

ON THE MODELLING OF THIN UNIPERIODIC CYLINDRICAL SHELLS

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The aim of this contribution is to propose a new averaged model of dynamic problems for thin linear-elastic cylindrical shells having a periodic structure along one direction tangent to the shell midsurface. In contrast with the known homogenized models, the proposed one makes it possible to describe the effect of the periodicity cell size on the global dynamic shell behavior (a length-scale effect). In order to derive governing equations with constant or slowly varying coefficients, the known tolerance averaging procedure is applied. The comparison between the proposed model and the model without the length-scale effect as well as the known length-scale model for cylindrical shells with the periodic structure in both directions tangent to the shell midsurface is presented.

Key words: shell, modelling, dynamics, cell

1. Introduction

In this paper a new model of cylindrical shells having a periodic structure (a periodically varying thickness and/or periodically varying elastic and inertial properties) along one direction tangent to the undeformed shell midsurface \mathcal{M} is presented.

Cylindrical shells under consideration are composed of a large number of identical elements which are periodically distributed along one direction tangent to \mathcal{M} . Moreover, every such element is treated as a shallow shell. It means that the period of inhomogeneity is very large compared with the maximum shell thickness and very small as compared to the midsurface curvature radius as well as the smallest characteristic length dimension of the shell midsurface. Structures like that are termed *uniperiodic*.

It should be noted that in the general case, on the shell midsurface we deal with not a periodic structure but with what is called a locally periodic structure in directions tangent to \mathcal{M} . Following Woźniak (1999), it means that every small piece of the shell constituting a shallow shell, with sufficient accuracy, can be described as having a periodic structure related to the Cartesian coordinates on a certain plane tangent to \mathcal{M} . Hence, to every point \mathbf{x} belonging to \mathcal{M} we assign the plane $\mathcal{T}_{\mathbf{x}}$ tangent to \mathcal{M} at this point and periods $l_{\alpha}(\mathbf{x})$, $\alpha = 1, 2$ in the direction of unit vectors $\mathbf{e}_{\alpha}(\mathbf{x})$ on $\mathcal{T}_{\mathbf{x}}$. On every plane $\mathcal{T}_{\mathbf{x}}$ a local periodicity cell spanned on the vectors $(l_{\alpha}\mathbf{e}_{\alpha})(\mathbf{x})$ is defined. For locally uniperiodic shells, the index α is equal to either 1 or 2. For cylindrical shells, the Gaussian curvature is equal to zero, and hence on the developable cylindrical surface we can separate a cell which can be referred to as the representative cell for the whole shell midsurface. It means that on the cylindrical surface we deal with not a locally periodic but with periodic structure.

Problems of periodic (or locally periodic) structures are investigated by means of different methods. The exact analysis of shells and plates of this kind within solid mechanics can be carried out only for a few special problems. In the most cases, the exact equations of the shell (plate) theory are too complicated to constitute the basis for investigations of most engineering problems because they involve highly oscillating and often discontinuous coefficients. Thus, many different approximated modelling methods for periodic (locally periodic) shells and plates have been formulated.

Structures of this kind are usually described using homogenized models derived by means of asymptotic methods. These models from a formal point of view represent certain equivalent structures with constant or slowly varying stiffnesses and averaged mass densities. In the case of periodic plates, these asymptotic homogenization methods were presented by Caillerie (1984) (in this contribution two small parameters – thickness of a plate and the characteristic size of a periodicity cell – are used to investigate periodic plates), Kohn and Vogelius (1984) (this paper deals with thin plates having a rapidly varying thickness), Lewiński (1992) (in this contribution homogenized stiffnesses are analyzed) and others. The asymptotic approach to periodic shells was proposed by Lutoborski (1985), Kalamkarov (1987), Lewiński and Telega (1988); the discussion of the above approach can be found in Woźniak (1999). The formulation of mathematical models of shells by using the asymptotic expansions is rather complicated from the computational point of view. That is why the asymptotic procedures are restricted to the first approximation. Within this approximation, we obtain models which neglect the effect of periodicity cell length dimensions on the global structure behavior (the length-scale effect).

This effect plays an important role mainly in the vibration and wave propagation analysis. To formulate the length-scale models in the framework of the asymptotic homogenization we could find higher-order terms of the asymptotic expansions, cf. Lewiński and Kucharski (1992). Models of this kind have a complicated analytical form, and applied to the investigation of boundary-value problems often lead to a large number of boundary conditions, which may be not well motivated from the physical viewpoint.

The alternative nonasymptotic modelling procedure based on the notion of tolerance and leading to the so-called length-scale (or tolerance) models of dynamic and stationary problems for micro-periodic structures was proposed by Woźniak in a series of papers, e.g. Woźniak (1993, 1997), Woźniak and Wierzbicki (2000). These tolerance models have constant coefficients and take into account the effect of the periodicity cell size on the global body behavior (the length-scale effect). This effect is described by means of certain extra unknowns called *internal or fluctuation variables* and by known functions which represent oscillations inside the periodicity cell, and are obtained as approximate solutions to special eigenvalue problems for free vibrations on the separated cell with periodic boundary conditions. The averaged models of this kind have been applied to analyze certain dynamic problems of periodic structures, e.g. for Hencky-Reissner periodic plates (Baron and Woźniak, 1995), for Kirchhoff periodic plates (Jędrzyński, 1998, 2000), for periodic beams (Mazur-Śniady, 1993), for periodic wavy-plates (Michalak, 1998, 2000), for cylindrical shells with a two-directional periodic structure (Tomczyk, 1999) and others.

A general modelling method based on the concept of internal variables and leading from 2D equations of thin shells with a two-directional locally periodic structure to the averaged equations with slowly varying coefficients depending on the local cell length dimensions has been proposed by Woźniak (1999). However, these internal variable models are not sufficient to analyze problems of shells with a locally periodic (or periodic) structure in only one direction tangent to the undeformed shell midsurface. Shells of this kind, called the locally uniperiodic shells, in general are not special cases of those with a locally periodic structure in both directions tangent to \mathcal{M} .

The aim of this contribution is three-fold:

- First, to derive an averaged model of a uniperiodic cylindrical shell which has constant coefficients in the direction of periodicity and describes the effect of a cell size on the overall shell behavior. The length scales will be introduced to the global description of both inertial and constitutive properties of the shell under consideration. This model will be derived by using the tolerance averaging procedure proposed by Woźniak and

Wierzbicki (2000), and hence will be called *the tolerance fluctuation variable model* for uniperiodic cylindrical shells.

- Second, to derive a simplified (homogenized) model in which the length-scale effect is neglected.
- Third, to compare the proposed here tolerance fluctuation variable model with the homogenized one and with the known tolerance model of cylindrical shells having a periodic structure in both directions tangent to \mathcal{M} .

Basic denotations and starting equations of the shell theory will be presented in Section 2. To make considerations more clear, the general line of the tolerance averaging approach, following the monograph by Woźniak and Wierzbicki (2000), will be presented in Section 3. In the subsequent section, the tolerance model with the fluctuation variables for dynamic problems in linear-elastic thin cylindrical shells with a periodic structure along one direction tangent to \mathcal{M} and a slowly varying structure along the perpendicular one tangent to \mathcal{M} will be shown. For comparison, the governing equations of a certain homogenized model will be presented in Section 5. Final remarks will be formulated in the last section.

2. Preliminaries

In this paper, we will investigate thin linear-elastic cylindrical shells with a periodic structure along one direction tangent to \mathcal{M} and a slowly varying structure along the perpendicular direction tangent to \mathcal{M} . Cylindrical shells of this kind will be termed *uniperiodic*. Examples of such shells are presented in Fig. 1.

