partial^3 F}{\partial x^3}+\frac{\partial^3 F}{\partial x\partial y^2}\Big)+\frac{1+\nu}{E}\Big(\frac{h''(x)h(x)-2h'^2(x)}{h^3(x)}\Big)\frac{\partial^2 F}{\partial y^2}=0
\end{aligned}$$

where $x \in [-L/2;L/2]$ and $y \in [0;2\pi R]$ are the axial and circimferential coordinates, respectively, $W(x,y)$ is the radial displacement (positive outward), $F(x,y)$ is the stress function, $h(x)$ is the shell thickness. It is necessary to note that the above given equations can be applied to shells of medium length ($1.5 \leqslant L/R \leqslant 4$) provided that the wave number in the circumferential direction is rather large.

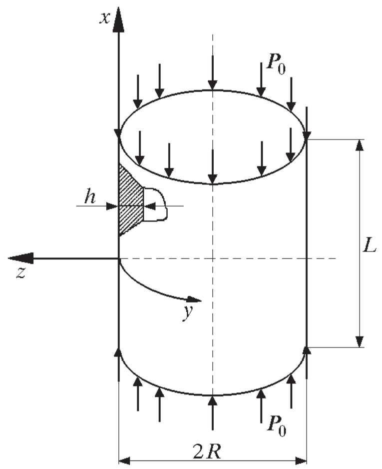

Fig. 1. Imperfect cylindrical shell under axial compression

By introducing the following non-dimensional parameters

$$\xi = \frac{x}{L} \qquad \eta = \frac{y}{L} \qquad H(\xi) = \frac{h(x)}{h_0}$$

$$W^*(\xi,\eta) = \frac{W(x,y)}{h_0} \qquad F^*(\xi,\eta) = \frac{F(x,y)}{D} \qquad D = \frac{Eh_0^3}{12(1-\nu^2)}$$

where h_0 is the nominal thickness of the shell, D is the bending stiffness, governing equations (2.1) can be rewritten into their non-dimensional form

$$\begin{aligned}
&H^3\Big(\frac{\partial^4 W^*}{\partial\xi^4} + 2\frac{\partial^4 W^*}{\partial\xi^2\partial\eta^2} + \frac{\partial^4 W^*}{\partial\eta^4}\Big) + 3(2HH'^2 + H^2H'')\Big(\frac{\partial^2 W^*}{\partial\xi^2} + \frac{\partial^2 W^*}{\partial\eta^2}\Big) + \\
&+\frac{L^2}{Rh_0}\frac{\partial^2 F^*}{\partial\xi^2} + 6H^2H'\Big(\frac{\partial^3 W^*}{\partial\xi^3} + \frac{\partial^3 W^*}{\partial\xi\partial\eta^2}\Big) - \\
&-3(1-\nu)(2HH'^2 + H^2H'')\frac{\partial^2 W^*}{\partial\eta^2} + \frac{P_0L^2}{D}\frac{\partial^2 W^*}{\partial\xi^2} = 0 \\
&H^2\Big(\frac{\partial^4 F^*}{\partial\xi^4} + 2\frac{\partial^4 F^*}{\partial\xi^2\partial\eta^2} + \frac{\partial^4 F^*}{\partial\eta^4}\Big) + 2H'^2\Big(\frac{\partial^2 F^*}{\partial\xi^2} - \nu\frac{\partial^2 F^*}{\partial\eta^2}\Big) - \\
&-HH''\Big(\frac{\partial^2 F^*}{\partial\xi^2} - \nu\frac{\partial^2 F^*}{\partial\eta^2}\Big) - 2HH'\Big(\frac{\partial^3 F^*}{\partial\xi^3} + \frac{\partial^3 F^*}{\partial\xi\partial\eta^2}\Big) - \\
&-\frac{12(1-\nu^2)L^2}{Rh_0}H^3\frac{\partial^2 W^*}{\partial\xi^2} = 0
\end{aligned} \tag{2.2}$$

We seek the solution to the initial equations in the following form (Koiter *et al.*, 1994)

$$\begin{aligned} F^*(\xi,\eta) &= f(\xi)\cos\Big(\frac{nL}{R}\eta\Big) \\ W^*(\xi,\eta) &= w(\xi)\cos\Big(\frac{nL}{R}\eta\Big) \end{aligned} \tag{2.3}$$

where n is the wave number in the circumferential direction.