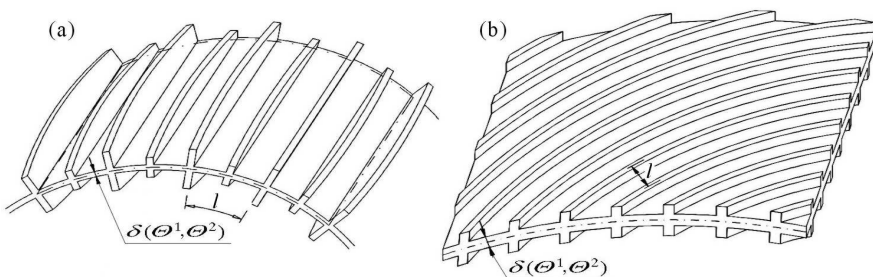


Fig. 1. Examples of uniperiodic shells

Denote by $\Omega \subset R^2$ a regular region of points $\Theta \equiv (\theta^1, \theta^2)$ on the $O\theta^1\theta^2$ -plane, θ^1, θ^2 being the Cartesian orthogonal coordinates on this plane, and let E^3 be the physical space parametrized by the Cartesian orthogonal coordinate system $Ox^1x^2x^3$. Let us introduce the orthogonal parametric representation of the undeformed smooth cylindrical shell midsurface \mathcal{M} by means of $\mathcal{M} := \{\mathbf{x} \equiv (x^1, x^2, x^3) \in E^3 : \mathbf{x} = \mathbf{x}(\theta^1, \theta^2), \Theta \in \Omega\}$, where $\mathbf{x}(\theta^1, \theta^2)$ is a position vector of a point on \mathcal{M} having coordinates θ^1, θ^2 .

Throughout the paper, the indices α, β, \dots run over 1,2 and are related to the midsurface parameters θ^1, θ^2 ; the indices A, B, \dots run over 1, 2, ..., N , the summation convention holds for all aforesaid indices.

To every point $\mathbf{x} = \mathbf{x}(\Theta)$, $\Theta \in \Omega$ we assign covariant base vectors $\mathbf{a}_\alpha = \mathbf{x}_{,\alpha}$ and covariant midsurface first and second metric tensors denoted by $a_{\alpha\beta}$, $b_{\alpha\beta}$, respectively, which are given as follows: $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$, $b_{\alpha\beta} = \mathbf{n} \cdot \mathbf{a}_{\alpha,\beta}$, where \mathbf{n} is a unit vector normal to \mathcal{M} .

Let $\delta(\Theta)$ stand for the shell thickness. We also define t as the time coordinate.

Taking into account that coordinate lines $\theta^2 = \text{const}$ are parallel on the $O\theta^1\theta^2$ -plane and that θ^2 is an arc coordinate on \mathcal{M} , we define l as the period of the shell structure in the θ^2 -direction. The period l is assumed to be sufficiently large compared with the maximum shell thickness and sufficiently small as compared to the midsurface curvature radius R as well as the characteristic length dimension L of the shell midsurface along the direction of shell periodicity, i.e. $\sup \delta(\cdot) \ll l \ll \min\{R, L\}$. On the given above assumptions for the period l , the shell under consideration will be referred to as a *mezostructured shell*, cf. Woźniak (1999), and the period l will be called *the mezostructured length parameter*.

We shall denote by $\Lambda \equiv \{0\} \times (-l/2, l/2)$ the straight line segment on the $O\theta^1\theta^2$ -plane along the $O\theta^2$ -axis direction, which can be taken as a representative cell of the shell periodic structure (the periodicity cell). To every $\Theta \in \Omega$ an arbitrary cell on the $O\theta^1\theta^2$ -plane will be defined by means of: $\Lambda(\Theta) + \Lambda$, $\Theta \in \Omega_\Lambda$, $\Omega_\Lambda := \{\Theta \in \Omega : \Lambda(\Theta) \subset \Omega\}$, where the point $\Theta \in \Omega_\Lambda$ is a center of a cell $\Lambda(\Theta)$ and Ω_Λ is a set of all the cell centers which are inside Ω .

A function $f(\Theta)$ defined on Ω_Λ will be called Λ -periodic if for arbitrary but fixed θ^1 and arbitrary $\theta^2, \theta^2 \pm l$ it satisfies the condition: $f(\theta^1, \theta^2) = f(\theta^1, \theta^2 \pm l)$ in the whole domain of its definition, and it is not constant.

It is assumed that the cylindrical shell thickness as well as its material and inertial properties are Λ -periodic functions of θ^2 and slowly varying

functions of Θ^1 . Shells like that are called *uniperiodic*, moreover, on the given above assumptions for the period l they are referred to *mezostructured shells*.

For an arbitrary integrable function $\varphi(\cdot)$ defined on Ω , following Woźniak and Wierzbicki (2000), we define *the averaging operation*, given by

$$\langle \varphi \rangle(\Theta) \equiv \frac{1}{l} \int_{\Lambda(\Theta)} \varphi(\Theta^1, \Psi^2) d\Psi^2 \quad \Theta = (\Theta^1, \Theta^2) \in \Omega_\Lambda \quad (2.1)$$

For a function φ , which is Λ -periodic in Θ^2 , formula (2.1) leads to $\langle \varphi \rangle(\Theta^1)$. If the function φ is Λ -periodic in Θ^2 and is independent of Θ^1 , its averaged value obtained from (2.1) is constant.

Our considerations will be based on the simplified linear Kirchhoff-Love theory of thin elastic shells in which terms depending on the second metric tensor of \mathcal{M} are neglected in the formulae for curvature changes.

Let $u_\alpha(\Theta, t)$, $w(\Theta, t)$ stand for the midsurface shell displacements in directions tangent and normal to \mathcal{M} , respectively. We denote by $\varepsilon_{\alpha\beta}(\Theta, t)$, $\kappa_{\alpha\beta}(\Theta, t)$ the membrane and curvature strain tensors and by $n^{\alpha\beta}(\Theta, t)$, $m^{\alpha\beta}(\Theta, t)$ the stress resultants and stress couples, respectively. The properties of the shell are described by 2D-shell stiffness tensors $D^{\alpha\beta\gamma\delta}(\Theta)$, $B^{\alpha\beta\gamma\delta}(\Theta)$, and let $\mu(\Theta)$ stand for the shell mass density per midsurface unit area. Let $f_\alpha(\Theta, t)$, $f(\Theta, t)$ be external force components per midsurface unit area, respectively tangent and normal to \mathcal{M} .

Functions $\mu(\Theta)$, $D^{\alpha\beta\gamma\delta}(\Theta)$, $B^{\alpha\beta\gamma\delta}(\Theta)$ and $\delta(\Theta)$ are Λ -periodic functions of Θ^2 and are assumed to be slowly varying functions of Θ^1 .

The equations of the shell theory under consideration consist of:

— the strain-displacement equations

$$\varepsilon_{\gamma\delta} = u_{(\gamma,\delta)} - b_{\gamma\delta}w \quad \kappa_{\gamma\delta} = -w_{,\gamma\delta} \quad (2.2)$$

— the stress-strain relations

$$n^{\alpha\beta} = D^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta} \quad m^{\alpha\beta} = B^{\alpha\beta\gamma\delta}\kappa_{\gamma\delta} \quad (2.3)$$

— the equations of motion

$$\begin{aligned} n_{,\alpha}^{\alpha\beta} - \mu a^{\alpha\beta}\ddot{u}_\alpha + f^\beta &= 0 \\ m_{,\alpha\beta}^{\alpha\beta} + b_{\alpha\beta}n^{\alpha\beta} - \mu\ddot{w} + f &= 0 \end{aligned} \quad (2.4)$$

In the above equations, the displacements $u_\alpha = u_\alpha(\Theta, t)$ and $w = w(\Theta, t)$, φ , are the basic unknowns.

For mezostructured shells, $\mu(\Theta)$, $D^{\alpha\beta\gamma\delta}(\Theta)$ and $B^{\alpha\beta\gamma\delta}(\Theta)$, $\Theta \in \Omega$, are highly oscillating Λ -periodic functions; that is why equations (2.2)-(2.4) cannot be directly applied to the numerical analysis of special problems. From (2.2)-(2.4), an averaged model of uniperiodic cylindrical shells having coefficients, which are independent of the Θ^2 -midsurface parameter, and are slowly varying functions of Θ^1 as well as describing the length-scale effect will be derived. In order to derive it, the tolerance averaging procedure given by Woźniak and Wierzbicki (2000), will be applied. To make the analysis more clear, in the next section we shall outline the basic concepts and the main kinematic assumption of this approach, following the monograph by Woźniak and Wierzbicki (2000).

3. Basic concepts

Following the monograph by Woźniak and Wierzbicki (2000), we outline below the basic concepts, which will be used in the course of the modelling procedure.

The fundamental concepts of the tolerance averaging approach are that of a certain tolerance system, slowly varying functions, periodic-like functions and periodic-like oscillating functions. These functions will be defined with respect to the Λ -periodic shell structure defined in the foregoing section.

By a *tolerance system* we shall mean a pair $T = (\mathcal{F}, \varepsilon(\cdot))$, where \mathcal{F} is a set of real valued bounded functions $F(\Theta)$ defined on $\overline{\Omega}$ and their derivatives (including also time derivatives), which represent the unknowns in the problem under consideration (such as unknown shell displacements tangent and normal to \mathcal{M}), and for which the tolerance parameters ε_F being positive real numbers and determining the admissible accuracy related to computations of values of $F(\cdot)$ are given; by ε the mapping $\mathcal{F} \ni F \rightarrow \varepsilon_F$ is denoted.

A continuous bounded differentiable function $F(\Theta, t)$ defined on $\overline{\Omega}$ is called Λ -*slowly varying* with respect to the cell Λ and the tolerance system T , $F \in SV_\Lambda(T)$, if roughly speaking, can be treated (together with its derivatives) as constant on an arbitrary periodicity cell Λ .

The continuous function $\varphi(\cdot)$ defined on $\overline{\Omega}$ will be termed a Λ -*periodic-like function*, $\varphi(\cdot) \in PL_\Lambda(T)$, with respect to the cell Λ and the tolerance system T , if for every $\Theta = (\Theta^1, \Theta^2) \in \Omega_\Lambda$ there exists a continuous Λ -periodic function $\varphi_\Theta(\cdot)$ such that $(\forall \Psi = (\Psi^1, \Psi^2) [\|\Theta - \Psi\| \leq l \Rightarrow \varphi_\Theta(\Psi)])$, $\Psi \in \Lambda(\Theta)$, and similar conditions are also fulfilled by all its derivatives. It means that the values of the periodic-like function $\varphi(\cdot)$ in an arbitrary cell $\Lambda(\Theta)$, $\Theta \in \Omega_\Lambda$,

can be approximated, with sufficient accuracy, by corresponding values of a certain Λ -periodic function $\varphi_{\Theta}(\cdot)$. The function $\varphi_{\Theta}(\cdot)$ will be referred to as a Λ -periodic approximation of $\varphi(\cdot)$ on $\Lambda(\Theta)$.

Let $\mu(\cdot)$ be a positive value Λ -periodic function. The periodic-like function φ is called Λ -oscillating (with the weight μ), $\varphi(\cdot) \in PL_{\Lambda}^{\mu}(T)$, provided that the condition $\langle \mu\varphi \rangle(\Theta) \cong 0$ holds for every $\Theta \in \Omega_{\Lambda}$.

If $F \in SV_{\Lambda}(T)$, $\varphi(\cdot) \in PL_{\Lambda}(T)$ and $\varphi_{\Theta}(\cdot)$ is a Λ -periodic approximation of $\varphi(\cdot)$ on $\Lambda(\Theta)$, then for every Λ -periodic bounded function $f(\cdot)$ and every continuous Λ -periodic differentiable function $h(\cdot)$ such that $\sup\{|h(\Psi^1, \Psi^2)|, (\Psi^1, \Psi^2) \in \Lambda\} \leq l$, the following *tolerance averaging relations* hold for every $\Theta \in \Omega_{\Lambda}$:

- (T1) $\langle fF \rangle(\Theta) \cong \langle f \rangle(\Theta)F(\Theta)$ for $\varepsilon = \langle |f| \rangle \varepsilon_F$
- (T2) $\langle f(hF)_{,2} \rangle(\Theta) \cong \langle fFh_{,2} \rangle(\Theta)$ for $\varepsilon = \langle |f| \rangle (\varepsilon_F + l\varepsilon_{F,2})$
- (T3) $\langle f\varphi \rangle(\Theta) \cong \langle f\varphi_{\Theta} \rangle(\Theta)$ for $\varepsilon = \langle |f| \rangle \varepsilon_{\varphi}$
- (T4) $\langle h(f\varphi)_{,2} \rangle(\Theta) \cong -\langle f\varphi h_{,2} \rangle(\Theta)$ for $\begin{cases} \varepsilon = \varepsilon_F + l\varepsilon_{F,2} \\ F = \langle hf\varphi \rangle \end{cases}$

where ε is a tolerance parameter which defines the pertinent tolerance \cong .

In the tolerance averaging procedure, the left-hand sides of formulae (T1)-(T4) will be approximated by their right-hand sides, respectively – this operation will be called the *Tolerance Averaging Assumption*.

In the subsequent considerations, the following lemma will be used:

- (L1) If $\varphi(\cdot) \in PL_{\Lambda}(T)$ and f is a bounded Λ -periodic function then $\langle f\varphi \rangle(\cdot) \in SV_{\Lambda}(T)$
- (L2) If $\varphi(\cdot) \in PL_{\Lambda}(T)$ then there exists the decomposition $\varphi(\cdot) = \varphi^0(\cdot) + \tilde{\varphi}(\cdot)$, where $\varphi^0(\cdot) \in SV_{\Lambda}(T)$ and $\tilde{\varphi}(\cdot) \in PL_{\Lambda}^{\mu}(T)$, moreover, it can be shown that $\varphi^0(\cdot) \cong \langle \mu\varphi \rangle(\cdot) \langle \mu \rangle^{-1}$
- (L3) If $F \in SV_{\Lambda}(T)$ and f is a bounded continuous Λ -periodic function then $\langle fF \rangle \in PL_{\Lambda}(T)$
- (L4) If $F \in SV_{\Lambda}(T)$, $G \in SV_{\Lambda}(T)$, $kF + mG \in F$ for some reals k, m , then $kF + mG \in SV_{\Lambda}(T)$.

The main kinematic assumption of the tolerance averaging method is called the *Conformability Assumption* and states that in every periodic solid the

displacement fields have to conform to the periodic structure of this solid. It means that the displacement fields are periodic-like functions and hence can be represented by a sum of the averaged displacements, which are slowly varying, and by highly oscillating periodic-like disturbances, caused by the periodic structure of the solid.

The aforementioned *Conformability Assumption* together with the *Tolerance Averaging Assumption* constitute the foundations of the tolerance averaging technique. Using this technique, the tolerance model of dynamic problems for uniperiodic cylindrical shells will be derived in the subsequent section.

4. The tolerance model of dynamic problems for uniperiodic cylindrical shells

Let us assume that there is a certain tolerance system $T = (\mathcal{F}, \varepsilon(\cdot))$, where the set \mathcal{F} consists of the unknown shell displacements tangent and normal to \mathcal{M} and their derivatives (also time derivatives). From the *Conformability Assumption*, it follows that the unknown shell displacements $u_\alpha(\cdot, t)$, $w(\cdot, t)$ in Eqs (2.2)-(2.4) have to satisfy the conditions: $u_\alpha(\cdot, t) \in PL_\Lambda(T)$, $w(\cdot, t) \in PL_\Lambda(T)$. It means that in every cell $\Lambda(\cdot)$, $\Theta \in \Omega_\Lambda$, the displacement fields can be represented, within a tolerance, by their periodic approximations.

Taking into account Lemma (**L2**), we obtain what is called the *modelling decomposition*

$$\begin{aligned} u_\alpha(\cdot, t) &= U_\alpha(\cdot, t) + d_\alpha(\cdot, t) & w(\cdot, t) &= W(\cdot, t) + p(\cdot, t) \\ U_\alpha(\cdot, t), W(\cdot, t) &\in SV_\Lambda(T) & d_\alpha(\cdot, t), p(\cdot, t) &\in PL_\Lambda^\mu(T) \end{aligned} \quad (4.1)$$

which becomes under the normalizing condition $\langle \mu d_\alpha(\cdot, t) \rangle = \langle \mu p(\cdot, t) \rangle = 0$ (in dynamic problems) or $\langle d_\alpha(\cdot, t) \rangle = \langle p(\cdot, t) \rangle = 0$ (in quasi-stationary problems).