We consider the local thickness defects in the form

$$H(\xi) = 1 - \varepsilon \exp\Big(-\frac{\mu^2}{2}\xi^2\Big) \tag{2.4}$$

where $\varepsilon \ll 1$ and μ^2 are parameters indicating the amplitude magnitude of the thickness defects and the character of the defect location, respectively. The above given form of local thickness defects is "local" in the sense that function (2.4) decays rapidly in the axial direction because of the exponential factor $\exp(-\mu^2\xi^2/2)$.

The buckling mode of the circular cylindrical shell can be written in the form

$$w(\xi) = \Big[A_1\cos(m\xi) + A_2\cos(3m\xi)\Big]\exp\Big(-\frac{\mu^2}{2}\xi^2\Big) \tag{2.5}$$

where A_1 and A_2 are undetermined constants, $m = pL/R$, $p = p_0/2$ is the number of half-waves along the shell length. Here

$$p_0 = \sqrt{2c\frac{R}{h_0}} \qquad\qquad c = \sqrt{3(1-\nu^2)}$$

The first term in the brackets is the buckling pattern that satisfies the boundary conditions of the simple support. The second term and modulating factor $\exp(-\mu^2\xi^2/2)$ are introduced in (2.5) as the local thickness defects may initiate the local buckling mode.

The unknown function $f(\xi)$ is sought as a two-term expansion in the small defect parameter ε

$$f(\xi) = \varphi_0(\xi)\exp\Big(-\frac{\mu^2}{2}\xi^2\Big) + \varepsilon\varphi_1(\xi)\exp(-\mu^2\xi^2) + \dots \tag{2.6}$$

Taking into account that the wave number p_0 is large enough for thin shells, and respectively $m \gg 1$, we may consider $\mu^2/m^2 \ll 1$ being a small parameter.

The function $\varphi_0(\xi)$ can be represented as an asymptotic expansion in the small parameter $\beta^2 = \mu^2/m^2$ such as

$$\varphi_0(\xi) = f_0(\xi) + \beta^2 f_1(\xi) + \dots \tag{2.7}$$

Substituting expansions (2.6) and (2.7) into the initial equation after collecting coefficients with equal orders of the parameters ε and β^2 we obtain a system of the unknown functions $f_0(\xi)$, $f_1(\xi)$, $\varphi_1(\xi)$. We neglect here the terms of higher order than ε and β^2.

It should be noted that there are many coincident modes and the corresponding wave numbers n and p are located on Koiter's circle. We consider the case when the buckling mode has the same wave numbers both in the axial and circumferential directions. Thus, in this case we have (Koiter *et al.*, 1994)

$$p = n = \frac{p_0}{2}$$

We apply the Boobnov-Galerkin procedure to equilibrium equation $(2.2)_1$. Admittedly, it should be pointed out that in order to obtain analytical forms of all integrals in the solution we replace the finite limits of the integrals by infinite ones. It is possible that the process of deformation has local character. All integrals from $(2.2)_1$ are as follows

$$\begin{aligned} &\int\limits_{-\infty}^{+\infty} \exp(-\beta^2\xi^2)\, d\xi = \frac{\sqrt{\pi}}{\beta} \\ &\int\limits_{-\infty}^{+\infty} \exp(-\beta^2\xi^2)\cos(g\xi)\, d\xi = \frac{\sqrt{\pi}}{\beta}\exp\Big(-\frac{g^2}{4\beta^2}\Big) \\ &\int\limits_{-\infty}^{+\infty} \xi\exp(-\beta^2\xi^2)\sin(g\xi)\, d\xi = \frac{g\sqrt{\pi}}{2\beta^3}\exp\Big(-\frac{g^2}{4\beta^2}\Big) \end{aligned} \tag{2.8}$$

where g is integer.

The Boobnov-Galerkin procedure yields the eigenvalue problem. Denoting

$$\lambda = \frac{P_0}{P_c} \qquad \text{where} \qquad P_c = \frac{Eh_0^2}{R\sqrt{3(1-\nu^2)}}$$

after some simplifications we obtain an asymptotic formula for the critical buckling load parameter λ

$$\lambda = 1 - \Big[\frac{\sqrt{2}\,\varepsilon(1+K)}{1+\beta^2}\Big] \tag{2.9}$$

where $K = \exp(-1/\beta^2)$.

As indicated in formula (2.9), the local thickness defects, see Eq. (2.4), reduce the critical buckling load parameter.