It can be shown, cf. Woźniak and Wierzbicki (2000), that the unknown Λ -slowly varying averaged displacements $U_\alpha(\cdot, t)$, $W(\cdot, t)$ in (4.1) are given by: $U_\alpha(\cdot, t) \equiv \langle \mu \rangle^{-1} \langle \Theta^1 \rangle \langle \mu u_\alpha \rangle(\cdot, t)$, $W(\cdot, t) \equiv \langle \mu \rangle^{-1} \langle \Theta^1 \rangle \langle \mu w \rangle(\cdot, t)$.

The unknown displacement disturbances $d_\alpha(\cdot, t)$, $p(\cdot, t)$ in (4.1) being oscillating periodic-like functions are caused by the highly oscillating character of the shell mezostructure.

Substituting the right-hand side of (4.1) into (2.4), and after the tolerance averaging of the resulting equations, we arrive at the equations

$$[\langle D^{\alpha\beta\gamma\delta} \rangle(\Theta^1)(U_{\gamma,\delta} - b_{\gamma\delta}W) + \langle D^{\alpha\beta\gamma\delta} d_{\gamma,\delta} \rangle(\Theta, t) + b_{\gamma\delta} \langle D^{\alpha\beta\gamma\delta} p \rangle(\Theta, t)]_{,\alpha} - \langle \mu \rangle(\Theta^1) a^{\alpha\beta} \dot{U}_\alpha = - \langle f^\beta \rangle(\Theta, t) \quad (4.2)$$

$$[\langle B^{\alpha\beta\gamma\delta} \rangle(\Theta^1)W_{,\gamma\delta} + \langle B^{\alpha\beta\gamma\delta} p_{,\gamma\delta} \rangle(\Theta, t)]_{,\alpha\beta} - b_{\alpha\beta} [\langle D^{\alpha\beta\gamma\delta} \rangle(\Theta^1)(U_{\gamma,\delta} - b_{\gamma\delta}W) + \langle D^{\alpha\beta\gamma\delta} d_{\gamma,\delta} \rangle(\Theta, t) - b_{\gamma\delta} \langle D^{\alpha\beta\gamma\delta} p \rangle] + \langle \mu \rangle(\Theta^1) \ddot{W} = \langle f \rangle(\Theta, t)$$

which must hold for every $\Theta \in \Omega_\Lambda$ and every time t .

By means of Lemma **(L4)**, the left-hand sides of Eqs (4.2) can be treated as slowly varying functions; hence from Lemma **(L1)** it follows that $\langle f^\beta \rangle(\Theta, t)$, $\langle f \rangle(\Theta, t) \in SV_\Lambda(T)$. This situation takes place if the shell external loadings satisfy the condition: $f^\beta(\Theta, t), f(\Theta, t) \in PL_\Lambda(T)$. This condition is called the *Loading Restriction*.

From the *Loading Restriction* and Lemma **(L2)** it follows that the shell external loadings can be presented as the sum of Λ -slowly varying loadings and Λ -oscillating periodic-like loadings, i.e.

$$\begin{aligned} f^\beta(\cdot, t) &= f_0^\beta + \tilde{f}^\beta(\cdot, t) & f(\cdot, t) &= f_0(\cdot, t) + \tilde{f}(\cdot, t) \\ f_0^\beta(\cdot, t), f_0(\cdot, t) &\in SV_\Lambda(T) & \tilde{f}^\beta(\cdot, t), \tilde{f}(\cdot, t) &\in PL_\Lambda^1(T) \end{aligned} \quad (4.3)$$

where $\langle \tilde{f}^\beta \rangle(\Theta, t) = \langle \tilde{f} \rangle(\Theta, t) \cong 0$.

Multiplying Eqs (2.4) by arbitrary Λ -periodic test functions d^*, p^* , respectively, such that $\langle \mu d^* \rangle = \langle \mu p^* \rangle = 0$, integrating these equations over $\Lambda(\Theta)$, $\Theta \in \Omega_\Lambda$, and using the *Tolerance Averaging Assumption*, as well as denoting by $\tilde{d}_\alpha, \tilde{p}$ the Λ -periodic approximations of d_α, p , respectively, on $\Lambda(\Theta)$, we obtain the periodic problem on $\Lambda(\Theta)$ for functions $\tilde{d}_\alpha(\Theta^1, \Theta^2, t), \tilde{p}(\Theta^1, \Theta^2, t), (\Theta^1, \Theta^2) \in \Lambda(\Theta) = \Lambda(\Theta^1, \Theta^2)$, given by the following variational conditions

$$\begin{aligned} & - \langle d^*_2 D^{2\beta\gamma\delta} \tilde{d}_{\gamma,\delta} \rangle + \langle d^*(D^{1\beta\gamma\delta} \tilde{d}_{\gamma,\delta})_{,1} \rangle - b_{\gamma\delta} [- \langle d^*_2 D^{2\beta\gamma\delta} \tilde{p} \rangle + \\ & + \langle d^*(D^{1\beta\gamma\delta} \tilde{p})_{,1} \rangle] - \langle d^* \mu \tilde{d} \rangle a^{\alpha\beta} = \\ & = - \langle d^* f^\beta \rangle + \langle d^*_{,\alpha} D^{\alpha\beta\gamma\delta} \rangle (U_{\gamma,\delta} - b_{\gamma\delta}W) - [\langle d^* D^{1\beta\gamma\delta} \rangle (U_{\gamma,\delta} - b_{\gamma\delta}W)]_{,1} \\ & \langle p^*_{,22} B^{22\gamma\delta} \tilde{p}_{,\gamma\delta} \rangle - 2 \langle p^*_{,2} (B^{21\gamma\delta} \tilde{p}_{,\gamma\delta})_{,1} \rangle + \langle p^*(B^{11\gamma\delta} \tilde{p}_{,\gamma\delta})_{,11} \rangle + \\ & - b_{\alpha\beta} [\langle p^* D^{\alpha\beta\gamma\delta} \tilde{d}_{\gamma,\delta} \rangle - b_{\gamma\delta} \langle p^* D^{\alpha\beta\gamma\delta} \tilde{p} \rangle] + \langle p^* \mu \tilde{p} \rangle = \\ & = \langle p^* f \rangle + b_{\alpha\beta} \langle p^* D^{\alpha\beta\gamma\delta} \rangle (U_{\gamma,\delta} - b_{\gamma\delta}W) - \langle p^*_{,22} B^{22\lambda\delta} \rangle W_{,\gamma\delta} + \\ & + 2[\langle p^*_{,2} B^{21\gamma\delta} \rangle_{,1} - \langle p^*_{,21} B^{21\gamma\delta} \rangle] W_{,\gamma\delta} + \langle p^*_{,2} B^{21\gamma\delta} \rangle W_{,\gamma\delta 1} + \\ & - \{[\langle p^* B^{11\gamma\delta} \rangle_{,1} - 2 \langle p^*_{,1} B^{11\lambda\delta} \rangle]_{,1} + \langle p^*_{,11} B^{11\gamma\delta} \rangle\} W_{,\gamma\delta} + 2 \langle p^* B^{11\gamma\delta} \rangle_{,1} + \\ & - \langle p^*_{,1} B^{11\gamma\delta} \rangle W_{,\gamma\delta 1} + \langle p^* B^{11\gamma\delta} \rangle W_{,\gamma\delta 11} \} \end{aligned} \quad (4.4)$$

Conditions (4.4) must hold for every Λ -periodic test function d^* and for every Λ -periodic test function p^* , respectively.

Equations (4.2) and (4.4) represent the basis for obtaining the tolerance models for analyzing quasi-stationary and dynamic problems of linear elastic uniperiodic cylindrical shells. In this work, the model of dynamic problems will be derived.

In order to obtain solutions to the periodic problems on $\Lambda(\Theta)$, given by variational equations (4.4), we can apply the orthogonalization method known in the dynamics of elastic shells and plates.

The right-hand sides of Eqs (4.4) can be interpreted as certain time dependent loadings on the cell $\Lambda(\Theta)$. In the absence of these loadings we obtain from (4.4) a periodic problem on $\Lambda(\Theta)$ given by

$$\begin{aligned}
 (D^{2\beta\gamma^2}\tilde{d}_{\gamma,2} - D^{2\beta 22}b_{22}\tilde{p}),_2 - \mu a^{\alpha\beta}\tilde{d}_\alpha &= 0 \\
 (B^{2222}\tilde{p}_{,22})_{,22} - b_{22}(D^{22\gamma^2}\tilde{d}_{\gamma,2} - b_{22}D^{2222}\tilde{p}) + \mu\tilde{p} &= 0
 \end{aligned}
 \tag{4.5}$$

which on the assumption that $\tilde{d}_\alpha(\Theta^1, \Psi^2, t) = h_\alpha(\Theta^1, \Psi^2) \cos(\omega t)$, $\tilde{p}(\Theta^1, \Psi^2, t) = g(\Theta^1, \Psi^2) \cos(\omega t)$, $(\Theta^1, \Psi^2) \in \Lambda(\Theta)$, $\Theta = (\Theta^1, \Theta^2) \in \Omega_\Lambda$, leads to the periodic eigenvalue problem of finding Λ -periodic functions h_α, g given by the equations

$$\begin{aligned}
 [D^{2\beta\gamma^2}(\Theta^1, \Psi^2)h_{\gamma,2}(\Theta^1, \Psi^2)]_{,2} + \mu(\Theta^1, \Psi^2)[\omega(\Theta^1)]^2 a^{\alpha\beta}h_\alpha(\Theta^1, \Psi^2) &= 0 \\
 [B^{2222}(\Theta^1, \Psi^2)g_{22}(\Theta^1, \Psi^2)]_{,22} - \mu(\Theta^1, \Psi^2)[\omega(\Theta^1)]^2 g(\Theta^1, \Psi^2) &= 0
 \end{aligned}
 \tag{4.6}$$

and by the periodic boundary conditions on the cell $\Lambda(\Theta)$ together with the continuity conditions inside $\Lambda(\Theta)$. By averaging the above equations over $\Lambda(\Theta)$, we obtain $\langle \mu h_\alpha \rangle(\Theta^1) = \langle \mu g \rangle(\Theta^1) = 0$.

Let $h_\alpha^1(\Theta^1, \Psi^2), g^1(\Theta^1, \Psi^2), h_\alpha^2(\Theta^1, \Psi^2), g^2(\Theta^1, \Psi^2), \dots$, be a sequence of eigenfunctions related to the sequence of eigenvalues $[\omega_\alpha^2, \omega^2]_1, [\omega_\alpha^2, \omega^2]_2, \dots$. For arbitrary Θ^1 and $(\Theta^1, \Psi^2) \in \Lambda(\Theta)$, $\Theta = (\Theta^1, \Theta^2) \in \Omega_\Lambda$ we can look for solutions to the periodic problem (4.4) in the form of the finite series

$$\begin{aligned}
 \tilde{d}_\alpha(\Theta^1, \Psi^2, t) &= h^A(\Theta^1, \Psi^2)Q_\alpha^A(\Theta^1, \Theta^2, t) \\
 \tilde{p}(\Theta^1, \Psi^2, t) &= g^A(\Theta^1, \Psi^2)V^A(\Theta^1, \Theta^2, t) \quad A = 1, 2, \dots, N
 \end{aligned}
 \tag{4.7}$$

in which the choice of the number N of terms determines different degrees of approximations.

The functions $h^A(\Theta^1, \cdot)$, $g^A(\Theta^1, \cdot)$, $A = 1, \dots, N$, are called *the mode-shape functions* and are assumed to be known in every problem under consideration. They are linear independent, l -periodic and such that h^A , $lh_{,2}^A$, $l^{-1}g^A$, $g_{,2}^A$, $lg_{,22}^A \in \mathcal{O}(l)$ and $\max |h^A(\Theta^1, \Psi^2)| \leq l$, $\max |g^A(\Theta^1, \Psi^2)| \leq l^2$ as well as $\langle \mu h^A \rangle(\Theta^1) = \langle \mu g^A \rangle(\Theta^1) = 0$ for every A and $\langle \mu h^A h^B \rangle(\Theta^1) = \langle \mu g^A g^B \rangle(\Theta^1) = 0$ for every $A \neq B$.

In most problems, the analysis will be restricted to the simplest case $N = 1$ in which we take into account only the lowest natural vibration modes (in the direction tangent and normal to \mathcal{M}) related to Eqs (4.6).

The functions $Q_\alpha^A(\Theta^1, \Theta^2, t)$, $V^A(\Theta^1, \Theta^2, t)$ in (4.7) represent new unknowns, called *the fluctuation variables*. Because the functions $\tilde{d}_\alpha(\Theta^1, \Psi^2, t)$, $\tilde{p}(\Theta^1, \Psi^2, t)$ are the Λ -periodic approximations of $d_\alpha(\Theta^1, \Psi^2, t)$, $p(\Theta^1, \Psi^2, t)$, respectively, on the cell Λ and $d_\alpha(\Theta^1, \Psi^2, t), p(\Theta^1, \Psi^2, t) \in PL_\Lambda^\mu(T)$, then from (4.7) and from Lemma **(L3)** it follows that the functions $Q_\alpha^A(\Theta^1, \Theta^2, t)$, $V^A(\Theta^1, \Theta^2, t)$, $A = 1, 2, \dots, N$, are Λ -slowly varying functions in Θ^2 , i.e. $Q_\alpha^A, V^A \in SV_\Lambda(T)$.

Substituting the right-hand sides of (4.7) into (4.2) and (4.4), setting $d^* = h^A(\Theta^1, \Psi^2)$, $p^* = g^A(\Theta^1, \Psi^2)$, $A = 1, 2, \dots, N$, in (4.4) and taking into account (4.3), on the basis of the *Tolerance Averaging Assumption* we arrive at the *tolerance fluctuation variable model of dynamic problems for unperiodic cylindrical shells*. Under extra denotations