3. Analysis of the critical buckling load parameter

In Table 1 we compare the results corresponding to the buckling load parameter λ for cylindrical shells with the local thickness defects given in form (2.4) and with periodical ones in the form $H(\xi) = 1 - \varepsilon\cos(2m\xi)$, which were considered by Koiter *et al.* (1994).

Table 1. Comparison of buckling load parameters for periodical and local thickness defects

ε	Koiter's formula	Eq. (2.9) $\beta^2 = 0.05$	Eq. (2.9) $\beta^2 = 0.1$
0.01	0.998	0.987	0.987
0.05	0.988	0.933	0.936
0.10	0.966	0.865	0.871
0.15	0.935	0.798	0.807

As the results show, the local form of thickness variation as well as the periodical one effects the buckling load parameter. In Table 1 it is easy to see that at fixed parameter β^2 the increasing amplitude of the local thickness defects (parameter ε) reduces the critical buckling load. In particular, when the amplitude of the local thickness defects increases by 15% ($\varepsilon = 0.15$) and the defect area amounts to about 45% (when $\beta^2 = 0.1$) of the shell length, the critical buckling load parameter decreases by 20% in comparison with the perfect shell. At the same time at fixed parameter ε the increasing parameter β^2 (which means narrowing of the defect area) lowers the shell sensitivity to this defect. Thus, the local thickness defects of a cylindrical shell are as harmful as those of the periodical type, Koiter *et al.* (1994).

The form of the deformed surface of the compressed cylindrical shell depending on the character of the defect location is given in Fig. 2.

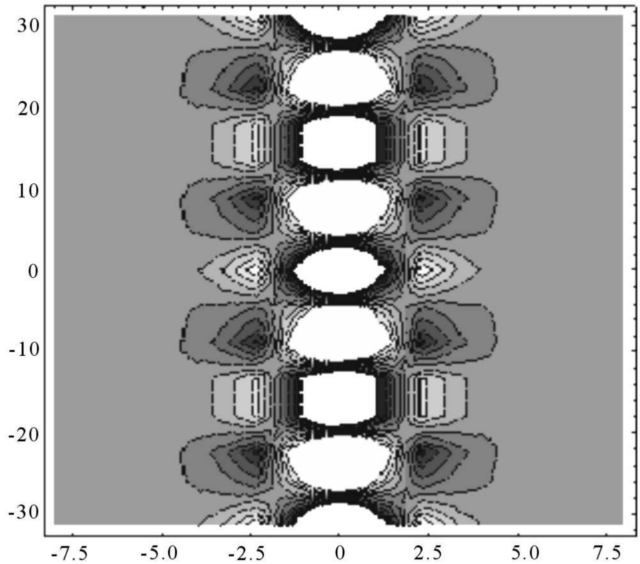

Fig. 2. Contour picture of the shell surface with the local thickness defects at $\beta^2 = 0.5$ and $\varepsilon = 0.1$

4. Analysis of the energy functional of the cylindrical shell

The energy functional of an isotropic cylindrical shell subject to axial compressive load has been evaluated by Gristchak (1976), Koiter (1978), and takes the form

$$\begin{aligned} P[u,v,w,w_0] = \frac{E}{2(1-\nu^2)R^2} \iint_\Omega \Big\{ h(u'_x)^2 + h(v'_y + w)^2 + \\ +2h\nu u'_x(v'_y + w) + \frac{1}{2}h(1-\nu)(u'_y + v'_x)^2 + \\ + \frac{h^3}{12R^2}[(w"_{xx})^2 + (w"_{yy})^2 + 2\nu w"_{xx}w"_{yy} + 2(1-\nu)(w"_{yx})^2] - \\ - \frac{(1-\nu^2)h\sigma}{E}[(w'_x)^2 + 2w'_x w'_{0x}] + \frac{h}{R}[u'_x + \nu(v'_y + w)](w'_x)^2 + \\ + \frac{h}{R}(v'_y + w + \nu u'_x)(w'_y)^2 + \frac{h}{R}(1-\nu)(u'_y + v'_x)w'_x w'_y \Big\}\, dxdy \end{aligned} \tag{4.1}$$

where Ω is the shell surface, E is Young's modulus of the shell material, ν is Poisson's ratio, R is the shell radius, $h(x,y)$ is the shell thickness, σ is the axial compressive stress, $x \in [-L/(2R); L/(2R)]$ and $y \in [0; 2\pi]$ are the non-dimensional axial and circumferential coordinates, $u(x,y)$, $v(x,y)$, $w(x,y)$ are the components of the displacements, $w_0(x,y)$ is the function of the local imperfections.