$$\begin{aligned}
 \tilde{D}^{\alpha\beta\gamma\delta} &\equiv \langle D^{\alpha\beta\gamma\delta} \rangle & D^{A\alpha\beta\gamma} &\equiv \langle D^{\alpha\beta\gamma\delta} h_{,\delta}^A \rangle \\
 \overline{D}^{A\alpha\beta\gamma} &\equiv l^{-1} \langle D^{\alpha\beta\gamma 1} h^A \rangle & L^{A\alpha\beta} &\equiv l^{-2} b_{\gamma\delta} \langle D^{\alpha\beta\gamma\delta} g^A \rangle \\
 \tilde{B}^{\alpha\beta\gamma\delta} &\equiv \langle B^{\alpha\beta\gamma\delta} \rangle & K^{A\alpha\beta} &\equiv \langle B^{\alpha\beta\gamma\delta} g_{,\gamma\delta}^A \rangle \\
 \overline{K}^{A\alpha\beta} &\equiv l^{-1} \langle B^{\alpha\beta 1\delta} g_{,\delta}^A \rangle & \check{K}^{A\alpha\beta} &\equiv l^{-2} \langle B^{\alpha\beta 11} g^A \rangle \\
 C^{AB\beta\gamma} &\equiv \langle D^{\alpha\beta\gamma\delta} h_{,\alpha}^A h_{,\delta}^B \rangle & \overline{C}^{AB\beta\gamma} &\equiv l^{-1} \langle D^{\alpha\beta\gamma 1} h_{,\alpha}^A h^B \rangle \\
 F^{AB\beta} &\equiv l^{-2} b_{\gamma\delta} \langle D^{\alpha\beta\gamma\delta} h_{,\alpha}^A g_{,\delta}^B \rangle & \tilde{C}^{AB\beta\gamma} &\equiv l^{-2} \langle D^{1\beta\gamma 1} h^A h^B \rangle \\
 \overline{F}^{AB\beta} &\equiv l^{-3} b_{\gamma\delta} \langle D^{1\beta\gamma\delta} h^A g_{,\delta}^B \rangle & R^{AB} &\equiv \langle B^{\alpha\beta\gamma\delta} g_{,\alpha\beta}^A g_{,\gamma\delta}^B \rangle \\
 \overline{L}^{AB} &\equiv l^{-4} b_{\alpha\beta} b_{\gamma\delta} \langle D^{\alpha\beta\gamma\delta} g^A g^B \rangle & \check{R}^{AB} &\equiv l^{-1} \langle B^{1\beta\gamma\delta} g_{,\beta}^A g_{,\gamma\delta}^B \rangle \\
 \tilde{R}^{AB} &\equiv l^{-2} \langle B^{11\gamma\delta} g_{,\gamma\delta}^A g^B \rangle & \overline{R}^{AB} &\equiv l^{-3} \langle B^{1\beta 11} g_{,\beta}^A g^B \rangle \\
 \hat{R}^{AB} &\equiv l^{-4} \langle B^{1111} g^A g^B \rangle & \tilde{S}^{AB} &\equiv l^{-2} \langle B^{1\gamma 1\delta} g_{,\gamma}^A g_{,\delta}^B \rangle \\
 \tilde{\mu} &\equiv \langle \mu \rangle & \tilde{\mu}^{AB} &\equiv l^{-2} \langle \mu h^A h^B \rangle \\
 \overline{\mu}^{AB} &\equiv l^{-4} \langle \mu g^A g^B \rangle & \tilde{P}^{A\beta} &\equiv l^{-1} \langle \tilde{f}^\beta h^A \rangle \\
 \tilde{P}^A &\equiv l^{-2} \langle \tilde{f} g^A \rangle
 \end{aligned} \tag{4.8}$$

this model is represented by:

— the constitutive equations

$$\begin{aligned}
 N^{\alpha\beta} &= \tilde{D}^{\alpha\beta\gamma\delta}(U_{\gamma,\delta} - b_{\gamma\delta}W) + D^{B\alpha\beta\gamma}Q_\gamma^B + \underline{\underline{D}}^{B\alpha\beta\gamma}Q_{\gamma,1}^B - \underline{\underline{l^2L}}^{B\alpha\beta}V^B \\
 M^{\alpha\beta} &= \tilde{B}^{\alpha\beta\gamma\delta}W_{,\gamma\delta} + K^{B\alpha\beta}V^B + 2\underline{\underline{lK}}^{B\alpha\beta}V_{,1}^B + \underline{\underline{l^2\check{K}}}^{B\alpha\beta}V_{,11}^B \\
 H^{AB} &= D^{A\beta\gamma\delta}(U_{\gamma,\delta} - b_{\gamma\delta}W) + C^{AB\beta\gamma}Q_\gamma^B + \underline{\underline{lC}}^{AB\beta\gamma}Q_{\gamma,1}^B - \underline{\underline{l^2F}}^{AB\beta\gamma}V^B \\
 \overline{H}^{AB} &\equiv \underline{\underline{lD}}^{A\beta\gamma\delta}(U_{\gamma,\delta} - b_{\gamma\delta}W) + \underline{\underline{lC}}^{AB\beta\gamma}Q_\gamma^B + \underline{\underline{l^2\check{C}}}^{AB\beta\gamma}Q_{\gamma,1}^B - \underline{\underline{l^3\overline{F}}}^{AB\beta\gamma}V^B \quad (4.9) \\
 G^A &\equiv -\underline{\underline{l^2L}}^{A\gamma\delta}(U_{\gamma,\delta} - b_{\gamma\delta}W) + K^{A\alpha\beta}W_{,\alpha\beta} - \underline{\underline{l^2F}}^{AB\gamma}Q_\gamma^B - \underline{\underline{l^3\overline{F}}}^{AB\gamma}Q_{\gamma,1}^B + \\
 &\quad + (R^{AB} + \underline{\underline{l^4\check{L}}}^{AB})V^B + 2\underline{\underline{l\check{R}}}^{AB}V_{,1}^B + \underline{\underline{l^2\check{R}}}^{AB}V_{,11}^B \\
 \tilde{G}^A &= \underline{\underline{l^2K}}^{A\alpha\beta}W_{,\alpha\beta} + \underline{\underline{l^2\check{R}}}^{AB}V^B + 2\underline{\underline{l^3\overline{R}}}^{AB}V_{,1}^B + \underline{\underline{l^4\hat{R}}}^{AB}V_{,11}^B \\
 \overline{G}^A &= \underline{\underline{lK}}^{A\alpha\beta}W_{,\alpha\beta} + \underline{\underline{l\check{R}}}^{AB}V^B + 2\underline{\underline{l^2\check{S}}}^{AB}V_{,1}^B + \underline{\underline{l^3\overline{R}}}^{AB}V_{,11}^B
 \end{aligned}$$

— the system of three averaged partial differential equations of motion for the averaged displacements $U_\alpha(\Theta, t)$, $W(\Theta, t)$

$$N_{,\alpha}^{\alpha\beta} - \tilde{\mu}a^{\alpha\beta}\ddot{U}_\alpha + f_0^\beta = 0 \tag{4.10}$$

$$M_{,\alpha\beta}^{\alpha\beta} - b_{\alpha\beta}N^{\alpha\beta} + \tilde{\mu}\ddot{W} - f_0 = 0$$

— the system of $3N$ partial differential equations for the fluctuation variables $Q_\alpha^B(\Theta, t)$, $V^B(\Theta, t)$, $B = 1, 2, \dots, N$, called the dynamic evolution equations

$$\underline{\underline{l^2\tilde{\mu}^{AB}}}a^{\gamma\beta}\ddot{Q}_\gamma^B + H^{AB} - \overline{H}_{,1}^{AB} - \underline{\underline{l\tilde{P}}}^{AB} = 0 \tag{4.11}$$

$$\underline{\underline{l^4\check{\mu}^{AB}}}\ddot{V}^B + G^A + \tilde{G}_{,11}^A - 2\overline{G}_{,1}^A - \underline{\underline{l^2\tilde{P}}}^A = 0 \quad A, B = 1, 2, \dots, N$$

The above model has a physical sense provided that the basic unknowns $U_\alpha(\Theta, t)$, $W(\Theta, t)$, $Q_\gamma^A(\Theta, t)$, $V^A(\Theta, t) \in SV_\Lambda(T)$, $A = 1, 2, \dots, N$, i.e. they are Λ -slowly varying functions of the Θ^2 -midsurface parameter.

Taking into account (4.1) and (4.7), the shell displacement fields can be approximated by means of formulae

$$u_\alpha(\cdot, t) \approx U_\alpha(\cdot, t) + h^A(\cdot)Q_\alpha^A(\cdot, t) \tag{4.12}$$

$$w(\cdot, t) \approx W(\cdot, t) + g^A(\cdot)V^A(\cdot, t) \quad A = 1, 2, \dots, N$$

where the approximation \approx depends on the number of terms $h^A(\cdot)Q_\alpha^A(\cdot, t)$, $g^A(\cdot)V^A(\cdot, t)$.

The characteristic features of the derived model are:

- The model takes into account the effect of the cell size on the overall dynamic shell behavior; this effect is described by the underlined coefficients dependent on the mezostructure length parameter l .
- The model equations involve averaged coefficients which are independent of the Θ^2 -midsurface parameter (i.e. they are constant in direction of periodicity), and are slowly varying functions of Θ^1 .
- The number and form of the boundary conditions for the averaged displacements $U_\alpha(\boldsymbol{\Theta}, t)$, $W(\boldsymbol{\Theta}, t)$ are the same as in the classical shell theory governed by equations (2.2)-(2.4). The boundary conditions for the fluctuation variables $Q_\gamma^A(\boldsymbol{\Theta}, t)$, $V^A(\boldsymbol{\Theta}, t)$ should be defined only on the boundaries $\Theta^1 = \text{const}$.
- It is easy to see that in order to derive governing equations (4.9)-(4.11), we have to obtain the mode-shape functions $h^A(\Theta^1, \Psi^2)$, $g^A(\Theta^1, \Psi^2)$, $A = 1, 2, \dots, N$, as solutions to the periodic eigenvalue problem given by (4.6). In practice, derivation of these exact solutions is possible only for cells with a structure which is not too complicated. In most cases, these eigenfunctions have to be obtained by using approximate methods. Moreover, for uniperiodic shells, the mode-shape functions are periodic in only one direction; in this work they are l -periodic functions only of the Θ^2 -midsurface parameter.

Assuming that the cylindrical shell under consideration has material and geometrical properties independent of Θ^1 , we obtain governing equations (4.9)-(4.11) with constant averaged coefficients. Moreover, in this case the mode-shape functions h^A , g^A , $A = 1, 2, \dots, N$, are also independent of the Θ^1 -midsurface parameter.

For a homogeneous shell, $\mu(\boldsymbol{\Theta})$, $D^{\alpha\beta\gamma\delta}(\boldsymbol{\Theta})$ and $B^{\alpha\beta\gamma\delta}(\boldsymbol{\Theta})$, $\boldsymbol{\Theta} \in \Omega$ are constant, and because $\langle \mu h^A \rangle = \langle \mu g^A \rangle = 0$ we obtain $\langle h^A \rangle = \langle g^A \rangle = 0$, and hence $\langle h_{,\alpha}^A \rangle = \langle g_{,\alpha}^A \rangle = \langle g_{,\alpha\beta}^A \rangle = 0$. In this case, equations (4.10) reduce to the well known linear-elastic shell equations of motion for the averaged displacements $U_\alpha(\boldsymbol{\Theta}, t)$, $W(\boldsymbol{\Theta}, t)$, and independently for $Q_\alpha^A(\boldsymbol{\Theta}, t)$, $V^A(\boldsymbol{\Theta}, t)$ we arrive at a system of N differential equations. In this case, under the condition $\tilde{f}^\beta = \tilde{f} = 0$ and for the initial conditions $Q_\alpha^A(\boldsymbol{\Theta}, t_0) = V^A(\boldsymbol{\Theta}, t_0) = 0$, $A = 1, 2, \dots, N$, we obtain $Q_\alpha^A = V^A = 0$; hence constitutive equations (4.9) and equations of motion (4.10) reduce to starting equations (2.3) and (2.4), respectively.

At the end of this section let us compare the obtained above *tolerance fluctuation variable model* of uniperiodic cylindrical shells with the *tolerance*

internal variable model of shells having a locally periodic structure in both directions tangent to \mathcal{M} , which were proposed by Woźniak (1999), and used to analyze dynamic problems of cylindrical shells with two-directional periodicity by Tomczyk (1999). In the sequel, cylindrical shells having a periodic structure in both directions tangent to \mathcal{M} will be termed *biperiodic*, cf. Woźniak and Wierzbicki (2000). An example of such a shell is presented in Fig. 2.

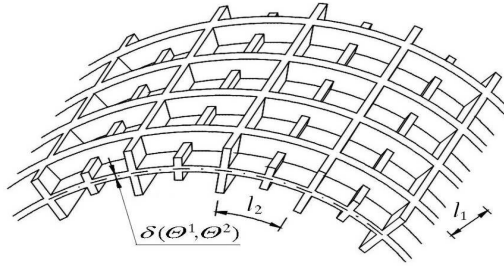


Fig. 2. Example of a biperiodic shell

Following Tomczyk (1999), the governing equations of the *tolerance internal variable model* of cylindrical biperiodic shells is represented by:

— the constitutive equations ($A, B = 1, 2, \dots, N$)

$$\begin{aligned}
 N^{\alpha\beta} &= \langle D^{\alpha\beta\gamma\delta} \rangle (U_{\gamma,\delta} - b_{\gamma\delta}W) + \langle D^{\alpha\beta\gamma\delta} h_{,\delta}^B \rangle Q_\gamma^B - \langle D^{\alpha\beta\gamma\delta} g^B \rangle b_{\gamma\delta}V^B \\
 M^{\alpha\beta} &= -\langle B^{\alpha\beta\gamma\delta} \rangle W_{,\gamma\delta} - \langle B^{\alpha\beta\gamma\delta} g_{,\gamma\delta}^B \rangle V^B \\
 H^{A\beta} &= \langle D^{\alpha\beta\gamma\delta} h_{,\alpha}^A \rangle (U_{\gamma,\delta} - b_{\gamma\delta}W) + \langle D^{\alpha\beta\gamma\delta} h_{,\alpha}^A h_{,\delta}^B \rangle Q_\gamma^B - b_{\gamma\delta} \langle D^{\alpha\beta\gamma\delta} h_{,\alpha}^A g^B \rangle V^B \\
 G^A &= -b_{\alpha\beta} \langle D^{\alpha\beta\gamma\delta} g^A \rangle (U_{\gamma,\delta} - b_{\gamma\delta}W) + \langle B^{\alpha\beta\gamma\delta} g_{,\gamma\delta}^A \rangle W_{,\alpha\beta} + \\
 &\quad - b_{\alpha\beta} \langle D^{\alpha\beta\gamma\delta} g^A h_{,\delta}^B \rangle Q_\gamma^B + (\langle B^{\alpha\beta\gamma\delta} g_{,\alpha\beta}^A g_{,\gamma\delta}^B \rangle + b_{\alpha\beta} \langle D^{\alpha\beta\gamma\delta} g^A g^B \rangle b_{\gamma\delta}) V^B
 \end{aligned}
 \tag{4.13}$$

— the system of three averaged partial differential equations of motion for the averaged displacements $U_\alpha(\Theta, t)$, $W(\Theta, t)$

$$\begin{aligned}
 N_{,\alpha}^{\alpha\beta} - \langle \mu \rangle a^{\alpha\beta} \ddot{U}_\alpha + f_0^\beta &= 0 \\
 M_{,\alpha\beta}^{\alpha\beta} + b_{\alpha\beta} N^{\alpha\beta} - \langle \mu \rangle \ddot{W} + f_0 &= 0
 \end{aligned}
 \tag{4.14}$$

— the system of $3N$ ordinary differential equations for the fluctuation variables $Q_\alpha^A(\Theta, t)$, $V^A(\Theta, t)$ called the dynamic evolution equations

$$\begin{aligned} \langle \underline{\mu h^A h^B} \rangle a^{\gamma\beta} \ddot{Q}_\gamma^B + H^{A\beta} + \langle \underline{\tilde{f}^\beta h^A} \rangle &= 0 \\ \langle \underline{\mu g^A g^B} \rangle \ddot{V}^B + G^A + \langle \underline{\tilde{f} g^A} \rangle &= 0 \quad A, B = 1, 2, \dots, N \end{aligned} \quad (4.15)$$

where the basic unknowns $U_\alpha(\boldsymbol{\Theta}, t)$, $W(\boldsymbol{\Theta}, t)$, $Q_\alpha^A(\boldsymbol{\Theta}, t)$, $V^A(\boldsymbol{\Theta}, t)$, $A = 1, 2, \dots, N$, are slowly varying functions (with respect to the two-dimensional periodicity cell and tolerance system) in Θ^1 and Θ^2 alike, and also the mode-shape functions h^A , g^A are l -periodic functions in both Θ^1 and Θ^2 . Equations (4.13)-(4.15) have constant coefficients; the underlined terms depend on the mezostructure length parameter l , and hence describe the effect of the cell size on the overall shell behavior.