Taking into consideration that the initial imperfections are localized, we assume that the displacements in the functional are given in the form proposed by Gristchak (1976), Koiter (1978) for more or less localized imperfections

$$\begin{aligned} u(x,y) &= b_0\Big[-\frac{\nu}{2m}\sin(2mx) + \frac{1-\nu}{m}\sin(mx)\cos(my)\Big]\exp\Big[-\frac{\mu^2}{2}(x^2+y^2)\Big] \\ v(x,y) &= b_0\Big[-\frac{3+\nu}{m}\cos(mx)\sin(my)\Big]\exp\Big[-\frac{\mu^2}{2}(x^2+y^2)\Big] \\ w(x,y) &= b_0\Big[\cos(2mx) + 4\cos(mx)\cos(my)\Big]\exp\Big[-\frac{\mu^2}{2}(x^2+y^2)\Big] \end{aligned} \tag{4.2}$$

where b_0 is the amplitude of the displacements, μ^2 is the parameter that effects the location of the initial geometric imperfections, $m = \sqrt{cR/(2h_0)}$ is the number of half-waves along the shell length, $c = \sqrt{3(1-\nu^2)}$.

It is worth noticing that in order to compare the obtained results with the established ones (Gristchak, 1976; Koiter, 1978) we use the same form of the energy functional and forms of displacements.

The forms of displacements (4.2) differ from buckling mode (2.3) as in the previous case we have considered a cylindrical shell with the thickness defects only, whereas in the case under consideartion functions (4.2) describe more exactly the shape of the deformation of the cylindrical shell with the initial geometric imperfections.

The initial imperfections in the shape of the buckling mode are assumed in the form

$$w_0(x,y) = kh_0[\cos(2mx) + 4\cos(mx)\cos(my)]\exp\Big[-\frac{\mu^2}{2}(x^2+y^2)\Big] \tag{4.3}$$

where k is the amplitude of the initial imperfections, h_0 is the nominal thickness of the shell.

We consider the case when the thickness defects are local in the axial direction

$$h(x) = h_0\Big[1 - \varepsilon\exp\Big(-\frac{N^2}{2}x^2\Big)\Big] \tag{4.4}$$

where $\varepsilon \ll 1$ and N^2 are parameters indicating the amplitude of the thickness defects and the character of the defect location, respectively.

Substituting displacements (4.2) and functions (4.3) and (4.4) into the expression of energy functional (4.1) we omit the terms of higher order than ε and μ^2/m^2. As the form of shell deformation has local character under axial compression, it was mentioned above that the integration over the surface Ω of the shell may be replaced by integration from $-\infty$ to $+\infty$ in both axial and circumferential directions.

5. Asymptotic formula for the critical buckling load parameter

Using the Rayleigh-Ritz method and replacing the finite limits of the integrals by infinite ones, after integration we get the total energy in the following form

$$\begin{aligned} P = \frac{3\pi}{2}\frac{Eh_0}{\mu^2 R^2}\Big[\Big(1+d\frac{\mu^2}{m^2}-\varepsilon B\Big)-\frac{\lambda}{\lambda_1}\Big(1+s\frac{\mu^2}{m^2}-\varepsilon C\Big)b_0^2\Big]+ \\ +\frac{3\pi}{2}\frac{Eh_0}{\mu^2 R^2}\Big[\frac{4m^2}{3R}\Big(1+q\frac{\mu^2}{m^2}-\varepsilon A\Big)b_0^3-2\frac{\lambda}{\lambda_1}h_0k\Big(1+s\frac{\mu^2}{m^2}-\varepsilon C\Big)b_0\Big] \end{aligned} \tag{5.1}$$

where $\lambda_1 = h_0/(cR)$ is the critical buckling load parameter for the perfect shell, $\lambda = \sigma/E$; d, s, q are some constants which depend on the elastic properties of the shell material

$$d = \frac{60+4\nu+7\nu^2-5\nu^3+80(1-\nu^2)}{96(1-\nu^2)}$$

$$s = \frac{3}{8} \qquad\qquad q = \frac{7+5\nu-6\nu^2}{18(1-\nu^2)}$$

A, B, C are parameters that depend on the character of the localization both the initial geometric imperfections and the thickness defects of the shell

$$A = \frac{3}{\sqrt{9+3\frac{N^2}{\mu^2}}} \qquad\qquad B = \frac{2\sqrt{2}}{\sqrt{2+\frac{N^2}{\mu^2}}} \qquad\qquad C = \frac{\sqrt{2}}{\sqrt{2+\frac{N^2}{\mu^2}}}$$