Comparing Eqs (4.13)-(4.15) and (4.9)-(4.11) it can be seen that Eqs (4.13)-(4.15) for biperiodic shells can be obtained from Eqs (4.9)-(4.11) by neglecting in (4.9) the singly underlined terms; it means that the tolerance model of biperiodic cylindrical shells is a special case of that for uniperiodic shells proposed in this paper. The main differences between both models are:

- in the model of a uniperiodic shell we deal with functions which are slowly varying or periodic-like (with respect to the cell and tolerance system) in only one direction, while in the other one these functions are slowly-varying or periodic-like in two directions
- within the framework of the model of uniperiodic shells, the unknowns $Q_\alpha^A(\boldsymbol{\Theta}, t)$, $V^A(\boldsymbol{\Theta}, t)$, $A = 1, 2, \dots, N$, are governed by the system of $3N$ partial differential equations (4.11), while within the framework of the model of biperiodic shells these unknowns are governed by the system of $3N$ ordinary differential equations involving only time derivatives; hence there are no extra boundary conditions for these functions, and that is why they play the role of kinematic internal variables, cf. Woźniak (1999).

In the next section the homogenized model of uniperiodic cylindrical shells will be derived as a special case of Eqs (4.9)-(4.11).

5. Homogenized model

The simplified model of uniperiodic cylindrical shells can be derived directly from the tolerance model, (4.9)-(4.11), by the limit passage $l \rightarrow 0$, i.e.

by neglecting the underlined terms which depend on the mezostructure length parameter l . Hence, Eqs (4.11) yield

$$\begin{aligned} C^{AB\beta\gamma} Q_\gamma^B &= -D^{A\beta\gamma\delta} (U_{\gamma,\delta} - b_{\gamma\delta} W) \\ R^{AB} V^A &= -K^{B\gamma\delta} W_{,\gamma\delta} \end{aligned} \tag{5.1}$$

From the positive definiteness of the strain energy it follows that the $N \times N$ matrix of the elements R^{AB} is non-singular, and the linear transformation determined by the components $C^{AB\beta\gamma}$ is invertible. Hence a solution to equations (5.1) can be written in the form

$$\begin{aligned} Q_\gamma^B &= -G_{\gamma\eta}^{BC} D^{C\eta\mu\vartheta} (U_{\mu,\vartheta} - b_{\mu\vartheta} W) \\ V^A &= -E^{AB} K^{B\gamma\delta} W_{,\gamma\delta} \end{aligned} \tag{5.2}$$

where $G_{\alpha\beta}^{AB}$ and E^{AB} are defined by

$$G_{\alpha\beta}^{AB} C^{BC\beta\gamma} = \delta_\alpha^\gamma \delta^{AC} \qquad E^{AB} R^{BC} = \delta^{AC}$$

Setting

$$\begin{aligned} D_{eff}^{\alpha\beta\gamma\delta} &\equiv \tilde{D}^{\alpha\beta\gamma\delta} - D^{A\alpha\beta\eta} G_{\eta\xi}^{AB} D^{B\xi\gamma\delta} \\ B_{eff}^{\alpha\beta\gamma\delta} &\equiv \tilde{B}^{\alpha\beta\gamma\delta} - K^{A\alpha\beta} E^{AB} K^{B\gamma\delta} \end{aligned}$$

and substituting expression (5.2) into constitutive equations (4.9)_{1,2}, in which the underlined terms are neglected, we arrive at the homogenized shell model governed by:

— *equations of motion*

$$D_{eff}^{\alpha\beta\gamma\delta} (U_{\gamma,\delta\alpha} - b_{\gamma\delta} W_{,\alpha}) - \langle \mu \rangle a^{\alpha\beta} \ddot{U}_\alpha + f_0^\beta = 0 \tag{5.3}$$

$$B_{eff}^{\alpha\beta\gamma\delta} W_{,\alpha\beta\gamma\delta} - b_{\alpha\beta} D_{eff}^{\alpha\beta\gamma\delta} (U_{\gamma,\delta} - b_{\gamma\delta} W) + \langle \mu \rangle \ddot{W} - f_0 = 0$$

— *constitutive equations*

$$N^{\alpha\beta} = D_{eff}^{\alpha\beta\gamma\delta} (U_{\gamma,\delta} - b_{\gamma\delta} W) \qquad M^{\alpha\beta} = -B_{eff}^{\alpha\beta\gamma\delta} W_{,\gamma\delta} \tag{5.4}$$

where $D_{eff}^{\alpha\beta\gamma\delta}$, $B_{eff}^{\alpha\beta\gamma\delta}$ are called the effective stiffnesses.

The obtained above homogenized model governed by Eqs (5.3), (5.4) is not able to describe the length-scale effect on the overall dynamic shell behavior being independent of the mezostructure length parameter l .

In order to show differences between the results obtained from the tolerance uniperiodic shell model, (4.9)-(4.11), and from the homogenized model, (5.3) and (5.4), free vibrations of the uniperiodic cylindrical shell will be analyzed in the second part of this paper.

6. Final remarks

The subject-matter of this contribution is a thin linear-elastic cylindrical shell having a periodic structure in one direction tangent to the undeformed shell midsurface \mathcal{M} . Shells of this kind are termed uniperiodic. For these shells, equations governed by the Kirchhoff-Love shell theory involve highly oscillating periodic coefficients.

In order to simplify the Kirchhoff-Love shell theory to the form which can be applied to engineering problems and also can take into account the effect of the periodicity cell on the overall dynamic shell behavior a new model of thin uniperiodic cylindrical shells has been proposed. In order to derive it, the *tolerance averaging procedure* given by Woźniak and Wierzbicki (2000), has been applied. This model, called the tolerance model, is represented by a system of partial differential equations (4.10) and (4.11) with coefficients which are constant in the direction of periodicity. The basic unknowns are: the *averaged displacements* U_α , W and the *fluctuation variables* Q_α^A , V^A , $A = 1, 2, \dots, N$, which have to be *slowly varying* functions with respect to the cell and certain tolerance system. This requirement imposes certain restrictions on the class of problems described by the model under consideration. In order to obtain the governing equations, the *mode-shape functions* h^A , g^A , $A = 1, 2, \dots, N$, should be derived as approximated solutions to eigenvalue problems on the periodicity cell with periodic boundary conditions.

In contrast with the homogenized models, the proposed one makes it possible to describe the effect of the periodicity cell on the overall shell behavior (the length-scale effect). The length scale is introduced to the global description of both inertial and constitutive properties of the shell under consideration.

Comparing the proposed here *tolerance fluctuation variable model* for uniperiodic cylindrical shells given by Eqs (4.9)-(4.11), and the known *tolerance internal variable model* for biperiodic cylindrical shells (i.e. shells with a periodic structure in both directions tangent to \mathcal{M}) governed by Eqs (4.13)-(4.15), it is seen that the equations for uniperiodic shells contain the singly underlined terms which have no counterparts in the equations for biperiodic shells. Moreover, for uniperiodic shells, the unknowns Q_α^A , V^A , $A = 1, 2, \dots, N$, are governed by a system of $3N$ partial differential equations (4.11), and hence do not play the role of kinematic internal variables, unlike the unknowns Q_α^A , V^A , $A = 1, 2, \dots, N$ in Eqs (4.15). It means that the tolerance model of biperiodic shells is a special case of that describing the uniperiodic shells proposed in this paper, and hence the biperiodic shell model is not sufficient to analyze dynamic problems of shells with a periodic structure in one direction tangent to \mathcal{M} .

The problems related to various applications of proposed Eqs (4.9)-(4.11) to the dynamics of uniperiodic cylindrical shells are reserved for a separate paper.

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Modelowanie cienkich powłok walcowych o jednokierunkowej periodycy

Streszczenie

Celem pracy jest wyprowadzenie uśrednionego modelu służącego do analizy dynamiki cienkich liniowo-sprężystych powłok walcowych mających periodyczną strukturę w jednym kierunku stycznym do powierzchni środkowej powłoki. Proponowany model, w przeciwieństwie do znanych modeli zhomogenizowanych, umożliwia badanie wpływu wielkości komórki periodyczności na dynamikę powłoki walcowej (wpływ ten zwany jest efektem skali). W celu wyprowadzenia równań o stałych lub wolnozmiennych współczynnikach zastosowano znaną metodę tolerancyjnego uśredniania. Wyprowadzony model porównano z modelem dla powłoki walcowej z periodyką w dwóch kierunkach wzajemnie prostopadłych i stycznych do powierzchni środkowej powłoki oraz z modelem bez efektu skali.

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