Using the critical balance conditions $\delta P = 0$ and $\delta^2 P = 0$, we obtain an asymptotic formula for the critical buckling load parameter λ^*/λ_1

$$\Big[1+(d-s)\frac{\mu^2}{m^2}+\varepsilon(C-B)-\frac{\lambda^*}{\lambda_1}\Big]^2 = -4ck\frac{\lambda^*}{\lambda_1}\Big[1+(q-s)\frac{\mu^2}{m^2}+\varepsilon(C-A)\Big] \tag{5.2}$$

We note that asymptotic formula (5.2), except for the terms with the parameter ε, is similar to Gristchak's form (1976).

The obtained formula enables us to get values of the critical buckling load parameter λ^*/λ_1 when the initial imperfections are absent. In this case, we put $ck = 0$.

Having calculated the parameter of the critical buckling load for the shell with the localized initial geometric imperfections, we obtain the upper bound for the buckling stress owing to the application of the Rayleigh-Ritz method. In this case, the error may reach about 11% (Gristchak, 1976).

Values of the critical buckling load parameter λ^*/λ_1 for the shell with the local initial geometric imperfections and the local thickness defects are given in Fig. 3.

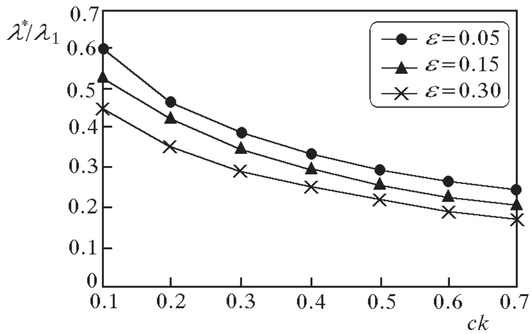

Fig. 3. The effect of the initial imperfections and the thickness defects of the shell on the critical buckling load parameter $(\mu^2/m^2 = 0.1)$

6. Concluding remarks

In the paper some formulas for the axial buckling of a circular cylindrical shell with local thickness defects and local initial geometric imperfections are presented.

The obtained results confirm the following conclusions:

- Analysis of the critical buckling load parameter shows that the effect of certain types of the thickness variation and initial geometric imperfections on the buckling load deserves special attention. Specifically, the results show that the local thickness defects in the absence of the initial geometric imperfections reduce the buckling load compared to the shell with the constant thickness. Even if the amplitude of the local thickness defects is as small as 0.1, the thickness variation reduces the buckling load by 13% from its counterpart of the shell with the constant thickness. When $\varepsilon = 0.15$ the classical buckling load is decreased by 20%. The comparison of the obtained results with those of Koiter's (1994) for the periodical thickness defects shows that the local thickness defects are as harmful as the periodical ones.

- Both the local thickness defects and the initial geometric imperfections of the cylindrical shell have more influence on the classical buckling load, reducing it by 40%-50%. Thus, the thickness and geometric defects may constitute the most important factors in the buckling load reduction.

- The proposed asymptotic method is inapplicable because of increasing error in the case when the amplitude of the thickness defects ε is rather large ($\varepsilon \gg 0.3$). For this reason, we intend to develop a hybrid asymptotic approach in order to obtain results within a wide variation interval of the parameter ε.

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Hybrydowa asymptotyczna metoda szacowania wpływu lokalnych zmian grubości i początkowych zaburzeń strukturalnych na wyboczenie powłok walcowych

Streszczenie

W pracy zaprezentowano podwójnie asymptotyczną metodę szacowania wpływu początkowych zaburzeń strukturalnych i lokalnych uszkodzeń zmieniających grubość na wyboczenie wybranych układów mechanicznych. Zastosowane wyrażenia asymptotyczne pozwalające na określenie krytycznych parametrów wyboczenia otrzymano techniką perturbacyjną.

Manuscript received December 2, 2002; accepted for print April 29, 2003